# Lecture 22

## Lie algebras and their representations.

§1. Definition. — A Lie algebra* $\mathfrak{g}$ (over a field $k$) is a $k$-vector space, together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (called Lie bracket)

(i) Skew-symmetry : $[X, X] = 0 \quad \forall X \in \mathfrak{g}$.

(or, equivalently - if $\mathrm{char}(k) \neq 2$. $[X,Y] = -[Y,X] \; \forall X, Y \in \mathfrak{g}$.)

(ii) Jacobi identity : $\forall X, Y, Z \in \mathfrak{g}$, we have:

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0.$$

§2. Main examples. — (i) Derivations — Let $\mathcal{A}$ be a $k$-algebra. (not necessarily comm., assoc. or unital)

$$\mathrm{Der}_k(\mathcal{A}) := \{ f : \mathcal{A} \to \mathcal{A} \ k\text{-linear s.t. } f(ab) = f(a)b + af(b) \ \forall a, b \in \mathcal{A} \}$$

Then $\mathrm{Der}_k(\mathcal{A})$ is a Lie algebra with Lie bracket given by

$$[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1.$$

Note: Composition of derivations is not necessarily a derivation, but commutator of two derivations is another derivation.

Proof. - $d_1(d_2(ab)) = d_1(d_2(a)b + a d_2(b))$

$= d_1(d_2(a))b + d_2(a)d_1(b) + d_1(a)d_2(b) + ad_1(d_2(b)) \quad [\forall a, b \in \mathcal{A};\ d_1, d_2 \in \mathrm{Der}(\mathcal{A})]$.

* Sophus Lie (Dec. 17, 1842 - Feb. 18, 1899)

$$\Rightarrow [d_1, d_2](ab) = ([d_1,d_2](a))b + a \cdot ([d_1,d_2](b)). \quad \square$$

Jacobi identity – $[d_1,[d_2,d_3]] = [d_1, d_2d_3 - d_3d_2]$

$$= d_1d_2d_3 - d_1d_3d_2 - d_2d_3d_1 + d_3d_2d_1.$$

$$\Rightarrow [d_1,[d_2,d_3]] + [d_3,[d_1,d_2]] + [d_2,[d_3,d_1]] = 0.$$

(ii) Assoc. alg's $\rightsquigarrow$ Lie alg's : If $A$ is any assoc. alg. $/ k$, then

$L(A) = A$ as vector space, together with

$$[a,b] = ab - ba, \quad \forall\, a,b \in A,$$

is a Lie algebra.

(iii) $\mathfrak{gl}_N(k)$ = Lie algebra of all $N \times N$ matrices over $k$.

( $A = M_{N \times N}(k)$ in the previous example. )

$\mathfrak{sl}_N(k) \subset \mathfrak{gl}_N(k)$ consisting of <u>all matrices of trace zero</u>,

is a Lie subalgebra of $\mathfrak{gl}_N(k)$.

(iv) Abelian Lie algebra. – $\mathfrak{a}$ is <u>abelian</u> if $[a,b] = 0$, $\forall\, a,b \in \mathfrak{a}$.

e.g. $\mathfrak{h}_N(k) \subset \mathfrak{gl}_N(k)$ consisting of <u>diagonal matrices</u>

is an <u>abelian Lie algebra</u>.

§2. Homomorphisms of Lie algebras. - Let $(\mathfrak{g}_1, [\cdot,\cdot]_1)$ and $(\mathfrak{g}_2, [\cdot,\cdot]_2)$ be two Lie algebras. A Lie alg. hom. is a $k$-linear map

$$f : \mathfrak{g}_1 \to \mathfrak{g}_2 \quad \text{s.t.} \quad f([x,y]_1) = [f(x), f(y)]_2 \quad \forall\, x, y \in \mathfrak{g}_1.$$

Example. - Adjoint Hom. - Let $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_k(\mathfrak{g})$ be given by

$$\mathrm{ad}(x) : y \longmapsto [x,y].$$

Jacobi identity can be viewed as a statement that
ad is a Lie alg. hom, and $\mathrm{ad}(x) \in \mathrm{Der}_k(\mathfrak{g})$ $\forall x \in \mathfrak{g}$.

