# Lecture 21



Recall- last time we proved :

Krull-Schmidt Theorem.- Let $A$ be an associative algebra over $K$. Let $V$ be a finite-dim'l representation of $A$. Then $V \cong V_1 \oplus \ldots \oplus V_n$ where each $V_i$ is indecomposable. This decomposition is unique up to reordering the direct summands.

Remark- The uniqueness part is false in infinite-dimensions.

e.g. $A = \{ f : \mathbb{R} \to \mathbb{R} \text{ cnts s.t. } f(x+1) = f(x) \}$

$$\overset{A \circlearrowright}{M} = \{ f : \mathbb{R} \to \mathbb{R} \text{ cnts s.t. } f(x+1) = -f(x) \}$$

Then $A$ and $M$ are both indecomposable $A$-repns, not isomorphic, but $A \oplus A \simeq M \oplus M$. (Exercise)

Artin-Wedderburn Theorem. - Let $A$ be a finite-dim'l (as $K$-vector sp.) associative algebra, $\{V_1, \ldots, V_N\}$ = set of iso. classes of irred. $A$-repns.

Then, the following are equivalent:

(1) $\operatorname{Ker}\left( \rho = \oplus \rho_i : A \to \bigoplus_{i=1}^{N} \operatorname{End} V_i \right) = 0$

(2) $A \cong \bigoplus_{i=1}^{N} \operatorname{End}(V_i)$ as $K$-algebras

(3) Every finite-dim'l $A$-repn. is semisimple.

(4) $A$ is semisimple as $A$-repn, via left multiplication.

## §1. Double commutant theorem.-

Let $E$ be a finite-dimensional vector space (over $K$).

Let $A, B \subset \mathrm{End}_{\mathbb{C}}(E)$ be two subalgebras.

**Theorem.-** If $A$ is semisimple and $B = \mathrm{End}_A(E)$, then

(i) $B$ is semisimple, (ii) $A = \mathrm{End}_B(E)$ and

(iii) $E \cong \bigoplus_{i=1}^{N} V_i \otimes W_i$, where

$\{V_1, \dots, V_N\}$ = set of iso. classes of irred $A$-repns

$\{W_1, \dots, W_N\}$ = " " " " " $B$-repns.

**Proof.-** As $A$ is finite-dimensional semisimple algebra,

$$A \cong \bigoplus_{i=1}^{N} \mathrm{End}_{\mathbb{C}}(V_i) \quad \text{where } \{V_1, \dots, V_N\} = \mathrm{Irred}(A).$$

and as an $A$-repn, $E \cong \bigoplus_{i=1}^{N} V_i \otimes \underbrace{W_i}_{\text{aux. space}}$ where

(A acts on $1^{st}$ $\otimes$ factor only)

$W_i \cong \mathrm{Hom}_A(V_i, E)$.

By Schur's lemma, $B = \mathrm{End}_A(E) \cong \bigoplus_{i=1}^{N} \mathrm{End}_{\mathbb{C}}(W_i)$

and the statement of the theorem follows. □

## §2. Another proof of Schur-Weyl duality.

Take $E = \underbrace{V \otimes \cdots \otimes V}_{n\text{-fold}}$ ($V$ : a f.d. $\mathbb{C}$-vector space)

$A$ = Image of $\mathbb{C} S_n \to \mathrm{End}_{\mathbb{C}}(E)$.

is semisimple subalgebra of $\mathrm{End}_{\mathbb{C}}(E)$.

**Theorem.-** $\mathrm{End}_A(E) = \text{Span of } \{ g \otimes \cdots \otimes g : g \in GL(V) \}$

**Proof.-** Let $B = \mathrm{End}_A(E) = \mathrm{End}_{S_n}(\underbrace{V \otimes \cdots \otimes V}_{n\text{-fold}})$

$$\cong \left( \mathrm{End}(V) \otimes \cdots \otimes \mathrm{End}(V) \right)^{S_n}$$

$$= \mathrm{Sym}^n(\mathrm{End}(V)).$$

[ Claim.- for any finite-dim'l space $U$, $\mathrm{Sym}^n(U)$ is spanned by $\{ u \otimes \cdots \otimes u : u \in U \}$.

