# Lecture 27

Our next example of Lie algebras – whose repn. th. is of interest in different branches of mathematics and physics – arise from the combinatorial data of Cartan matrices /. root systems.

$$\begin{tikzcd}
 & \text{Root Systems} \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow] \\
\text{Cartan Matrices} \arrow[r, leftrightarrow] & \begin{array}{c}\text{Finite-dim'l simple} \\ \text{Lie alg's over } \mathbb{C}\end{array}
\end{tikzcd}$$

[Sketch of the famous Cartan-Killing* classification thm.]

§1. <u>Cartan matrices</u>: Let $I$ be a finite indexing set. A <u>Cartan matrix</u> $A = (a_{ij})_{i,j\in I} \in M_{I\times I}(\mathbb{Z})$ is a square $(I\times I)$ integer matrix satisfying:

(1) $a_{ii} = 2 \quad \forall\, i \in I$

(2) $a_{ij} \in \mathbb{Z}_{\leq 0} \quad \forall\, i \neq j$

(3) $\exists\ d_i \in \mathbb{Z}_{>0}\ (i\in I)$ s.t. $d_i a_{ij} = d_j a_{ji} \quad \forall\, i,j$

(i.e., $D\cdot A$ is a symmetric matrix – where $D = \text{diagonal}(d_i : i \in I)$.)

(4) $A$ is positive-definite.

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* Élie Joseph Cartan (April 9, 1869 – May 6, 1951); Wilhelm Karl Joseph Killing (May 10, 1847 – Feb. 11, 1923)

Remark. – Victor Kac (Infinite-dim'l Lie alg's – Chapter 1) has introduced various relaxed versions of axioms (1) – (4) – leading to the notion of "generalized Cartan matrices". Axiom (3) is referred to as <u>symmetrizability</u> and (4) is a <u>finite type condition</u>. Dropping (4), one gets "symmetrizable generalized C.M.'s" – in Kac's definitions.

Example : (a) $|I| = 1$ : $A = (2)$.

(b) $|I| = 2$. $A = \begin{bmatrix} 2 & -b \\ -c & 2 \end{bmatrix}$ $b, c \in \mathbb{Z}_{\geq 0}$.

Symmetrizability means $d_1 b = d_2 c$ for some $d_1, d_2 \in \mathbb{Z}_{>0}$.

i.e. either $b = c = 0$ ; or $bc \neq 0$.

Positive-definite-ness $\iff$ $\det(A) = 4 - bc > 0$

Assuming $b \geq c$ ; we get : $c = 0, 1$ &

$$\left.\begin{array}{ll} b = 0, & c = 0 \\ b = 1, & c = 1 \\ b = 2, & c = 1 \\ b = 3, & c = 1 \end{array}\right\} \text{all } \overbrace{\text{rank}}^{|I|} 2 \text{ possibilities}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$A_1 \times A_1$ $\quad$ $A_2$ $\quad$ $B_2$ $\quad$ $G_2$

[Dynkin labelling of rank 2 C.M.'s]

## §2. Dynkin diagram of a Cartan matrix.

Let $A = (a_{ij})_{i,j \in I}$ be a Cartan matrix. The Dynkin diagram associated to $A$ is a graph on $I$, with 3 type of edges.

$i - j$ if $a_{ij} = a_{ji} = -1$

$i \Leftarrow j$ if $a_{ij} = -2$ ; $a_{ji} = -1$

$i \Lleftarrow j$ if $a_{ij} = -3$ ; $a_{ji} = -1$.

( $a_{ij} = a_{ji} = 0$ means $i$ & $j$ are not connected )

We say $A$ is indecomposable (or just connected) if the associated Dynkin diagram is connected.

Classification of finite type Cartan matrices. - The following is the complete list of connected Dynkin diagrams, associated to (finite type) Cartan matrices.

$A_n$ : $1 - 2 - \cdots - n$

$B_n$ : $1 - 2 - \cdots - n\text{-}1 \Leftarrow n$

$C_n$ : $1 - 2 - \cdots - n\text{-}1 \Rightarrow n$

$D_n$ : $1 - 2 - \cdots - n\text{-}2 \begin{smallmatrix} \diagup\, n\text{-}1 \\ \diagdown\, n \end{smallmatrix}$

$E_{6,7,8}$ : $\begin{array}{c} 1 - 3 - 4 - 5 - 6\ (-7)\ (-8) \\ \qquad\quad\ \ | \qquad\qquad\qquad\qquad \\ \qquad\quad\ \ 2 \qquad\qquad\qquad\qquad \end{array}$

$F_4$ : $1 - 2 \Leftarrow 3 - 4$

$G_2$ : $1 \Lleftarrow 2$

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* Eugene Borisovitch Dynkin (May 11, 1924 - Nov. 14, 2014)

## § 3. Root system assoc. to Cartan matrix.

$A = (a_{ij})_{i,j \in I}$ Cartan Matrix

$E :=$ $\mathbb{R}$ - vector space of dim $|I|$ $\quad$ $E^*$ : dual vector space.
- basis $\{h_i : i \in I\}$.