( Proof. - $\mathrm{ad}(x) \in \mathrm{Der}_k(\mathfrak{g}) \iff \mathrm{ad}(x)([y,z]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]$

$\forall\, y, z \in \mathfrak{g}$

$\iff [x,[y,z]] = [[x,y],z] + [y,[x,z]]$ ( Jacobi id - assuming skew-symmetry)

Similarly, ad is a Lie alg. hom.

$\iff \mathrm{ad}([x,y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$ $\forall x,y \in \mathfrak{g}$ ← matrix commutator

$\iff \mathrm{ad}([x,y]) \cdot z = \mathrm{ad}(x) \cdot (\mathrm{ad}(y) \cdot z) - \mathrm{ad}(y) \cdot (\mathrm{ad}(x) \cdot z)$

$\forall\, x, y, z \in \mathfrak{g}$.

$\iff [[x,y],z] = [x,[y,z]] - [y,[x,z]]$ : Jacobi id. $\square$ )

Example. - Free Lie alg - Let $A = \{a_j\}_{j \in J}$ be a set.

$\mathrm{Free}_{(LA)}(A)$ is defined by the following universal property:

There is a canonical (set) ~~inclusion~~ map $A \to \mathrm{Free}(A)$.

For any Lie algebra $\mathfrak{g}$ and a (set) map $A \to \mathfrak{g}$, $\exists!$ Lie alg. hom. $\mathrm{Free}(A) \to \mathfrak{g}$ s.t.

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ \mathrm{Free}(A) \arrow[rr] & & \mathfrak{g} \end{tikzcd}$$

commutes.

– i.e., $\mathrm{Hom}_{LA}(\mathrm{Free}(A), \mathfrak{g}) = \mathrm{Hom}_{set}(A, \mathfrak{g}) \quad \forall \mathfrak{g}$.

(a) If $|A| = 1$, then $\mathrm{Free}(A) = k$ with zero bracket - necessarily.

(b) If $|A| = 2$, say $A = \{x, y\}$; then $\mathrm{Free}(A)$ is infinite-dim'l.

$\mathrm{Free}(A) =$ $k$- span of all parenthesized (non-empty) words in $x, y$ quotiented by subspace generated by $(w_1, w_2) + (w_2, w_1)$ and $(w_1(w_2 w_3)) + (w_3(w_1 w_2)) + (w_2(w_3 w_1))$

e.g. $\mathrm{Free}(A)$ degree 1 part is 2-dim'l - $\{x, y\}$ basis.

———— 2 ——— 1-dim'l - $\{(xy)\}$ basis.

———— 3 ——— 2 dim'l - $\{x(xy),\ y(xy) = -y(yx)\}$

Exercise - What is the dim. of $\mathrm{Free}(A)_{deg = N}$?

§3. Representations of Lie algebras. - Let $\mathfrak{g}$ be a Lie algebra.

A representation of $\mathfrak{g}$ is a Lie alg. hom $\mathfrak{g} \xrightarrow{\pi_V} \mathfrak{gl}(V)$ where $V$ is a $k$-vector space. Often denoted by $\mathfrak{g} \circlearrowright V$, we usually omit $\pi_V$ for ease of notation. : $X \cdot v = \pi_V(X)(v)$.

The usual notions of <u>kernel / image</u> of <u>$\mathfrak{g}$-intertwiners</u>, <u>sub / quotient repns</u>, $\oplus$ of 2 repns., <u>irreducible / indecomposable repns</u> - carry over to this setting of Lie algebras.

Rep $(\mathfrak{g})$ has $\otimes$ and duals - similar to the case of groups. :

Let $V_1, V_2$ be two $\mathfrak{g}$-repns. $\underset{\text{L.A. homs.}}{\pi_j : \mathfrak{g} \to \mathfrak{gl}(V_j)}$ $(j = 1, 2)$.