Proof of the claim.- $F = \mathrm{Span}\{ u \otimes \cdots \otimes u : u \in U \} \subset \mathrm{Sym}^n(U)$ is a non-zero $GL(U)$-subrepn. But $\mathrm{Sym}^n(U)$ is irreducible (Problem 21), hence $F = \mathrm{Sym}^n(U)$. $\square$
as $GL(U)$-repn ]

So, $B = \mathrm{Span}\{ f^{\otimes n} : f \in \mathrm{End}(V) \}$

Let $B_1 \subset B$ be the span of $\{ g^{\otimes n} : g \in GL(V) \}$.

For any $f \in \operatorname{End} V$, $\quad t \cdot Id + f \in GL(V)$ for almost every $t \in \mathbb{C}$.

Hence $(t \cdot Id + f)^{\otimes n} \in B_1$ for almost every $t \in \mathbb{C}$.

As $B_1 \subset B$ is a subspace, hence closed, we conclude that $(t \cdot Id + f)^{\otimes n} \in B_1$ for every $t \in \mathbb{C}$. In particular, for $t = 0$, we get $f^{\otimes n} \in B_1$, which implies that $B_1 = B$ as claimed. $\square$

In other words, $S_n \circlearrowright V^{\otimes n} \circlearrowleft GL(V)$ centralize each other and we get - as a corollary of the previous two theorems.

<u>Cor.</u> $V^{\otimes n}$ is a semisimple $GL(V)$-repn. and

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V_\lambda \otimes L_\lambda \quad \text{as } S_n \times GL(V)\text{-repns.}$$

Here $L_\lambda = \operatorname{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ is irred. $GL(V)$-repn, or zero.

<u>Remark.</u> $L_\lambda = 0 \iff \ell(\lambda) > m \quad (m = \dim(V))$ was proved using $\chi_{L_\lambda}(x_1, \dots, x_m) = S_\lambda(x_1, \dots, x_m)$ Schur polynomials.

where, recall that, for $GL_m(\mathbb{C}) \circlearrowright L$,

$$\chi_L(x_1, \dots, x_m) = \text{Trace of } \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix} \text{ acting on } L.$$

§3. Another proof of $\chi_{L_\lambda}(x_1, \dots, x_m) = s_\lambda(x_1, \dots, x_m)$.

$$S_n \circlearrowright V^{\otimes n} \circlearrowleft GL(V) \qquad (V \cong \mathbb{C}^m)$$

Let $w \in S_n$ and $g = \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_m \end{bmatrix} \in GL_m(\mathbb{C})$. Then,

from $V^{\otimes n} \cong \bigoplus_\lambda V_\lambda \otimes L_\lambda$, we get

$$\text{Trace of } (w, g) \text{ acting on } V^{\otimes n} = \sum_\lambda \chi_{V_\lambda}(w)\, \chi_{L_\lambda}(x_1, \dots, x_m).$$

We compute this trace directly as follows.

Let $\{e_1, \dots, e_m\}$ be the standard basis of $\mathbb{C}^m$, so that

$$\text{Basis of } (\mathbb{C}^m)^{\otimes n} = \{ e_{i_1} \otimes \dots \otimes e_{i_n} \mid i_1, \dots, i_n \in \{1, \dots, m\} \}$$

and
$$g \cdot (e_{i_1} \otimes \dots \otimes e_{i_n}) = x_{i_1} \cdots x_{i_n} (e_{i_1} \otimes \dots \otimes e_{i_n})$$
$$w \cdot (e_{i_1} \otimes \dots \otimes e_{i_n}) = e_{i_{w(1)}} \otimes \dots \otimes e_{i_{w(n)}}.$$

