Lecture 29.

Universal enveloping algebras

§1. <u>Definition</u>. — Let $\mathfrak{g}$ be a Lie algebra over a field k. The <u>universal enveloping algebra of $\mathfrak{g}$</u>, denoted by $\mathcal{U}(\mathfrak{g})$, is a <u>unital, assoc alg. / k</u>, satisfying the following universal property.

- There is a k-linear map $\mathfrak{g} \xrightarrow{i} \mathcal{U}(\mathfrak{g})$ s.t.
$$i([x,y]) = i(x)\,i(y) - i(y)\,i(x) \quad \forall\, x,y \in \mathfrak{g}.$$
- If A is any unital assoc. algebra (over k), and $f : \mathfrak{g} \to A$ is a k-linear map satisfying $f([x,y]) = f(x)f(y) - f(y)f(x)$ $\forall x,y \in \mathfrak{g}$,

Then $\exists!$ (unital) alg. hom. $\tilde{f} : \mathcal{U}(\mathfrak{g}) \to A$ s.t. the following diagram commutes

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, "i"'] \arrow[dr, "f"] & \\ \mathcal{U}(\mathfrak{g}) \arrow[rr, "\tilde{f}"'] & & A \end{tikzcd}$$

In less words, $\operatorname{Hom}_{\text{Lie Alg}}(\mathfrak{g}, L(A)) = \operatorname{Hom}_{k\text{-alg}}(\mathcal{U}(\mathfrak{g}), A)$

for every unital, assoc. alg. A. (Recall: $L(A) = A$ as vector spaces with Lie bracket defined as commutator).

i.e.
$$\begin{tikzcd} \text{Lie Algebras over } k \arrow[r, bend right, "\mathcal{U}"'] & \text{Unital assoc. alg's over } k \arrow[l, bend right, "L"'] \end{tikzcd}$$
are adjoint functors.

§2. <u>Construction of $\mathcal{U}(\mathfrak{g})$</u> Given $\mathfrak{g}$ as above, define the tensor algebra of $\mathfrak{g}$: $\quad T^{\bullet}(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^{n}(\mathfrak{g})$,

$T^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{n\text{-fold}}$. Multiplication in $T^{\bullet}(\mathfrak{g})$ is nothing but concatenation of tensors.

$T^0(\mathfrak{g}) = k$

Let $J \subset T^{\bullet}(\mathfrak{g})$ be the 2-sided ideal generated by

$$x \otimes y - y \otimes x - [x,y] \quad (x, y \in \mathfrak{g}).$$

Then

$$\mathcal{U}(\mathfrak{g}) := T^{\bullet}(\mathfrak{g})/J.$$

The map $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ arises as the following composition:

$$\mathfrak{g} = T^1(\mathfrak{g}) \hookrightarrow T^{\bullet}(\mathfrak{g}) \twoheadrightarrow \mathcal{U}(\mathfrak{g}) = T^{\bullet}(\mathfrak{g})/J.$$

<u>Verification of the universal property</u>: If A is a unital, assoc. alg. and $f : \mathfrak{g} \to A$ is a k-linear map, then — by universal property of $T^{\bullet}(\mathfrak{g})$. [$\mathrm{Hom}_{Alg}(T^{\bullet}(V), B) = \mathrm{Hom}_{v.s.}(V, B)$ $\forall B$: unital, assoc. alg.]

we get : there is a unique alg. hom. $f_1 : T^{\bullet}(\mathfrak{g}) \to A$ s.t. $f_1(x) = f(x)$ $\forall x \in \mathfrak{g} = T^1(\mathfrak{g})$.

If f satisfies $f([x,y]) = f(x) f(y) - f(y) f(x)$ $(\forall x, y \in \mathfrak{g})$ then $J \subset \mathrm{Ker}(f)$. Hence, we have a unique alg. hom

$$\tilde{f} : T^{\bullet}(\mathfrak{g})/J = \mathcal{U}(\mathfrak{g}) \longrightarrow A$$

s.t.

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl] \arrow[dr, "f"] & \\ \mathcal{U}(\mathfrak{g}) \arrow[rr, "\tilde{f}"'] & & A \end{tikzcd}$$

commutes.

§3. Remarks and examples. –

(i) Again by universal property of $\mathcal{U}(\mathfrak{g})$, we have

$$\mathrm{Hom}_{L.A.}(\mathfrak{g}, \mathfrak{gl}(V)) = \mathrm{Hom}_{Alg.}(\mathcal{U}(\mathfrak{g}), \mathrm{End}(V))$$

(recall: $\mathfrak{gl}(V) = L(\mathrm{End}\,V)$).

i.e. $\boxed{\mathfrak{g}\text{-repns} \longleftrightarrow \mathcal{U}(\mathfrak{g})\text{-repns.}}$.