Then $\boxed{\pi_{V_1 \otimes V_2}(x) := \pi_1(x) \otimes Id_{V_2} + Id_{V_1} \otimes \pi_2(x)}$ $\forall x \in \mathfrak{g}$

is a Lie alg. hom $\mathfrak{g} \longrightarrow \cancel{End}_k\ \mathfrak{gl}(V_1 \otimes V_2)$.

Hence $\mathfrak{g} \circlearrowright V_1 \otimes V_2$ by $x \cdot (v_1 \otimes v_2) = (x \cdot v_1) \otimes v_2 + v_1 \otimes (x \cdot v_2)$.

Similarly $\mathfrak{g} \circlearrowright V \rightsquigarrow \mathfrak{g} \circlearrowright V^*$ $\quad (x \cdot \xi)(v) = -\xi(x \cdot v)$
$\forall\ \xi \in V^*,\ x \in \mathfrak{g},\ v \in V.$

$\mathfrak{g} \circlearrowright \mathrm{Hom}_k(V_1, V_2)$ by $(x \cdot f)(v_1) = x \cdot (f(v_1)) - f(x \cdot v_1)$
$\forall\ \underset{k\text{-linear}}{f : V_1 \to V_2}$ ; $v_1 \in V_1$, $x \in \mathfrak{g}$.

<u>Ex.</u> $V_1^* \otimes V_2 \longrightarrow \mathrm{Hom}_k(V_1, V_2)$ is $\mathfrak{g}$-intertwiner.

§4. Examples. - (Abelian Lie alg's).

(i) Let $\mathfrak{a}_N$ be abelian Lie algebra of dimension $N$.

Let $\{x_1, \dots, x_N\}$ be a basis of $\mathfrak{a}_N$. A repn. of $\mathfrak{a}_N$, is completely given by $N$ commuting matrices.

Assuming $k$ is alg. closed (say $k = \mathbb{C}$) - if $\{X_1 = \rho(x_1), \ldots, X_N = \rho(x_N)\} \subset End(V)$

Then we can find a joint eigenvector for $X_1, \ldots, X_N$; say $v_1$.

($\rho : \mathfrak{a}_N \to \mathfrak{gl}(V)$ a f.d. repn.)

(i.e. $\exists\ \gamma_1 : \mathfrak{a}_N \to \mathbb{C}$ linear form, $0 \neq v_1 \in V$ s.t. $\rho(x) \cdot v_1 = \gamma_1(x) \cdot v_1$.)

Quotient by subrepn $\mathbb{C} \cdot v_1 \subset V$ and repeat to get a basis $\{v_1, \ldots, v_\ell\}$ of $V$, and linear forms $\gamma_1, \ldots, \gamma_\ell \in \mathfrak{a}_N^*$ s.t.

$$\rho(x) \cdot v_j = \gamma_j(x)\, v_j + \sum_{i=1}^{j-1} c_{ij}(x)\, v_i .$$

~~[Finitely many~~ (i.e., commuting family of linear operators on a f.d. vector space over alg. closed field, can be simultaneously "triangularized".)

Moreover $\mathrm{Irr}_{fd}(\mathfrak{a}_N) \longleftrightarrow \mathfrak{a}_N^*$ (all 1-dim'l).

(ii) Let $\mathfrak{g}$ be the Lie alg. generated by $x, y$ subject to $[x,y] = \gamma y$ ($\gamma \in k \setminus \{0\}$).

Claim - (Char $k = 0$, $k = \bar{k}$) - Every f.d. irred. $\mathfrak{g}$-repn. is 1-dim'l.

Pf. - Let $\mathfrak{g} \circlearrowright V$ be a f.d. irred. $\mathfrak{g}$-repn. Let $0 \neq v \in V$ be an eigenvector for $x$, with eigenvalue $c \in k$.

