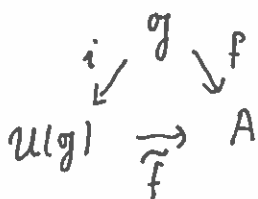


# Lecture 29.

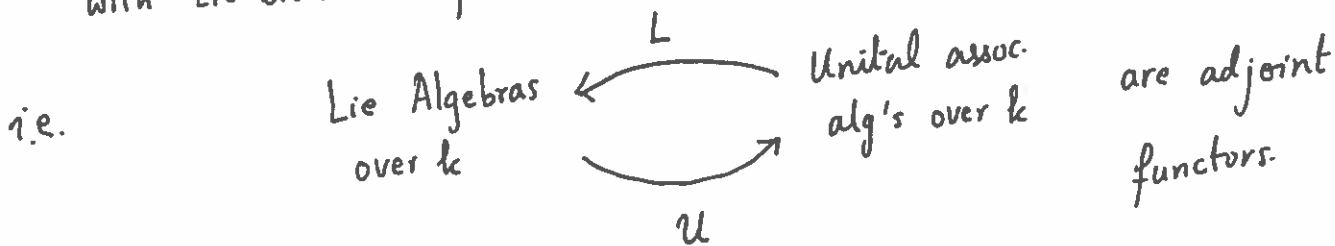
## Universal enveloping algebras

§1. Definition. - Let  $\mathfrak{g}$  be a Lie algebra over a field  $k$ . The universal enveloping algebra of  $\mathfrak{g}$ , denoted by  $U(\mathfrak{g})$ , is a unital, assoc. alg. /  $k$ , satisfying the following universal property.

- There is a  $k$ -linear map  $\mathfrak{g} \xrightarrow{i} U(\mathfrak{g})$  s.t.
 
$$i([\mathfrak{x}, \mathfrak{y}]) = i(\mathfrak{x})i(\mathfrak{y}) - i(\mathfrak{y})i(\mathfrak{x}) \quad \forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{g}.$$
- If  $A$  is any unital assoc. algebra (over  $k$ ), and  $f: \mathfrak{g} \rightarrow A$  is a  $k$ -linear map satisfying  $f([\mathfrak{x}, \mathfrak{y}]) = f(\mathfrak{x})f(\mathfrak{y}) - f(\mathfrak{y})f(\mathfrak{x}) \quad \forall \mathfrak{x}, \mathfrak{y} \in \mathfrak{g}$ , then  $\exists!$  (unital) alg. hom.  $\tilde{f}: U(\mathfrak{g}) \rightarrow A$  s.t. the following diagram commutes



In less words,  $\text{Hom}_{\text{Lie Alg}}(\mathfrak{g}, L(A)) = \text{Hom}_{k\text{-alg}}(U(\mathfrak{g}), A)$  for every unital, assoc. alg.  $A$ . (Recall:  $L(A) = A$  as vector space, with Lie bracket defined as commutator).



§2. Construction of  $U(\mathfrak{g})$  Given  $\mathfrak{g}$  as above, define the tensor algebra of  $\mathfrak{g}$ :  $T^\bullet(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^n(\mathfrak{g})$ ,

(2)

$T^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \dots \otimes \mathfrak{g}}_{n\text{-fold}}$ . Multiplication in  $T^\bullet(\mathfrak{g})$  is nothing but concatenation of tensors.  
 $T^0(\mathfrak{g}) = k$

Let  $\mathcal{I} \subset T^\bullet(\mathfrak{g})$  be the 2-sided ideal generated by

$$x \otimes y - y \otimes x - [x, y] \quad (x, y \in \mathfrak{g}). \text{ Then}$$

$$U(\mathfrak{g}) := T^\bullet(\mathfrak{g}) / \mathcal{I}.$$

The map  $\mathfrak{g} \rightarrow U(\mathfrak{g})$  arises as the following composition:

$$\mathfrak{g} = T^1(\mathfrak{g}) \hookrightarrow T^\bullet(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) = T^\bullet(\mathfrak{g}) / \mathcal{I}.$$

