Lecture 32.

Recall : $A = (a_{ij})_{i,j\in I}$ Cartan Matrix.

$\mathfrak{g}$ = f.d. simple Lie algebra assoc. to A.

$= \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$; $\mathfrak{n}_\pm = \bigoplus_{\alpha\in R_+} \mathfrak{g}_{\pm\alpha}$

$\dim \mathfrak{g}_\alpha = 1 \quad \forall \alpha \in R.$

Generators : $\{h_i, e_i, f_i\}_{i\in I}$ Relations : $[x,x'] = 0 \ \forall x,x' \in \mathfrak{h}.$

$(\mathfrak{h} = \bigoplus_{i\in I} \mathbb{C} h_i)$

$$[x, e_i] = \alpha_i(x)\, e_i \qquad \forall \begin{smallmatrix} x\in\mathfrak{h} \\ i\in I \end{smallmatrix}$$
$$[x, f_i] = -\alpha_i(x)\, f_i$$

$[e_i, f_j] = \delta_{ij} h_i$ and $\mathrm{ad}(e_i)^{1-a_{ij}} \cdot e_j = 0$ $(\forall\ i\neq j)$.

$\mathrm{ad}(f_i)^{1-a_{ij}} \cdot f_j = 0$

$\mathrm{Irr}_{fd}(\mathfrak{g}) = \{L_\lambda : \lambda \in P_+\}$ $(P_+ = \{\gamma\in\mathfrak{h}^*,\ \gamma(h_i)\in\mathbb{Z}_{\geq 0} \forall i\in I\})$.

Description of L_λ : L_λ is generated (as $\mathfrak{g}$-repn) by a (non-zero) vector $\mathbb{1}_\lambda$ subject to the following rel$^{\underline{n}}$s :

$e_i \cdot \mathbb{1}_\lambda = 0 \ (\forall i\in I)$ $\qquad h.\, \mathbb{1}_\lambda = \lambda(h).\, \mathbb{1}_\lambda \quad \forall h \in \mathfrak{h}.$

$f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda = 0 \quad (\forall\ i\in I).$

<u>Remark</u>.- We haven't yet proved that the list of rel$^{\underline{n}}$s given above is complete. That is, if $\widetilde{L}_\lambda$ is the $\mathfrak{g}$-repn. generated by Ω_λ subject to rel$^{\underline{n}}$s given above, then we have shown

(see Thm §5 - pages 6,7 of Lecture 31) :

$\tilde{L}_\lambda \twoheadrightarrow L_\lambda$ and $\tilde{L}_\lambda$ is integrable and hence, $\dim \tilde{L}_\lambda < \infty$.

$(\Omega_\lambda \mapsto 1_\lambda)$

§1. Invariant forms and Casimir element.-

Lemma. - Let $\mathfrak{a}$ be a f.d. Lie algebra over $\mathbb{C}$ and $B : \mathfrak{a} \times \mathfrak{a} \to \mathbb{C}$ be a symmetric, invariant, bilinear form.

[Invariance : $B([x,y],z) = B(x,[y,z]) \quad \forall\, x,y,z \in \mathfrak{a}$.]

(i) $\operatorname{Rad} B = \{x \in \mathfrak{a} : B(x,y) = 0 \ \forall\, y \in \mathfrak{a}\}$ is an ideal of $\mathfrak{a}$.

(ii) If B is non-degenerate; then its canonical tensor, and its associated Casimir element in $\mathcal{U}(\mathfrak{a})$, commute with $\mathfrak{a}$-action. Meaning: let $\{x_a\}_{a=1}^n$ be a basis of $\mathfrak{a}$, and $\{x^a\}_{a=1}^n$ the basis dual to $\{x_a\}$ under B.

Then : (a) $\left[\sum_{a=1}^n x_a \otimes x^a \,,\, x \otimes 1 + 1 \otimes x\right] = 0$ in $\mathfrak{a} \otimes \mathfrak{a}$.

(b) $\left[\sum_{a=1}^n x_a x^a \,,\, x\right] = 0$ in $\mathcal{U}(\mathfrak{a})$.

($\sum_{a=1}^n x_a \otimes x^a$ is called the Casimir tensor

$\sum_{a=1}^n x_a x^a$ ——— Casimir operator.)

Proof of (i): If $x \in \mathrm{Rad}(B)$ and $t \in \mathfrak{g}$, then

$$\forall y \in \mathfrak{g}, \quad B([x,t],y) = B(x,[t,y]) = 0$$

$$\Rightarrow \quad [x,t] \in \mathrm{Rad}(B).$$

Proof of (ii): