Lecture 17

§1. Some special cases of general results proved so far:

(a) $\quad \mathrm{Poly}_n \cong S_n\text{-repns} \qquad \left[\begin{array}{l} F \overset{\varphi}{\longmapsto} L_F^{(n)}(\mathbb{C}) \\ \{X \mapsto (M \otimes X^{\otimes n})^{S_n}\} \underset{\psi}{\longleftarrow\!\shortmid} M \end{array} \right]$

$$T^n \longleftrightarrow \mathbb{C} S_n$$

(Schur functors) $\quad \mathbb{S}^\lambda \longleftrightarrow V_\lambda \quad$ (irred. repns.) $\qquad (\lambda \vdash n)$

$$\mathrm{Sym}^{\lambda_1} \otimes \cdots \otimes \mathrm{Sym}^{\lambda_\ell} \longleftrightarrow U_\lambda = \mathrm{Ind}_{S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}}^{S_n} (\text{Trivial})$$

(b) $\quad \mathrm{Tr}_{(m)} : K(\mathrm{Poly}_n) \to \mathbb{C}[x_1, \ldots, x_m]^{S_m}_{\deg = n}$

$$F \longmapsto \text{Trace of } F\left(\begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix}\right) \text{ on } F(\mathbb{C}^m)$$

Then, $\quad \mathrm{Tr}(\psi(M)) = \mathrm{ch}(\chi_M)$

(Frobenius characteristic map)

i.e., the following diagram commutes:

$$\begin{tikzcd} K(\mathrm{Poly}_n) \arrow[rr, shift left, "\varphi"] \arrow[dr, "\mathrm{Tr}"'] & & R(S_n) \arrow[ll, shift left, "\psi"] \arrow[dl, "\mathrm{ch}"] \\ & \mathbb{Z}[x_1, \ldots, x_m]^{S_m}_{\deg = n} & \end{tikzcd}$$

(Recall: $K(\mathrm{Poly}_n)$ = Grothendieck group of the category Poly_n

= abelian group on $[F]$ - iso classes of $F \in \mathrm{Poly}_n$

subject to $[F \oplus G] = [F] + [G]$)

(c) For every $V \in \underline{\mathrm{Vect}}$ (f.d. $\mathbb{C}$-vector space) of dim m

$$\text{Trace of } \mathbb{S}^\lambda\left(\begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix}\right) \text{ on } \mathbb{S}^\lambda(\mathbb{C}^m) = s_\lambda(x_1, \dots, x_m)$$

(Schur poly.)

Note: $s_\lambda(x_1, \dots, x_m) = 0$ iff $m < \ell(\lambda)$

(By Young's rule - if $m < \ell(\lambda)$ then there are no semi-standard Young tableaux of shape λ - filled with numbers $1, \dots, m$.)

Hence $\boxed{\mathbb{S}^\lambda(V) = \{0\} \text{ if and only if } \ell(\lambda) > \dim(V)}$

$$\dim(\mathbb{S}^\lambda(V)) = s_\lambda(x_1, \dots, x_m)\Big|_{x_1 = \dots = x_m = 1}$$

$= \#$ SSYT of shape λ, with entries from $\{1, \dots, m\}$.

e.g. $\lambda = (2,1)$.

$$\dim(\mathbb{S}^{(2,1)}(V)) = 2\binom{m}{2} + 2\binom{m}{3}$$

$$= 2\binom{m+1}{3}.$$

§2. <u>Decomposition of T^n</u>: As $T^n \longleftrightarrow \mathbb{C}S_n$ under $\mathrm{Poly}_n \cong S_n$-reps,

and $\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus d_\lambda}$ $\quad$ ($d_\lambda = \dim V_\lambda$)

We get: $\boxed{T^n(V) \cong \bigoplus_{\lambda \vdash n} \mathbb{S}_\lambda(V)^{\oplus d_\lambda}}$

Remarks. - (i) Since each $\mathbb{S}^\lambda$ is a functor, $\mathbb{S}^\lambda(V)$ is naturally a repn. of $GL(V)$:

$$g \in GL(V) \rightsquigarrow \mathbb{S}^\lambda(g) \in GL(\mathbb{S}^\lambda(V)).$$

Hence, the decomposition $T^n(V) = \bigoplus_{\lambda \vdash n} \mathbb{S}^\lambda(V)^{\oplus d_\lambda}$ is of $GL(V)$-repns.

