Recall, last time we proved the following results about representations of an associative algebra A over an algebraically closed field K.

(1) Subrepresentations of semisimple representations are semisimple (Lecture 19, Thm §3)

(2) If $V_1, \dots, V_N$ are distinct, finite-dimensional, irreducible representations of A, with $\rho_i : A \to \mathrm{End}_K(V_i)$ the action hom's, then
$$\rho = \bigoplus_{i=1}^{N} \rho_i : A \longrightarrow \bigoplus_{i=1}^{N} \mathrm{End}_K(V_i) \quad \text{is surjective}$$
(Lecture 19, Thm §5).

§1. Classification of semisimple, finite-dim'l algebras

Assume $\dim_{K\text{-v.s.}}(A) < \infty$. As a corollary of (2) above, we get
$$\dim_{K\text{-v.s.}}(A) \geq \sum_{i=1}^{N} (\dim V_i)^2$$
in particular, A has only finitely many irreducible repns.

Example. (a) $A = K[x]/(x^N)$ has a unique irreducible repn, namely K with A-action via $x \mapsto 0$.

So, the inequality is strict $N > 1^2$.

Example (b) $A = M_{n\times n}(K) \circlearrowright K^n$ is irreducible (check this)

Since $\dim_{K\text{-v.s.}}(A) = n^2 = \dim(K^n)^2$, we conclude

that A has no other irreducible representation.

<u>Prop.</u> - $A = M_{n\times n}(K)$. Let U be a finite-dim'l A-repn.

Then $U \cong V^{\oplus r}$, where $V = K^n$ and $r \geq 0$.

Proof. - $1_A = E_{11} + E_{22} + \dots + E_{nn}$

(Notation: let $\{|1\rangle, |2\rangle, \dots, |n\rangle\}$ denote the usual basis of K^n

$E_{ij} : K^n \to K^n$ $\quad E_{ij}\,|l\rangle = \delta_{jl}\,|i\rangle$.)

elementary matrices

and $E_{ii}^2 = E_{ii}$, $\quad E_{ii}E_{jj} = 0$ for $i \neq j$.

i.e. $\{E_{11}, \dots, E_{nn}\}$ is a complete system of pairwise orthogonal idempotents.

So $U = E_{11}(U) \oplus \dots \oplus E_{nn}(U)$ as K-vector space

Note: $E_{ii}(U) \xrightarrow{\sim} E_{jj}(U)$.

$u \longmapsto E_{ji}(u)$

Hence, for any non-zero $u \in E_{11}(U)$,

$$\mathrm{Span}\left\{ u, E_{21}(u), \ldots, E_{n1}(u) \right\} \cong K^n \text{ as } A\text{-repn.}$$

which implies $U \cong \underbrace{K^n \oplus \cdots \oplus K^n}_{\dim E_{11}(U) \text{ terms}}$. $\square$

§2. Again, let A be a $\underbrace{\text{f.d.}}_{\text{as } K\text{-vector space}}$ associative K-algebra

$\{V_1, \ldots, V_N\}$ = set of (iso. classes of) irred. repns. of A.

$$\rho : A \longrightarrow \bigoplus_{i=1}^{N} \mathrm{End}_K(V_i) \qquad \left(\begin{array}{l}\text{surjective as proved} \\ \text{in Thm §5 of L.19}\end{array}\right)$$

Define : $\mathrm{Rad}(A) = \mathrm{Ker}(\rho)$

(two-sided ideal)

Theorem. - The following are equivalent (in which case, we say A is semisimple)

(1) $\mathrm{Rad}(A) = \{0\}$

(2) $A \cong \bigoplus_{i=1}^{N} M_{d_i \times d_i}(K)$ $\quad (d_i = \dim V_i)$

(3) Every f.d. repn. of A is semisimple

(4) $A \xrightarrow[\lambda]{} \mathrm{End}_K(A)$ is semisimple.

$\lambda(a)(b) = ab$

Proof.- (1) $\iff$ (2) by Jacobson's density theorem.

