Lecture 19

Representations of associative algebras.

§1. Let K be a field. Let A be a unital, associative algebra over K.

That is, A is a K-vector space, together with a distinguished element $1 \neq 0$, and a bilinear $A \times A \to A$, $(a,b) \mapsto ab$ s.t. (1) $(ab)c = a(bc)$ $\forall a,b,c \in A$
(2) $1a = a1 = a$ $\forall a \in A$.

An A-representation is a K-vector space V together with a unital K-algebra hom. $\rho : A \to \mathrm{End}_K(V)$. Thus $A \circlearrowright V$ by $a \cdot v = \rho(a)(v)$ $\forall a \in A, v \in V$.

An A-representation is nothing but a left A-module.

We have the usual notions of subrepresentations, A-intertwiners, direct sum of two representations. As opposed to group representations, in general we do not have tensor product or duals for general A-repns.
For these operations to coherently exist in the category of A-repns, A must have a structure of a Hopf algebra - for later.

Examples. - (a) G: finite group $\rightsquigarrow$ $A = K[G]$ group algebra of G.

(b) $A = K\langle x_1, \dots, x_n \rangle$ polynomial algebra in n non-commuting variables.

(c) $A = K[x_1, \dots, x_n]$ n commuting variables.

In practice, many associative algebras are given in terms of generators and relations. For example,

(d) Weyl algebra $\quad A = K\langle X, D\rangle \big/ (DX - XD - 1)$

Two generators X, D and one relation $DX - XD = 1$.

(e) $\mathcal{U}(sl_2) = \mathbb{C}\langle e, f, h\rangle \Big/ \begin{pmatrix} he - eh = 2e \\ hf - fh = -2f \\ ef - fe = h \end{pmatrix}$

§2. Semisimple representations. An A-representation V is said to be semisimple if it is isomorphic to a direct sum of irreducible A-representations. ~~That is,~~ $\cancel{\exists\, V_1, V_2, \ldots, V_\ell}$.

The exact same proof of Schur's lemma for group repns. applies.

Schur's lemma. - Let V, W be two A-repns. and $f: V \to W$ an A-intertwiner.

(1) If V is irreducible, then either $f = 0$ or f is injective

(2) If W is irreducible, then either $f = 0$ or f is surjective.

(3) Assume $V = W$ is finite-dim'l, irreducible, and K is alg. closed.

Then $f = \lambda \cdot \mathrm{Id}_V$ for some $\lambda \in K$.

Some remarks and corollaries of Schur's lemma.

(a) Multiplicity of an irred. repn. Assume $V_1, V_2, \dots, V_N$ are pairwise non-iso., finite-dim'l, irreducible A-repns.

Let $V = \bigoplus_{i=1}^{N} V_i^{\oplus d_i}$ be a semisimple A-repn ($d_i \in \mathbb{Z}_{\geq 0}$).

Then $d_i = \dim \mathrm{Hom}_A(V_i, V) = \dim \mathrm{Hom}_A(V, V_i)$.

In other words, the natural A-intertwiner is an isomorphism:

$$\bigoplus_{i=1}^{N} V_i \otimes \underbrace{\mathrm{Hom}_A(V_i, V)}_{\text{(auxiliary space)}} \longrightarrow V$$

$$v_i \otimes X \longmapsto X(v_i) \quad \text{on } i\text{-th summand}$$

$$(v_i \in V_i, \quad X : V_i \to V \text{ an } A\text{-intertwiner})$$

Here, A acts on the 1^{st} tensor factor only - in $V_i \otimes \underbrace{\mathrm{Hom}(V_i, V)}_{\text{auxiliary space}}$

(b) Let V be a f.d. irred A-repn. Then

$$\mathrm{Hom}_A(V^{\oplus n}, V^{\oplus m}) = \mathrm{Hom}_A(V \otimes \underline{K^n}, V \otimes \underline{K^m}) \quad \text{aux. spaces}$$

$$\cong \mathrm{Hom}_K(K^n, K^m) = \mathrm{Mat}_{m\times n}(K)$$

Given $X \in \mathrm{Mat}_{m\times n}(K)$, $X = (x_{ij})$, the corresponding A-intertwiner $V^{\oplus n} \to V^{\oplus m}$ is :

$$(v_1, \dots, v_n) \longmapsto \left(\sum_{j=1}^{n} x_{ij} v_j \right)_{1 \leq i \leq m}.$$