(ii) Viewing $T^n(V)$ as a repn. of S_n, we have:

$$T^n(V) \cong \bigoplus_{\lambda \vdash n} \underset{\substack{\uparrow \\ S_n \text{ acts} \\ \text{here.}}}{V_\lambda} \otimes \underbrace{\mathrm{Hom}_{S_n}(V_\lambda, T^n(V))}_{\text{multiplicity space}}$$

$$\mathrm{Hom}_{S_n}(V_\lambda, X) \cong (V_\lambda^* \otimes X)^{S_n} \cong (V_\lambda \otimes X)^{S_n}$$

(for S_n, w, w^{-1} are conjugate to each other $\forall\, w \in S_n$.
$\Rightarrow \chi_{V_\lambda^*} = \chi_{V_\lambda}$; hence $V_\lambda^* \cong V_\lambda$.)

$\Rightarrow$ The multiplicity space above is same as

$$\mathrm{Hom}_{S_n}(V_\lambda, T^n(V)) \cong (V_\lambda \otimes V^{\otimes n})^{S_n} = \mathbb{S}^\lambda(V).$$

Combining these observations, we get

$$\boxed{T^n(V) \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes \mathbb{S}^\lambda(V)}$$ as $S_n \times GL(V)$-repns.

We will now show that

- Irred. f.d. $GL(V)$ - repns (polynomial) $\longleftrightarrow \{ \mathbb{S}^\lambda(V) : \ell(\lambda) \leq \dim V \}$.
- Category of f.d. polynomial repns. of $GL(V)$ is semisimple.

§3. Polynomial representations of $GL_m(\mathbb{C})$. - Recall that a representation $\alpha : GL_m(\mathbb{C}) \to GL_N(\mathbb{C})$ is polynomial if $\forall\, 1 \leq a,b \leq N$; $g \in GL_m(\mathbb{C})$ $\alpha(g)_{a,b}$ ((a,b)-th entry of $\alpha(g)$) is a polynomial in the entries g_{ij} $(1 \leq i,j \leq m)$.

Using the same proof as for Lemma §4 of Lecture 17, we have:

$$\boxed{\text{every polynomial repn. of } GL_m(\mathbb{C}) \text{ is a direct sum of homogeneous polynomial repns.}}$$

Let $R = \mathbb{C}[y_{ij} : 1 \leq i,j \leq m] = \bigoplus_{n=0}^{\infty} R_n$ ← hgs. poly. of degree n.

Note: $GL_m(\mathbb{C}) \circlearrowright L$ is f.d. hgs of degree n

$\iff$ Matrix coefficients of L are elements of R_n.

[viewing y_{ij} as a function on $GL_m(\mathbb{C})$; $y_{ij}(g) = g_{ij}$.]

In more detail - let $\{e_a : 1 \leq a \leq N\}$ be a basis of L.
($N = \dim L$)

$\forall\ 1 \leq a, b \leq N$, let $\alpha_{a,b} : GL_m(\mathbb{C}) \to \mathbb{C}$ be defined as:

$$\alpha_{a,b}(\sigma) = \text{coefficient of } e_b \text{ in } \sigma^{-1}(e_a).$$

L : hgs of degree $n \iff \alpha_{a,b} \in R_n\ \forall\ 1 \leq a, b \leq N$.

