Friday, Feb 23:

Recap: $GL_n(\mathbb{C})$ polynomial repns. - summary

(1)* Every f.d. poly repn of $GL_m(\mathbb{C})$ is semisimple
missing a piece

(2) Irreducible poly repn. of $GL_n(\mathbb{C})$ (up to isomorphism) $\longleftrightarrow \{\lambda = (\lambda_1 \geq \cdots \geq \lambda_m)\} \subseteq \mathbb{Z}^m_{\geq 0}$

$(n = |\lambda|)$ $\mathrm{Hom}_{S_n}(V_\lambda, (\mathbb{C}^m)^{\otimes n}) = L_\lambda \quad \longleftrightarrow \quad \lambda$

(3) (Schur)·Weyl character formula : Trace of $\begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix}$

acting on $L_\lambda = S_\lambda(x_1, \ldots, x_m)$.

Lie algebra perspective:

Let $\alpha : GL_m(\mathbb{C}) \longrightarrow GL(V)$ be a polynomial repn.

We define $\rho_{ij} : V \to V$ as follows:

For $i \neq j$

$$(1) \quad \alpha\left(\begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{cc} & j \\ & \downarrow \\ \end{array} \right)$$

$$(1) \quad \alpha\left(\begin{bmatrix} 1 & & 0 & \\ & \ddots & t & \leftarrow i \\ 0 & & \ddots & 0 \\ & & & 1 \end{bmatrix} \right)$$

ρ_{ij} = coeff. of t in ⤴

$$\rho_{ij} = \frac{d}{dt}\alpha(I_{m\times m} + tE_{ij})\Big|_{t=0}$$

where E_{ij} is the elementary matrix:

$$\begin{cases} 1 & \text{at } (i,j) \\ 0 & \text{everywhere else} \end{cases}$$

(2) $i = j$

$$\rho_{ii} = \frac{d}{dt}\alpha\left(\begin{array}{c|c} & i \\ \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & e^t & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} & \leftarrow i \end{array}\right)\Bigg|_{t=0}$$

Eg: if $V = \mathbb{C}^m$

$\alpha = id : GL_m(\mathbb{C}) \longrightarrow GL(\mathbb{C}^m)$

$\rho_{ij} = E_{ij} \quad |j\rangle \longmapsto |i\rangle$

<u>Lemma</u>: Let $\alpha_\ell : GL_m(\mathbb{C}) \longrightarrow GL(V_\ell)$ $(\ell = 1, 2)$

be two polynomial repn. Let $\rho_{ij}^{(\ell)} : V_\ell \longrightarrow V_\ell$ as defined. Let $\alpha = \alpha_1 \otimes \alpha_2 : GL_m(\mathbb{C}) \to GL(V_1 \otimes V_2)$ and $\rho_{ij} : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$ corresponding operator as defined. <u>Then</u>,

$$\rho_{ij}(v_1 \otimes v_2) = \rho_{ij}^{(1)}(v_1) \otimes v_2 + v_1 \otimes \rho_{ij}^{(2)}(v_2)$$

(Leibniz rule)

<u>Proof for $i \neq j$ case</u>:

$$(Id + t E_{ij}) \otimes (Id + t E_{ij}) = id \otimes id + t(E_{ij} \otimes id + id \otimes E_{ij}) + \dots$$

<u>Eg</u>: $Sym^{\ell}(\mathbb{C}^m) = $ Span of $\{x_{i_1} \dots x_{i_\ell} : 1 \le i_1 \le \dots \le i_\ell \le m\}$

Then, ρ_{ij} acts by $x_i \dfrac{\partial}{\partial x_j} : x_k \longmapsto \delta_{jk} x_i$

<u>Eg</u>: $\Lambda^2(\mathbb{C}^m) = $ Span $\{x_{i_1} \wedge x_{i_2} : 1 \le i_1 < i_2 \le m\}$

The rest is the same argument as before.