Therefore, as $S_n$-repns,

$$(\mathbb{C}^m)^{\otimes n} = \mathrm{Fun}(I_{m,n}; \mathbb{C}) \text{ where}$$
$$I_{m,n} = \{ \underline{i} = (i_1, \dots, i_n) : i_j \in \{1, \dots, m\} \ \forall j \}$$

Note: (i) $I_{m,n} \longleftrightarrow \mathcal{X}_{m,n} = \left\{(J_1, \ldots, J_m) : \{1,\ldots,n\} = \bigsqcup_{i=1}^{m} J_i \right\}$

$$\underline{i} = (i_1, \ldots, i_n) \rightsquigarrow \left(J_s := \{k : i_k = s\}\right)_{1 \le s \le m}$$

$$(i_1, \ldots, i_n) \leftsquigarrow (J_1, \ldots, J_m)$$

$i_k = s$ if $k \in J_s$.

(ii) $$\mathcal{X}_{m,n} = \bigsqcup_{\substack{\alpha = (\alpha_1, \ldots, \alpha_m) \\ \in \mathbb{Z}_{\ge 0}^m : \sum \alpha_j = n}} \mathcal{X}_{m,n}(\alpha) \qquad \left(\mathcal{X}_{m,n}(\alpha) = \{(J_1, \ldots, J_m) \in \mathcal{X}_{m,n} \text{ s.t. } |J_i| = \alpha_i\}\right)$$

Every basis vector corresponding to $(J_1, \ldots, J_m) \in \mathcal{X}_{m,n}(\alpha)$ is an eigenvector for $g$, with eigenvalue $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$. (Check).

So. Trace of $(w, g)$ acting on $(\mathbb{C}^m)^{\otimes n}$

$$= \sum_{\underline{\alpha} \in \mathbb{Z}_{\ge 0}^m} \left(\text{Trace of } w \text{ acting on } \mathrm{Fun}(\mathcal{X}_{m,n}(\alpha); \mathbb{C})\right) \cdot x_1^{\alpha_1} \cdots x_m^{\alpha_m}$$

$$= \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \le m}} \chi_{U_\lambda}(w) \cdot m_\lambda(x_1, \ldots, x_m)$$

($K_{\mu\lambda}$ = Kostka numbers – see Lecture 11, §1.)

Using $U_\lambda = \bigoplus_\mu V_\mu^{\oplus K_{\mu\lambda}}$ and $s_\mu = \sum_\lambda K_{\mu\lambda} m_\lambda$

we get that this trace $= \sum_{\mu \vdash n} \chi_{V_\mu}(w) \, s_\mu(x_1, \ldots, x_m)$

Comparing the two answers gives:

$$\chi_{L_\lambda}(x_1, \ldots, x_m) = S_\lambda(x_1, \ldots, x_m)$$

$$= \frac{\det\left(x_i^{\lambda_j + m - j}\right)}{\prod_{1 \leq i < j \leq m} x_i - x_j}$$

$$\boxed{\chi_{L_\lambda}(x_1, \ldots, x_m) = \frac{\sum_{w \in S_m} \operatorname{sgn}(w)\, \underline{x}^{w(\lambda+\rho)}}{\prod_{i<j} x_i - x_j}}$$

$\rho = (m-1, m-2, \ldots, 1, 0)$

Weyl character formula for $GL_m(\mathbb{C})$.

<u>Exercise</u> - Show that

$$S_\lambda(1, t, t^2, \ldots, t^{m-1}) = \prod_{1 \leq i < j \leq m} \frac{t^{\lambda_j + m - j} - t^{\lambda_i + m - i}}{t^{i-1} - t^{j-1}}$$

$$\rightarrow \prod_{i<j} \frac{\lambda_i - \lambda_j - i + j}{j - i} = \dim(L_\lambda)$$

(Weyl dimension formula)