§4. Prop. - The linear map $\eta : L \to R_n \otimes \underline{L}$ (auxiliary space) is an injective

$$e_a \mapsto \sum_{b=1}^{N} \alpha_{a,b} \otimes e_b$$

$GL_m(\mathbb{C})$ - intertwiner. (L : arbitrary f.d. poly. repn; hgs of degree n).
[see below for $GL_m(\mathbb{C})$ action on $R_n \otimes L$]

Proof. (Injectivity) If $\eta(v) = 0$, then $\forall\ b$, the fn.

$\alpha_{v,b} : \sigma \mapsto$ coefficient of e_b in $\sigma^{-1}(v)$ is identically zero implies $v = 0$.

Setting $\sigma = Id \in GL_m(\mathbb{C})$

→ $GL_m(\mathbb{C})$ action on $R_n \otimes \underline{L}$ is on the first tensor factor:
(L is aux. / mult. space).

Let $v \in L$, $g \in GL_m(\mathbb{C})$. Then

$$\eta(g \cdot v) = \sum_{b=1}^{N} \alpha_{gv,b} \otimes e_b.$$ We have to prove the following -

Claim. - $\alpha_{gv,b} = g \cdot (\alpha_{v,b})$

Proof of the claim.- For every $\sigma \in GL_m(\mathbb{C})$, we have

$$(g \cdot \alpha_{v,b})(\sigma) = \alpha_{v,b}(g^{-1}\sigma) = \text{coeff. of } e_b \text{ in } (g^{-1}\sigma)^{-1} v.$$

$$\alpha_{gv,b}(\sigma) = \text{coeff. of } e_b \text{ in } \sigma^{-1}(gv).$$

Hence the two are equal. $\square$

§5. In light of Prop §4, every f.d., hgs degree n repn L of $GL_m(\mathbb{C})$ is a subrepresentation of $R_n^{\oplus \dim L}$. We will now prove that R_n is semisimple and $\{L_\lambda = \mathbb{S}^\lambda(\mathbb{C}^m)\}_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}}$ is the complete set of irred. f.d. hgs. degree n repns. of $GL_m(\mathbb{C})$.

Theorem.- Matrix coefficients of $\{L_\lambda\}_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}}$ form a basis of R_n.

Proof.- Since $T^n(\mathbb{C}^m) \cong \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} L_\lambda^{\oplus \dim V_\lambda}$ as $GL_m(\mathbb{C})$-repns; matrix coefficients of $T^n(\mathbb{C}^m)$ are in the linear span of matrix coefficients of L_λ's.

But matrix coefficients of $T^n(\mathbb{C}^m)$ contain all degree n monomials in $\{y_{ij}\}_{1 \leq i,j \leq m}$ variables; which span R_n.

So, it suffices to check that

$$\dim R_n = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} (\dim L_\lambda)^2 .$$

Note :

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} \dim (L_\lambda)^2 = \sum_{\lambda \vdash n} s_\lambda(x_1, \dots, x_m)\, s_\lambda(y_1, \dots, y_m) \Big|_{\substack{x_1 = \dots = x_m = 1 \\ y_1 = \dots = y_m = 1}}$$

Recall :

$$\sum_{\lambda \vdash n} s_\lambda(x_1, \dots, x_m)\, s_\lambda(y_1, \dots, y_m) = \text{degree } n \text{ part of } \prod_{1 \leq i,j \leq m} \frac{1}{1 - x_i y_j}$$

$$\sum_{\lambda \vdash n} s_\lambda(1, \dots, 1)\, s_\lambda(t, \dots, t)\Big|_{t=1} = \text{coefficient of } t^n \text{ in } \frac{1}{(1-t)^{m^2}} .$$

Ex. $$\dim R_n = \binom{m^2+n-1}{n} = \text{coeff. of } t^n \text{ in } \frac{1}{(1-t)^{m^2}} .$$

$\square$

<u>Ex.</u> If $\{L_1, \dots, L_r\}$ are f.d. repns. of a group G s.t. $\{L_1, \dots, L_r\}$ are pairwise, non-iso., irred. repns. then matrix coefficients of L_j's are linearly independent,