§3. Theorem. Let $V_1, \dots, V_N$ be pairwise non-isomorphic, finite-dim'l, irreducible representations of $\mathcal{A}$. Let $V = \bigoplus_{i=1}^{N} V_i^{\oplus n_i}$, and $W \subset V$ be a subrepresentation. Then W is semisimple – i.e., $\exists\ r_i \in \{0, 1, \dots, n_i\}$ ($1 \leq i \leq N$) s.t. $W \simeq \bigoplus_{i=1}^{N} V_i^{\oplus r_i}$.

[We are assuming $\mathbb{k}$ is algebraically closed.]

Proof. – Argue by induction on $\sum_{i=1}^{N} n_i$. Base case is trivial, so we proceed with carrying out the induction step. Assume, $W \neq \{0\}$.

– if W is irreducible, then for some projection $V = \bigoplus_{i=1}^{N} V_i^{\oplus n_i} \xrightarrow{\pi_j^{(s)}} V_j$ $(1 \leq s \leq n_j)$

$$\pi_j^{(s)}\Big|_W : W \to V_j \quad \text{is non-zero}$$

$$\left(\text{since}\quad \pi_j^{(s)}(w) = 0 \ \ \forall\, \begin{matrix} 1 \leq j \leq N \\ 1 \leq s \leq n_j \end{matrix} \quad \Rightarrow \quad w = 0\right)$$

Schur's Lemma implies $W \simeq V_j$; and we are done

– if W is not irreducible. Let $P \subset W$ be an irreducible subrepn. of W (exists since W is finite-dim'l – so we can take P to be a non-zero (sub-)repn of W with smallest possible dimension).

Same argument as above shows that $P \simeq V_j$ for some $1 \leq j \leq N$.

The inclusion $P \simeq V_j \subset W \subset V = \bigoplus_{i=1}^{N} V_i^{\oplus n_i}$ is given by

$$v \in V_j \longmapsto \begin{matrix} (\lambda_1 v, \dots, \lambda_{n_j} v) & \text{in } V_j^{\oplus n_j} \text{ piece} \\ 0 & \text{in } V_i^{\oplus n_i} \text{ piece } (i \neq j) \end{matrix}$$

(see §2 (a), (b) - above)

here, $\lambda_1, \dots, \lambda_{n_j} \in \mathbb{k}$ are not all zero.

Let $X \in GL_{n_j}(\mathbb{k})$ be such that $X \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n_j} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and

let $\varphi_X \in \mathrm{Aut}_{\mathcal{A}}(V)$ given by $Id_{V_i^{\oplus n_i}}$ on i-th piece $(i \neq j)$

and $(v_1, \dots, v_{n_j}) \longmapsto \left(\sum_{\ell=1}^{n_j} x_{p\ell} v_\ell \right)_{1 \leq p \leq n_j}$ on j-th piece.

Then,
$$\begin{aligned} V_j &\hookrightarrow W \hookrightarrow V \\ v &\longmapsto (v, 0, \dots, 0) \text{ in } V_j^{\oplus n_j}\text{-piece} \\ & \qquad 0 \quad \text{ in } V_i^{\oplus n_i}\text{-piece } i \neq j \end{aligned}$$

$$\Rightarrow W \simeq V_j \oplus K \text{ where } K \subset V_1^{\oplus n_1} \oplus \dots \oplus V_j^{\oplus n_j - 1} \oplus \dots$$
$$= \mathrm{Ker}\left(W \hookrightarrow V \xrightarrow[\pi_j^{(1)}]{} V_j \right).$$

and we are done by induction. $\square$

Remark.- This theorem is valid for non-alg.-closed fields as well. Moreover, $V_1, \dots, V_N$ do not have to be finite-dim'l. (see Remark 3.1.5 of [Etingof et al]). In this generality, $\mathrm{End}_{\mathcal{A}}(V)$ need not be $\mathbb{k}$, but some division algebra (not necessarily commutative field) over $\mathbb{k}$.