<u>Remark</u>: If $W \subset V$ is a $GL_m(\mathbb{C})$-subrepn, then $\rho_{ij}(v) \in W$, $\forall v \in W$

Hence, if 0 and V are the only subspaces <u>invariant</u> under $\{\rho_{ij}\}_{i,j}$, then V is irreducible.

$\hookrightarrow$ W st $\rho_{ij}(W) \subset W$, $\forall i,j$

[This a hint for problem 21 on Homework]

Exercise: Let A be an algebra. A derivation of A is a linear map $\partial : A \to A$ s.t.

$$\partial(ab) = \partial(a)b + a\partial(b). \quad \text{[Derivation]}$$

Show that ∂_1, ∂_2 derivations $\Rightarrow \partial_1 \circ \partial_2 - \partial_2 \circ \partial_1$ is again a derivation

Further, assume $\partial : A \to A$ is a locally nilpotent* derivation. Show that $\exp(\partial) : A \to A$ is an algebra homomorphism.

* means $\forall a \in A, \exists N_a >> a$ st $\partial^n a = 0$ $\forall n \geq N_a$

Eg: $\partial_x \circlearrowright \mathbb{C}[x]$ is locally nilpotent but not nilpotent.

Remark: $\{ \rho_{ij} \}_{i,j}$ satisfy relations of $m \times m$ matrix

$$[\rho_{ij}, \rho_{kl}] = \delta_{jk} \rho_{il} - \delta_{il} \rho_{kj}$$

Lie algebra $gl_m(\mathbb{C})$

Let's now look at $Gl_2(\mathbb{C})$ case

$e = \rho_{12}$, $\quad f = \rho_{21}$, $\quad h = \rho_{11} - \rho_{22}$,

$I = \delta_{11} + \delta_{22}$

Lie algebra $gl_2(\mathbb{C})$

$[e, f] = h \quad , \quad [h, e] = 2e \quad ,$

$[h, f] = -2f, \quad [I, x] = 0 \quad , \forall x$

$sl_2(\mathbb{C})$: Span of $e, f, h \subset gl_2(\mathbb{C})$

Recall, for the case of $GL_2(\mathbb{C})$

Irred poly repns $\longleftrightarrow \{\lambda_1 \geq \lambda_2 \mid \lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 0}\}$

$L_\lambda \longleftrightarrow \lambda$

Trace of $\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$ acting on L_λ

$= S_\lambda(x_1, x_2)$

Further recall, $\underset{l(\lambda) \leq m}{S_\lambda(x_1, \dots, x_m)} = \dfrac{\det(x_i^{\lambda_j + m - j})_{1 \leq i,j \leq m}}{\det(x_i^{m-j})_{1 \leq i,j \leq m}}$

So, in our case $m = 2$, so we get

$$S_\lambda(x_1, x_2) = \frac{\begin{vmatrix} x_1^{\lambda_1+1} & x_1^{\lambda_2} \\ x_2^{\lambda_1+1} & x_2^{\lambda_2} \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}$$

$\lambda_1 \geq \lambda_2$

$\lambda_1 + 1 > \lambda_2 + 0$

$$= \frac{x_1^{\lambda_1+1} x_2^{\lambda_2} - x_1^{\lambda_2} x_2^{\lambda_1+1}}{x_1 - x_2}$$

$$= x_1^{\lambda_2} x_2^{\lambda_2} \left(\frac{x_1^{\lambda_1-\lambda_2+1} - x_2^{\lambda_1-\lambda_2+1}}{x_1 - x_2} \right)$$

$$= x_1^{\lambda_1} x_2^{\lambda_2} \left(\sum x_1^{\lambda_1-\lambda_2+1-j} x_2^{j} \right)$$

$\to$ set $x_1 = x_2 = 1$, $\dim L_\lambda = \lambda_1 - \lambda_2 + 1$

using the fact $\lim_{t\to 1} \frac{t^N - 1}{t-1} = N$

For $SL_2(\mathbb{C})$, we get that the irreducible polynomial repn° $\longleftrightarrow$ $\mathbb{Z}_{\geq 0}$