(3) $\Rightarrow$ (4) since A is finite-dim'l as a K-vector space

(2) $\Rightarrow$ (3) : Let $A_i = \mathrm{End}_K(V_i) \circlearrowright V_i$ (its only irred. repn.)

As $A = A_1 \oplus \cdots \oplus A_N$ as K-alg's, $1_A = (1_{A_1}, \ldots, 1_{A_N})$

Hence, if U is a f.d. A-repn. then $U \cong \bigoplus_{j=1}^{N} 1_{A_j}(U)$ as A-repns, where

A acts on $1_{A_j}(U)$ via j-th projection $A \twoheadrightarrow A_j \circlearrowright 1_{A_j}(U)$

As A_j has only one irreducible repn. and is semisimple (Prop. §1 above)

$1_{A_j}(U) = V_j^{\oplus n_j}$ for some $n_j \in \mathbb{Z}_{\geq 0}$. That is, U is semisimple.

(4) $\Rightarrow$ (2) If $A \circlearrowright A$ is semisimple, then, as A-repns,

$$A \cong \bigoplus_{j=1}^{N} V_j^{\oplus n_j} \quad , \text{ where, by Schur's lemma,}$$

$$n_j = \dim \mathrm{Hom}_A(A, V_j) = \dim V_j$$

$$\Rightarrow \quad \dim A = \sum_{j=1}^{N} (\dim V_j)^2 \text{ , i.e. } \rho : A \to \bigoplus_{i=1}^{N} \mathrm{End}\, V_i$$

is an isomorphism (surj. map b/w vector spaces of the same dimension). $\square$

§3. Indecomposable representations and Krull-Schmidt Theorem.

Lemma. – Let V be an indecomposable A-representation of finite dimension; $\varphi : V \to V$ be an A-intertwiner. Then either φ is nilpotent, or φ is an iso.

Proof. – Consider the chains of A-repns:

$$\mathrm{Ker}(\varphi) \subset \mathrm{Ker}(\varphi^2) \subset \cdots \qquad \text{These stabilize since } \dim V < \infty.$$

$$\mathrm{Im}(\varphi) \supset \mathrm{Im}(\varphi^2) \supset \cdots$$

i.e., $\exists\, N > 0$ s.t. $K = \mathrm{Ker}(\varphi^N) = \mathrm{Ker}(\varphi^{N+j}) \quad \forall j \geq 0$

$$I = \mathrm{Im}(\varphi^N) = \mathrm{Im}(\varphi^{N+j}) \quad \forall j \geq 0.$$

<u>Claim.</u> $V = K \oplus I$ (as A-repns.)

Proof. K and I are clearly sub-A-repns. of V.

If $v \in V$, then pick $w \in V$ s.t. $\varphi^N(v) = \varphi^{2N}(w)$ and write

$$v = \underbrace{(v - \varphi^N(w))}_{\in \mathrm{Ker}(\varphi^N) = K} + \underbrace{\varphi^N(w)}_{\in \mathrm{Im}(\varphi^N) = I}.$$

It remains to see that $K \cap I = \{0\}$. If $u \in K \cap I$, then $u = \varphi^N(v)$ for some $v \in V$. But $\varphi^N(u) = \varphi^{2N}(v) = 0$ $\left(u \in \underset{= \mathrm{Ker}\varphi^N}{K}\right)$

i.e. $v \in \mathrm{Ker}(\varphi^{2N}) = \mathrm{Ker}(\varphi^N) \Rightarrow u = \varphi^N(v) = 0.$ □

As V is indecomposable either $K = V$ (so $\varphi^N \equiv 0$, φ is nilpotent)

or ~~$K=$~~ $I = V$ (i.e. φ^N is an iso, hence φ is) □

Cor. [A, V as in Lemma above].

If $\varphi_1, \dots, \varphi_n \in \mathrm{End}_A(V)$ are nilpotent then

$\varphi_1 + \dots + \varphi_n$ is nilpotent.

