Lecture 25

Recall - we proved the following results about repns. of $\mathcal{U}(sl_2)$:

$$\left[\begin{array}{l} \mathcal{U}(sl_2) = \text{(unital, associative) } \mathbb{C}\text{-algebra generated by } \{h, e, f\} \\ \text{subject to 3 relations:} \quad he = eh + 2e \\ \qquad\qquad\qquad\qquad\qquad hf = fh - 2f \\ \qquad\qquad\qquad\qquad\qquad ef = fe + h \end{array}\right]$$

(i) $\mathrm{Irred}_{fd}(sl_2) = \{L_n : n \in \mathbb{Z}_{\geq 0}\}$ where L_n is $(n+1)$-dim'l irred. repn. of sl_2 given explicitly as: L_n has a basis $\{v_0, \ldots, v_n\}$; rel. to which, the sl_2-action is given by

$$h v_j = (n-2j) v_j \quad ; \quad e.v_j = (n-j+1) v_{j-1} \quad ; \quad f.v_j = (j+1) v_{j+1}.$$

(ii) Every f.d. repn. of sl_2 is completely reducible.

Therefore, we obtain the following results for an arbitrary f.d. repn. $sl_2 \circlearrowright V$:

(a) h-action on V is diagonalizable - i.e.,

$$V = \bigoplus_{\gamma \in \mathbb{C}} V_\gamma[\gamma] \quad ; \quad V[\gamma] = \{v \in V : h.v = \gamma v\}$$

(called <u>weight space</u> of weight γ).

Let $P(V) = \{\gamma \in \mathbb{C} : V[\gamma] \neq \{0\}\} \subset \mathbb{C}$

(set of weights of V)

(b) $P(V) \subset \mathbb{Z}$ (weights of f.d. repns. are integral).

(c) If $\exists\ 0 \neq v \in V[\ell]$ s.t. $e.v = 0$, then $\ell \geq 0$ and $f^{\ell+1}.v = 0$.

(d) (Weyl symmetry) $\dim V[\ell] = \dim V[-\ell] \quad \forall\, \ell \in P(V)$.

(e) # of irred. summands in V = dim Ker(e acting on V).

(f) $\forall\, \ell \in P(V)$; $\ell \geq 0$; the linear map obtained from the action of $e: V[\ell] \to V[\ell+2]$ is surjective.

Hence, # of irred. summands in V with even highest weight $= \dim V[0]$ ~~XXXXXX~~

# of irred. summands in V with odd highest weight $= \dim V[1]$.

§1. Weyl symmetry in f.d. repns. of sl_2.

[Character of V].

Definition: Given a f.d. repn. V of sl_2, let $\chi_V(z) := \sum_{\ell \in \mathbb{C}} \dim V[\ell] \cdot z^\ell$.

Integrality of weights : $\chi_V(z) \in \mathbb{Z}_{\geq 0}[z, z^{-1}]$

Weyl symmetry : $\chi_V(z) = \chi_V(z^{-1})$

e.g. $$\chi_{L_n}(z) = z^n + z^{n-2} + \dots + z^{-n+2} + z^{-n}$$

$$= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}$$

Ex: Verify the usual properties of characters:

$$\chi_{V_1 \otimes V_2}(z) = \chi_{V_1}(z) \cdot \chi_{V_2}(z)$$

$$\chi_{V_1 \oplus V_2}(z) = \chi_{V_1}(z) + \chi_{V_2}(z).$$

Weyl symmetry for weights & characters plays a crucial role in repn. theory of simple (or more generally Kac-Moody) Lie algebras. It is, thus, important to obtain it directly - without relying on classification of irred. f.d. repns and/or complete reducibility.

§2. Let V be a f.d. sl_2 - repn. We will use that h-action on V is diagonalizable.

Theorem.- (a) e and f act nilpotently on V.

(b) Let $S = \exp(e)\exp(-f)\exp(e) : V \to V$ invertible linear map.

Then $S : V[\ell] \xrightarrow{\sim} V[-\ell] \quad \forall\, \ell \in P(V)$.