§4. Some remarks and "coordinate-free" description of Thm §3.

In the set up of the theorem, V is a semisimple repn.

$$\bigoplus_{i=1}^{N} V_i \otimes \underbrace{\mathrm{Hom}_A(V_i, V)}_{\text{aux. space}} \longrightarrow V \quad \text{is an } A\text{-linear isomorphism}$$

(meaning A only acts on 1st tensor factor)

$$\begin{array}{c} V \\ \cup \\ W \end{array} \quad \text{a subrepn.}$$

Theorem §3 implies that $\displaystyle\bigoplus_{i=1}^{N} V_i \otimes \underline{\mathrm{Hom}_A(V_i, W)} \xrightarrow{\sim} W$

which can be conveniently stated as:

$$\begin{tikzcd} \bigoplus_{i=1}^{N} V_i \otimes \mathrm{Hom}_A(V_i, V) \arrow[r, "\cong"] & V \\ \bigoplus_{i=1}^{N} V_i \otimes \mathrm{Hom}_A(V_i, W) \arrow[u, hook] \arrow[r, "\cong"'] & W \arrow[u, hook, "j"'] \end{tikzcd}$$

left vertical map is

$\mathrm{Id} \otimes (j \circ X)$

$(X : V_i \to W \overset{j}{\hookrightarrow} V)$

§5. Jacobson's density theorem.

– Let $V_1, \dots, V_N$ be pairwise non-iso, finite-dim'l, irreducible representations of A. Then

$$\rho : A \longrightarrow \bigoplus_{i=1}^{N} \mathrm{End}_K(V_i) \quad \text{is surjective.}$$

(if $\rho_i : A \to \mathrm{End}_K(V_i)$ is the action hom. then $\rho(a) = (\rho_i(a))_{i=1}^{N}$.

Proof.- Step 1. -

$$A \xrightarrow{\rho} \bigoplus_{j=1}^{N} \mathrm{End}(V_j) \quad \text{and}$$

(easy check)

$$\bigoplus_{j=1}^{N} V_j \otimes \underline{V_j^*} \xrightarrow[\psi]{} \bigoplus_{j=1}^{N} \mathrm{End}\, V_j \qquad (V_j^* \text{ is auxiliary space})$$

(tensor-hom adjointness)

are A-intertwiners, where $A \circlearrowright \bigoplus_{j=1}^{N} \mathrm{End}(V_j)$ by

$$a \cdot (X_j)_{j=1}^{N} = (\rho_j(a)\, X_j)_{j=1}^{N}.$$

Step 2.- Let $U = \rho(A) \subset \bigoplus_{j=1}^{N} \mathrm{End}(V_j) \cong \bigoplus_{j=1}^{N} V_j \otimes \underbrace{V_j^*}_{\text{aux. space}}$

(semisimple repn. $\cong \bigoplus_{j=1}^{N} V_j^{\oplus \dim V_j}$.)

So, by Thm §3,

$$U = \bigoplus_{j=1}^{N} V_j \otimes L_j \quad \text{where } L_j \subset V_j^* \ \forall j.$$

Step 3.- If $\exists\, i \in \{1, \dots, N\}$ s.t. $L_i \subsetneq V_i^*$, then $\exists\, 0 \neq v \in V_i$ such that $ev_v(\eta) = \eta(v) = 0 \quad \forall\, \eta \in L_i$. Meaning, the following composition is identically zero.

$$\begin{array}{ccccc} U = \rho(A) & \hookrightarrow & \bigoplus_{j=1}^{N} V_j \otimes V_j^* & \twoheadrightarrow & V_i \otimes V_i^* \\ \uparrow & & & & \downarrow{\scriptstyle \mathrm{Id} \otimes ev_v} \\ A & & \xrightarrow{\quad = 0 \quad} & & V_i \end{array}$$

But $1 \in A$ under this composition goes to v, assumed to be non-zero.
This contradiction proves that $L_j = V_j^* \ \forall j$ and the theorem follows. $\square$