Define $\alpha_i \in E^*$ $(i \in I)$ by $\boxed{\alpha_j(h_i) = a_{ij} \quad \forall\, i,j \in I.}$

Positive-definite symmetric bilinear form on $E^*$ : defined by DA (axiom (3) of §1)

i.e. $\boxed{(\alpha_i, \alpha_j) = d_i a_{ij}}$ $\forall\, i,j \in I$.

Let $\phi : E^* \to E$ be the corresponding iso. - i.e., given by
$$\lambda(\phi(\alpha)) = (\lambda, \alpha) \quad \forall\, \lambda, \alpha \in E^*.$$

<u>Check:</u> $\phi(\alpha_i) = d_i h_i$. The transported $(\cdot\,,\cdot)$ on $E$ is $\boxed{(h_i, h_j) = a_{ij}\, d_j^{-1}}$.

<u>Orthogonal reflections</u>. - For $\gamma \in E^*$ $(\gamma \neq 0)$, let $s_\gamma : E \to E$ be the orthogonal reflection in the hyperplane $H_\gamma = Ker(\gamma) \subset E$.

Explicitly $$\boxed{s_\gamma(h) = h - \frac{2\,\gamma(h)}{(\gamma, \gamma)}\,\phi(\gamma)}$$

( easy check : $s_\gamma(h) = h \quad \forall\, h \in Ker(\gamma)$.
$s_\gamma(x) = -x$ for $x$ proportional to $\phi(\gamma)$ )

We will continue to denote by $s_\gamma$, the operator on $E^*$ - obtained from the identification $\phi : E^* \to E$. That is,

$S_\gamma : E^* \to E^*$ is given by

$$\boxed{S_\gamma(\lambda) = \lambda - \frac{2(\lambda,\gamma)}{(\gamma,\gamma)} \cdot \gamma} \quad . \qquad \forall \lambda \in E^*.$$

Weyl group $W :=$ subgroup of $GL(E)$ or $GL(E^*)$ generated by $s_i = S_{\alpha_i}$ $(i \in I)$.

<u>Set of roots</u> : $R \subset E^*$ defined as $W$-orbit of $\{\alpha_i\}_{i\in I}$

$$R := \bigcup_{i \in I} W \cdot \alpha_i$$

<u>Set of coroots</u> : $R^\vee \subset E$ is the $W$-orbit of $\{h_i\}_{i \in I}$.

$$R^\vee = \bigcup_{i\in I} W \cdot h_i = \left\{ \alpha^\vee = \frac{2\phi(\alpha)}{(\alpha,\alpha)} : \alpha \in R \right\}$$

Summarizing the above definitions. -

$A$ : Cartan matrix $\rightsquigarrow$

- $\phi : E^* \to E$ (W-intertwiner), with $W$ acting on $E^*$ and on $E$
- $\{\alpha_i\}_{i\in I} \subset E^*$
  $\{h_i\}_{i\in I} \subset E$
- $R \subset E^*$ and $R^\vee \subset E$.

§4. Lemma. – $|R| < \infty$ and $|W| < \infty$.

Proof. – Using $s_i(\alpha_j) = \alpha_j - \alpha_j(h_i)\alpha_i = \alpha_j - a_{ij}\alpha_i$

we conclude that $R \subset \sum_{i \in I} \mathbb{Z}\alpha_i$.

Moreover, $W$-action preserves length – (check : $W$-preserves $(\cdot,\cdot)$ on $E^*$) –

$$\Rightarrow \quad R \subset \underbrace{\left(\sum_{i\in I} \mathbb{Z}\alpha_i\right)}_{\text{discrete}} \cap \underbrace{\left\{\beta \in E^* : (\beta,\beta) = (\alpha_i,\alpha_i) \text{ for some } i \in I\right\}}_{\text{compact}}$$

hence $R$ is finite.

Since $\{\alpha_i\}_{i\in I} \subset R$, $R$ spans $E^*$. As $W$ permutes elements of $R$, $W <$ Permutations of $R$, so $|W| < \infty$. $\square$

§5. Rank 2 examples.

(i) $A_1 \times A_1$ $\qquad R = \{\pm\alpha_1, \pm\alpha_2\}$

Cartan matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\qquad (\alpha_1, \alpha_2) = 0$.

$$W = \left\langle s_1, s_2 \;\middle|\; \begin{array}{c} s_1^2 = s_2^2 = e \\ s_1 s_2 = s_2 s_1 \end{array} \right\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

(ii) $A_2$ $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$s_i(\alpha_i) = -\alpha_i$

$s_1(\alpha_2) = \alpha_2 + \alpha_1 = s_2(\alpha_1)$.

$$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}.$$

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = e,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle \cong S_3.$$

(iii) $B_2$ $\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$

$s_1(\alpha_2) = \alpha_2 + 2\alpha_1$

$s_2(\alpha_1) = \alpha_1 + \alpha_2$

$$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$$

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = e,\ s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \rangle \cong D_4$$

(dihedral gp.)

(iv) $G_2$ $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$s_1(\alpha_2) = \alpha_2 + 3\alpha_1$

$s_2(\alpha_1) = \alpha_1 + \alpha_2$

$$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}$$

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = e,\ s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1 \rangle$$

$$\cong D_6.$$