Verification of the universal property: If  $A$  is a unital, assoc. alg.

and  $f: \mathfrak{g} \rightarrow A$  is a  $k$ -linear map, then - by universal

$$\text{property of } T^\bullet(\mathfrak{g}) \cdot \left[ \text{Hom}_{\text{Alg}}(T^\bullet(V), B) = \text{Hom}_{\text{v.s.}}(V, B) \right. \\ \left. \forall B, \text{ unital, assoc. alg.} \right]$$

we get : there is a unique alg. hom.  $f_1: T^\bullet(\mathfrak{g}) \rightarrow A$  s.t.  
 $f_1(x) = f(x) \quad \forall x \in \mathfrak{g} = T^1(\mathfrak{g}).$

If  $f$  satisfies  $f([x, y]) = f(x)f(y) - f(y)f(x) \quad (\forall x, y \in \mathfrak{g})$

then  $\mathcal{I} \subset \text{Ker}(f)$ . Hence, we have a unique alg. hom

$$\tilde{f}: T^\bullet(\mathfrak{g}) / \mathcal{I} = U(\mathfrak{g}) \longrightarrow A \text{ s.t.}$$

$\swarrow$   
 $U(\mathfrak{g})$

$\searrow$   
 $\tilde{f}$

$\swarrow$   
 $A$

$\nearrow$   
 $f$

$\searrow$   
 $A$

commutes.

§3. Remarks and examples. -

(i) Again by universal property of  $U(\mathfrak{g})$ , we have

$$\text{Hom}_{L.A.}(\mathfrak{g}, \mathfrak{gl}(V)) = \text{Hom}_{Alg.}(U(\mathfrak{g}), \text{End}(V))$$

(recall:  $\mathfrak{gl}(V) = L(\text{End}V)$ ).

i.e.  $\mathfrak{g}\text{-reps} \leftrightarrow U(\mathfrak{g})\text{-reps.}$

(ii) Gradings and filtrations. - Note:  $T^*(\mathfrak{g})$  is  $\mathbb{N}$ -graded by:

$$\text{degree of } T^n(\mathfrak{g}) = n \quad \forall n \in \mathbb{N}. \quad (\mathbb{N} = \mathbb{Z}_{\geq 0}).$$

(recall: an  $\mathbb{N}$ -graded algebra is an alg.  $A$ , together with subspaces  $A_n \subset A \quad \forall n \in \mathbb{N}$  s.t.  $A = \bigoplus_{n \in \mathbb{N}} A_n$  as a vector space,

and  $a \in A_n, b \in A_m \Rightarrow a \cdot b \in A_{n+m}$ .)

However the ideal  $\mathcal{J} = \langle \overbrace{x \otimes y - y \otimes x}^{\text{in deg 2}} - [x, y] \rangle$  is not homogeneous.

So,  $U(\mathfrak{g})$  does not inherit  $\mathbb{N}$ -grading from  $T^*(\mathfrak{g})$ . It only gets a filtration - defined as follows -

$$F_n(U(\mathfrak{g})) := \text{Image of } \left( \bigoplus_{j=0}^n T^j(\mathfrak{g}) \hookrightarrow T^*(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}) \right).$$