(ii) Gradings and filtrations. – Note: $T^{\cdot}(\mathfrak{g})$ is $\mathbb{N}$-graded by:

$$\text{degree of } T^n(\mathfrak{g}) = n \quad \forall\, n \in \mathbb{N}. \quad (\mathbb{N} = \mathbb{Z}_{\geq 0}).$$

(recall: an $\mathbb{N}$-graded algebra is an alg. $/\Bbbk$ A, together with subspaces $A_n \subset A$ $\forall n \in \mathbb{N}$ s.t. $A = \bigoplus_{n \in \mathbb{N}} A_n$ as a vector space, and $\left.\begin{array}{l} a \in A_n \\ b \in A_m \end{array}\right\} \Rightarrow a \cdot b \in A_{n+m}.$)

However the ideal $\mathcal{J} = \langle \overbrace{x \otimes y - y \otimes x}^{\text{In deg 2}} - \underbrace{[x,y]}_{\text{in deg 1.}} \rangle$ is not homogeneous.

So, $\mathcal{U}(\mathfrak{g})$ does not inherit $\mathbb{N}$-grading from $T^{\cdot}(\mathfrak{g})$. It only gets a filtration – defined as follows.

$$\mathcal{F}_n(\mathcal{U}(\mathfrak{g})) := \text{Image of} \left(\bigoplus_{j=0}^{n} T^j(\mathfrak{g}) \hookrightarrow T^{\cdot}(\mathfrak{g}) \twoheadrightarrow \mathcal{U}(\mathfrak{g}) \right).$$

Check: $\mathcal{F}_{-1} = \{0\}$; $\mathcal{F}_0 = \Bbbk$.

$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ $\qquad \mathcal{U}(\mathfrak{g}) = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n(\mathcal{U}(\mathfrak{g}))$.

$a \in \mathcal{F}_n$, $b \in \mathcal{F}_m \Rightarrow a \cdot b \in \mathcal{F}_{n+m}$.

Easy exercise : $\bigoplus_{n=0}^{\infty} \mathcal{F}_n / \mathcal{F}_{n-1}$ is a commutative algebra

(i.e. given $\begin{array}{l} a \in \mathcal{F}_i \\ b \in \mathcal{F}_j \end{array}$, $ab - ba \in \mathcal{F}_{i+j-1}$.)

(iii) If $\mathfrak{g} = \text{Free}(V)$ (V : f.d. $\Bbbk$-vector space) - defined by the universal property $\text{Hom}_{LA}(\text{Free}(V), \mathfrak{a}) = \text{Hom}_{v.s.}(V, \mathfrak{a})$ -

($\mathfrak{a}$: Lie alg. / $\Bbbk$).

then $\mathcal{U}(\mathfrak{g}) \cong T^\bullet(V)$.

$$\begin{tikzcd} \text{Lie Alg.} \arrow[rr, "\mathcal{U}"] & & \text{Assoc. Alg.} \\ & \text{Vector Spaces} \arrow[ul, "\text{Free}"] \arrow[ur, "\text{Free} = T^\bullet"'] & \end{tikzcd}$$ commutes.

(Proof : $\text{Hom}_{Alg}(\mathcal{U}(\text{Free}(V)), A) = \text{Hom}_{LA}(\text{Free}(V), L(A))$

$= \text{Hom}_{v.s.}(V, A)$

$= \text{Hom}_{Alg}(T^\bullet(V), A)$ $\square$)

- ~~Application to computing graded dim. of Free Lie alg.~~ (later)

(iv) - Generalizing (iii) - if $\mathfrak{g}$ is given by generators and relations - then $\mathcal{U}(\mathfrak{g})$ has the "same" presentation.

e.g. $\mathfrak{sl}_2$: generators h, e, f.
rel^{n}s: $[h,e] = 2e$, $[h,f] = -2f$
$[e,f] = h$.

$\mathcal{U}(\mathfrak{sl}_2)$: unital, assoc. alg. generated by h, e, f - subject to relations $he - eh = 2e$, $hf - fh = -2f$, $ef - fe = h$.

§4. Poincaré - Birkhoff - Witt (PBW) Theorem. - If $\mathfrak{g}$ has a basis $\{x_i\}_{i\in I}$ (I : a totally ordered indexing set), then

$\mathcal{U}(\mathfrak{g})$ has a basis $\left\{ x_{i_1} x_{i_2} \cdots x_{i_n} : \begin{array}{l} n \geq 0 ;\ i_1, \dots, i_n \in I \\ i_1 \leq i_2 \cdots \leq i_n \end{array} \right\}$

(Alternately written as : $\bigoplus_{n\in\mathbb{N}} \mathcal{F}_n / \mathcal{F}_{n-1} \cong \operatorname{Sym}^{\bullet}(\mathfrak{g})$)
(see §3 (ii) for $\mathcal{F}_n(\mathcal{U}(\mathfrak{g}))$.)

e.g. $\mathcal{U}(\mathfrak{sl}_2)$ has the following basis $\{ f^a h^b e^c : a, b, c \geq 0 \}$.