Then $\forall\ \ell \geq 0$; $y^\ell . v$ is an eigenvector for $x$ with eigenvalue $c + \ell \cdot \gamma$ ( Proof - $[x, y^\ell] = \sum_{i=1}^{\ell} y^{i-1} [x,y]\, y^{\ell - i} = \gamma \cdot \ell \cdot y^\ell$

(This calculation is in $End(V)$)

$$\Rightarrow \quad x\cdot(y^{\ell}\cdot v) = y^{\ell}\cdot(x\cdot v) + [x, y^{\ell}]\cdot v$$
$$= (c + \ell\gamma)\, y^{\ell} v \,. \;)$$

Since all these eigenvalues are distinct (char $k = 0$), by finite-dim of $V$,
$\exists\, N$ s.t. $y^N v \neq 0$ and $y^n \cdot v = 0 \quad \forall\, n > N$.
$(N \geq 0)$

$\Rightarrow \quad k\cdot(y^N \cdot v) \subset V$ is a subrepn. By irred. $V$ is 1-dim'l $\square$

§5. Very important example. – $\mathfrak{g} = \mathfrak{sl}_2(k)$ ($k$ alg. closed, char$(k) = 0$).
2×2 matrices of trace zero.

Basis of $\mathfrak{g}$ : $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Lie bracket : $[h, e] = 2e$ ; $[h, f] = -2f$ , $[e, f] = h$.

( Abstract presentation of $\mathfrak{sl}_2$ - three generators & three rel$^{\underline{n}}$s.)

Theorem. – $\mathrm{Irr}_{fd}(\mathfrak{sl}_2) \longleftrightarrow \mathbb{Z}_{\geq 0}$
$L_n \longleftrightarrow n$ where $L_n$ is $(n+1)$-dim'l

irred. repn. given explicitly as :

Basis of $L_n$ : $\{v_0, \ldots, v_n\}$

$\mathfrak{sl}_2$-action : $\begin{cases} h\, v_j = (n - 2j)\, v_j \\ f\, v_j = (j+1)\, v_{j+1} \\ e\, v_j = (n - j + 1)\, v_{j-1} \end{cases}$ $\quad (v_{-1} = v_{n+1} = 0)$

Proof.- Let $V$ be a f.d. irred. repn. of $sl_2$.

By argument of Example (ii) §4, there exists $0 \neq v_0 \in V$ s.t. and $\lambda \in k$

$$h \cdot v_0 = \lambda v_0 \qquad e \cdot v_0 = 0 .$$

Define: $v_j = \dfrac{f^j}{j!} v_0 \quad \forall\, j \geq 0.$

(so $h \cdot v_j = (\lambda - 2j) v_j$)

Note: $$[e, f^r] = \sum_{i=1}^{r} f^{i-1} [e,f] f^{r-i} = \sum_{i=1}^{r} f^{i-1} h f^{r-i}$$

[all this happens in End $V$.]

as $hf = fh - 2f = f(h-2)$

we get $h \cdot f^{r-i} = f^{r-i}(h - 2(r-i))$

$$\Rightarrow [e, f^r] = \sum_{i=1}^{r} f^{i-1+r-i}(h - 2(r-i)) = r \cdot f^{r-1} h - r(r-1) f^{r-1}$$

$$= r \cdot f^{r-1}(h - r + 1)$$

Hence, $$e \cdot v_j = \frac{e f^j}{j!} v_0 = \frac{\cancel{f^j e} + [e, f^j]}{j!} \cdot v_0 \qquad (e \cdot v_0 = 0)$$

$$= j \cdot \frac{f^{j-1}}{j!} (\lambda - j + 1) v_0 = (\lambda - j + 1) v_{j-1}.$$

As $\dim V < \infty$, let $N \in \mathbb{Z}_{\geq 0}$ be s.t. $v_N \neq 0$ and $v_{N+1} = 0$. Then

$$0 = e \cdot v_{N+1} = (\lambda - N) v_N \Rightarrow \lambda = N \text{ and } \dim V = N+1$$

$V \cong L_N$ from Thm. $\square$