Check:  $F_{-1} = \{0\}; F_0 = k.$

$$F_0 \subset F_1 \subset \dots$$

$$U(\mathfrak{g}) = \bigcup_{n \in \mathbb{N}} F_n(U(\mathfrak{g})).$$

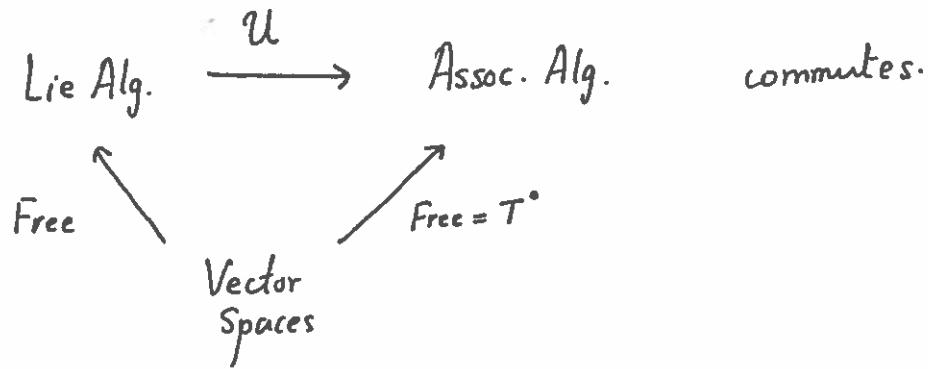
$$a \in F_n, b \in F_m \Rightarrow a \cdot b \in F_{n+m}.$$

Easy exercise :  $\bigoplus_{n=0}^{\infty} \mathcal{F}_n / \mathcal{F}_{n-1}$  is a commutative algebra (4)

(i.e. given  $a \in \mathcal{F}_i$ ,  $b \in \mathcal{F}_j$ ,  $ab - ba \in \mathcal{F}_{i+j-1}$ .)

(iii) If  $\mathfrak{g} = \text{Free}(V)$  ( $V$ : f.d.  $k$ -vector space) - defined by the universal property  $\text{Hom}_{\text{LA}}(\text{Free}(V), \mathfrak{a}) = \text{Hom}_{\text{v.s.}}(V, \mathfrak{a})$  - ( $\mathfrak{a}$ : Lie alg.  $(k)$ ).

then  $\mathcal{U}(\mathfrak{g}) \cong T^*(V)$ .



(Proof :  $\text{Hom}_{\text{Alg}}(\mathcal{U}(\text{Free}(V)), A) = \text{Hom}_{\text{LA}}(\text{Free}(V), L(A))$   
 $= \text{Hom}_{\text{v.s.}}(V, A)$   
 $= \text{Hom}_{\text{Alg}}(T^*(V), A) \quad \square$ )

~~- Application to computing graded dim. of Free Lie alg. (later)~~

(iv) - Generalizing (iii) - if  $\mathfrak{g}$  is given by generators and relations - then  $\mathcal{U}(\mathfrak{g})$  has the "same" presentation.

e.g.  $\mathfrak{sl}_2$  : generators  $h, e, f$ .  
 rel<sup>s</sup>:  $[h, e] = 2e, [h, f] = -2f$   
 $[e, f] = h$ .

$U(\mathfrak{sl}_2)$  : unital, assoc. alg. generated by  $h, e, f$  - subject to relations  $he - eh = 2e, hf - fh = -2f, ef - fe = h$ .

§4. Poincaré-Birkhoff-Witt (PBW) Theorem. - If  $\mathfrak{g}$  has a basis

$\{x_i\}_{i \in I}$  ( $I$  : a totally ordered indexing set), then

$U(\mathfrak{g})$  has a basis  $\left\{ x_{i_1} x_{i_2} \dots x_{i_n} : \begin{matrix} n \geq 0; i_1, \dots, i_n \in I \\ i_1 \leq i_2 \leq \dots \leq i_n \end{matrix} \right\}$

(Alternately written as:  $\bigoplus_{n \in \mathbb{N}} \mathcal{F}_n / \mathcal{F}_{n-1} \cong \text{Sym}(\mathfrak{g})$ )  
 (see §3 (ii) for  $\mathcal{F}_n(U(\mathfrak{g}))$ .)

e.g.  $U(\mathfrak{sl}_2)$  has the following basis  $\{ f^a h^b e^c : a, b, c \geq 0 \}$ .

1, 1, 1, 1