Proof.- (a) As $V = \bigoplus_{\ell\in P(V)} V[\ell]$ and $e : V[\ell] \to V[\ell+2]$, $f : V[\ell] \to V[\ell-2]$ we get that $e^N = 0$ for $N \gg 0$. (e.g. $N > |P(V)|$).
& $f^N = 0$ for $N \gg 0$

(b). We will directly check that
$$S(h\cdot v) = -h\cdot(S(v)) \quad \forall\, v\in V.$$

This is equivalent to : $\boxed{S h S^{-1} = -h}$

Using $\boxed{\exp(a)\, y\, \exp(-a) = \exp(\mathrm{ad}(a))\cdot y}$ $(\mathrm{ad}(a)\cdot y = ay - ya)$.

we get :

$\exp(X) = 1 + X + \frac{X^2}{2!} + \cdots$ makes sense for every nilpotent operator X.

- $$\exp(e)\, h\, \exp(-e) = \exp(\mathrm{ad}(e)) \cdot h$$
$$= h + \underbrace{[e,h] + \frac{1}{2}[e,[e,h]] + \cdots}_{\text{all zero.}} \qquad \begin{pmatrix} \mathrm{ad}(e)\cdot h = -2e \\ \mathrm{ad}(e)^2 \cdot h \\ = -2[e,e] = 0 \\ \cdots \end{pmatrix}$$
$$= h - 2e.$$

- $$\exp(-f)\,(h-2e)\,\exp(f) = \exp(-\mathrm{ad}(f)) \cdot (h-2e)$$
$$= (h-2e) \mp [f, h-2e] + \frac{[f,[f,h-2e]]}{2} \overbrace{\mp \frac{\mathrm{ad}(f)^3 \cdot (h-2e)}{6} + \cdots}^{\text{all zero}}$$

$$\left(\begin{array}{lll} \mathrm{ad}(f)\, h = 2f & \text{and} & \mathrm{ad}(f)\cdot e = -h \\ \mathrm{ad}(f)^2\, h = 0 \cdots & & \mathrm{ad}(f)^2 \cdot e = -\mathrm{ad}(f)\cdot h \quad \text{and subsequent} \\ & & \quad = -2f \qquad \mathrm{ad}(f)^r \cdot e = 0 \;;\; r \geq 3 \end{array} \right)$$

$$= h - 2e \mp 2f - 2h + \frac{1}{2}(-2)(-2f) = -h - 2e$$

- $$\exp(e)\,(-h-2e)\,\exp(-e) = -\exp(\mathrm{ad}(e)) \cdot (h+2e)$$
$$= -\left\{ (h+2e) + \underbrace{[e, h+2e] + \frac{\mathrm{ad}(e)^2 \cdot (h+2e)}{2} + \cdots}_{\text{zero.}} \right\}$$
$$= -h - 2e + 2e = -h.$$

Hence $s \cdot h \cdot s^{-1} = -h$ as claimed. Apply both sides to

$w = s(v)$ $(v \in V[\ell])$ to get $h \cdot w = s.(h \cdot v)$ $= -\ell\, w$. Hence $s : V[\ell] \to V[-\ell]$. □

<u>Exercise</u>- Verify: $s \cdot e \cdot s^{-1} = -f$

$s \cdot f \cdot s^{-1} = -e$

§3. Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ and Verma modules.

Definition :- BGG category $\mathcal{O}$ (for sl_2) consists of repns. (not necessarily finite-dim'l) of sl_2 satisfying the following conditions:

M : sl_2-repn is in $\mathcal{O}$ provided

(i) h-acts diagonalizably w/ f.d. eigenspaces.

i.e., $$M = \bigoplus_{\mu \in \mathbb{C}} M[\mu] \quad ; \quad M[\mu] = \{m \in M : hm = \mu m\}$$

and $\dim M[\mu] < \infty \quad \forall \mu \in P(M) := \{\gamma \in \mathbb{C} : M[\gamma] \neq \{0\}\}$

(ii) $\exists\ \lambda_1, \ldots, \lambda_r \in \mathbb{C}$ s.t. "highest weight condition".

$$P(M) \subset \bigcup_{j=1}^{r} (\lambda_j - 2\mathbb{Z}_{\geq 0})$$

Examples. - $\mathrm{Rep}_{fd}(sl_2) \subset \mathcal{O}$.