$(\lambda_1 - \lambda_2)$

$L_n \longleftrightarrow n$

$$\boxed{\dim L_n = n+1}$$

A concrete model for L_n

$L_n =$ degree n polynomials in $\mathbb{C}[x,y]$

$= \text{span}\{\underbrace{x^n, x^{n-1}y, \ldots, y^n}_{n+1}\}$

So, $\dim L_n = n+1$

$sl_2(\mathbb{C})$ - action on L_n

$$e = x\frac{\partial}{\partial y}, \quad f = y\frac{\partial}{\partial x}$$

$$h = x\partial_x - y\partial_y = \text{deg wrt } x - \text{deg wrt } y$$

So in this basis:

$$\begin{tikzcd}
0 & x^n \arrow[l, bend right, "e"'] \arrow[r, bend right, "f"'] & x^{n-1}y \arrow[l, bend right, "e"'] \arrow[r, bend right, "f"'] & \cdots \arrow[l, bend right] \arrow[r, bend right] & xy^{n-1} \arrow[l, bend right, "e"'] \arrow[r, bend right, "f"'] & y^n \arrow[l, bend right, "e"']
\end{tikzcd}$$

$$\begin{array}{lccccc}
h: & n & n-2 & \cdots & -n+2 & -n
\end{array}$$

acts diagonally

<u>Exercise</u>. This representation is irreducible.

<u>Geometric perspective</u>:

$$GL_2(\mathbb{C}) \circlearrowright \mathbb{P}^1(\mathbb{C})$$

$$L_n = \Gamma(\mathbb{P}^1, \mathcal{L}_n)$$

$$\mathbb{P}^1 = (\mathbb{C}\times\mathbb{C})\setminus\{0\} \Big/ \mathbb{C}^\times \qquad (1)$$

$$\mathbb{P}^1 = (\mathbb{C}, z) \underset{\substack{z \leftrightarrow w^{-1} \\ \text{on } \mathbb{C}^\times}}{\bigcup} (\mathbb{C}, w) \qquad (2)$$

In model (2) of $\mathbb{P}^1$,

$\mathcal{L}_n$ is a sheaf on $\mathbb{P}^1$,

$$\mathcal{U} \longmapsto \{ f : \mathcal{U} \to \mathbb{C} \mid \sum_a v_f(a) + n[\infty] \geq 0 \}$$

We are looking at functions,

$$\Gamma(\mathbb{P}^1, \mathcal{L}_n) = \{ f : \mathbb{C} \to \mathbb{C} \text{ st } f \text{ has at most pole of ord } n \text{ at } \infty \}$$

$$= \{ \text{poly's of deg} \leq n \} \quad n+1 \text{ dim}^{\ell}$$

In model (1) of $\mathbb{P}^1$,

$\mathbb{C}^\times \circlearrowright \mathbb{C}$ by

$z \mapsto z^n$

$$\begin{tikzcd}
(\mathbb{C}\times\mathbb{C}\setminus\{\underline{0}\}) \times \mathbb{C} \arrow[r] \arrow[d] & \underbrace{(\mathbb{C}\times\mathbb{C}\setminus\{\underline{0}\}) \times_{\mathbb{C}^\times} \mathbb{C}}_{} \arrow[d] \\
\mathbb{C}\times\mathbb{C}\setminus\underline{0} \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

$\mathcal{L}_n = (\mathbb{C}\times\mathbb{C}\setminus\{\underline{0}\}) \times_{\mathbb{C}^\times} \mathbb{C}$

Borel-Weil-Bott Theorem

If $n \geq 0$, then $H^0(\mathbb{P}^1, \mathcal{L}_n)$ is irreducible $SL_2(\mathbb{C})$ repn.

$H^1(\mathbb{P}^1, \mathcal{L}_n) = 0$. If $n < 0$, $H^0(\mathbb{P}^1, \mathcal{L}_n) = 0$,

$H^1(\mathbb{P}^1, \mathcal{L}_n) \cong L_{-n}$ irreducible.