(Induction on n, $n=1$ is obviously true)

Proof. If $\varphi = \varphi_1 + \dots + \varphi_n$ is not nilpotent, then it is invertible by Lemma above. So $Id_V = \varphi^{-1}\circ\varphi_1 + \dots + \varphi^{-1}\circ\varphi_n$.

$$Id_V - \varphi^{-1}\circ\varphi_n = \varphi^{-1}\circ\varphi_1 + \dots + \varphi^{-1}\circ\varphi_{n-1} \quad - (*)$$

$\forall j \in \{1, \dots, n\}$, $\varphi^{-1}\circ\varphi_j$ is not invertible (as φ_j is nilpotent) hence nilpotent by Lemma.

L.H.S. of $(*)$ is then invertible $\left((1-X)^{-1} = 1 + X + X^2 + \dots + X^l \text{ if } X^{l+1} = 0 \right)$

RHS of $(*)$ is sum of $n-1$ nilpotent operators on V - nilpotent by induction hypothesis.

a contradiction. $\square$

§4. **Theorem** – Let V be a finite-dim'l A-repn. Then $\exists!$ (up to relabelling) indecomposable A-repns. $V_1, \dots, V_n$ s.t

$$V \cong V_1 \oplus \dots \oplus V_n.$$

Proof.- Existence of such a decomposition is an easy induction argument on dimension. We will ~~[illegible]~~ prove uniqueness here

Assume $V \simeq V_1 \oplus \cdots \oplus V_n \simeq W_1 \oplus \cdots \oplus W_m$

where V_j and W_k are indecomposable.

$$\begin{tikzcd}
V_1 \arrow[r, hook, "\imath_1"] \arrow[ddd, dashed, "\theta_j"'] & \bigoplus_{s=1}^{n} V_s \arrow[r, "g", "\cong"'] & \bigoplus_{t=1}^{m} W_t \arrow[d, "\pi_j'"] \\
 & & W_j \arrow[dd, "\imath_j'"] \\
 & & \\
V_1 & \bigoplus_{s=1}^{n} V_s \arrow[l, "\pi_1"] & \bigoplus_{t=1}^{m} W_t \arrow[l, "g^{-1}"', "\cong"]
\end{tikzcd}$$

($\imath_s, \pi_s$ inclusion and projection for V; $\imath_t', \pi_t'$: the same for W)

For each $1 \le j \le m$, define $\theta_j \in \mathrm{End}_A(V_1)$ by the diagram above

$$\theta_j = \pi_1 \circ g^{-1} \circ (\imath_j' \circ \pi_j') \circ g \circ \imath_1 : V_1 \to V_1.$$

As $\sum_{j=1}^{m} \imath_j' \circ \pi_j' = Id$ on $\bigoplus_{t=1}^{m} W_t$, we get $\sum_{j=1}^{m} \theta_j = Id_{V_1}$.

$\Rightarrow$ some θ_j is an iso. ($b \circ a = \theta_j$ is an iso.)

by Cor. above

<u>Claim.</u> $V_1 \xrightarrow[a]{\pi_j' \circ g \circ \imath_1} W_j \xrightarrow[b]{\pi_1 \circ g^{-1} \circ \imath_j'} V_1$ are isomorphisms.

(Pf. Note: $W_j \cong \mathrm{Im}(a) \oplus \mathrm{Ker}(b)$

for every $w \in W_j$, $\exists\, v \in V_1$ s.t. $b(w) = \theta_j(v)$

$W_j \simeq \mathrm{Im}(a) \oplus \mathrm{Ker}(b)$ is the desired decomposition

$w \mapsto (a(v),\ w - a(v))$)

By indecomposability of W_j, injectivity of a and surjectivity of b, we get $Im(a) = W_j$ and $Ker(b) = \{0\}$,)

Relabelling j to 1, we have $V_1 \cong W_1$ and upon identifying the two,

$$V_1 \oplus \underbrace{\left(\bigoplus_{s=2}^{n} V_s \right)}_{B} \xrightarrow[\cong]{g} V_1 \oplus \underbrace{\left(\bigoplus_{t=2}^{m} W_t \right)}_{B'}$$

$h : B \to B'$ defined by $(0, h(v)) = g(0, v)$ for $v \in B$ is injective, hence an iso by dimension reasons.

Now we apply induction on n to conclude the theorem $\square$