Verma Modules : Let $\lambda \in \mathbb{C}$. The Verma module M_λ is an sl_2-repn from $\mathcal{O}$, satisfying the following universal property:

$$\mathrm{Hom}_{sl_2}(M_\lambda, V) = V[\lambda] \cap \mathrm{Ker}(e) = \left\{ v \in V : \begin{matrix} h.v = \lambda v \\ \& \ e.v = 0 \end{matrix} \right\}$$

- i.e., ~~$\forall$ sl_2-repn V and $0 \neq v$~~

(i) $\exists$ non-zero vector $0 \neq \Omega_\lambda \in M_\lambda$ s.t. $\begin{matrix} h\Omega_\lambda = \lambda \Omega_\lambda \\ e\,\Omega_\lambda = 0 \end{matrix}$.

(ii) For any sl_2-repn V and $0 \neq v \in V$ s.t. $hv = \lambda v$ & $e.v = 0$,

$\exists!$ sl_2-intertwiner $\varphi : M_\lambda \to V$.
$\varphi(\Omega_\lambda) = v$

Explicitly, $M_\lambda = \mathbb{C}$-span of $\{\Omega_{\lambda,r} : r \geq 0\}$ w/ sl_2-action

$$h.\Omega_{\lambda,r} = (\lambda - 2r)\,\Omega_{\lambda,r}.$$
$$e.\Omega_{\lambda,r} = (\lambda - r + 1)\,\Omega_{\lambda,r-1} \qquad \left(\begin{array}{l} \Omega_\lambda = \Omega_{\lambda,0}. \\ \Omega_{\lambda,r} = \dfrac{f^r}{r!}\,\Omega_\lambda \end{array} \right).$$
$$f.\Omega_{\lambda,r} = (r+1)\,\Omega_{\lambda,r+1}$$

Exercise: (i) $M_\lambda \cong \mathcal{U}(sl_2) \Big/ \begin{array}{l}\text{left ideal gen. by} \\ e \text{ and } h - \lambda.1\end{array}$.

(ii) Solve problem 33 - $\forall\, \lambda \in \mathbb{Z}_{\geq 0}$, we have a non-split short exact sequence in $\mathcal{O}$ of sl_2:

$$0 \to M_{-\lambda-2} \to M_\lambda \to L_\lambda \to 0.$$

M_λ is irreducible $\forall\, \lambda \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$.

§4. The notion of character extends to repns. from category $\mathcal{O}$.

$$M \in \mathcal{O} \rightsquigarrow \chi_M(t) = \sum_{\gamma \in \mathbb{C}} \dim M[\gamma] \cdot t^\gamma \leftarrow \text{no longer a poly. in } t, t^{-1}$$

$$\chi_{M_\lambda}(t) = t^\lambda\left(1 + t^{-2} + t^{-4} + \dots\right) = \frac{t^\lambda}{1 - t^{-2}} = \frac{t^{\lambda+1}}{t - t^{-1}}$$

(appearance of "Weyl denominator")

§5. The following characterization of f.d. repns. in $\mathcal{O}$ will be generalized later:

$L \in \mathcal{O}$ is f.d. $\iff$ e and f act locally nilpotently on L

(i.e. $\forall\, v \in L$, $\exists\, n$ (depending on v - perhaps) s.t. $e^n . v = 0$ & similarly $f^n . v = 0$)

Proof.- ($\Rightarrow$) See Thm. §2 (a) above.

($\Leftarrow$) If e and f act locally nilpotently on L, then

$S = \exp(e)\exp(-f)\exp(e) : L \to L$ makes sense

$\Rightarrow$ L has Weyl symmetry (i.e. $\chi_L(t) = \chi_L(t^{-1})$).

As $L \in \mathcal{O}$, ~~$\exists\, \lambda \in \mathbb{C}$ s.t. $L[\lambda]$~~ $\exists\, \lambda_1, \dots, \lambda_r \in \mathbb{C}$ s.t.

$$P(L) \subset \bigcup_{j=1}^{r} \lambda_j - 2\mathbb{Z}_{\geq 0} .$$

By Weyl symmetry - $P(L) \subset \underbrace{\left(\bigcup_{j=1}^{r} \lambda_j - 2\mathbb{Z}_{\geq 0}\right) \cap \left(\bigcup_{j=1}^{r} -\lambda_j + 2\mathbb{Z}_{\geq 0}\right)}_{\text{finite set of points.}}$

Hence $L = \bigoplus_{\underset{\text{finite set}}{\gamma \in P(L)}} \overset{\text{f.d.}}{L[\gamma]}$ is finite-dim'l. $\square$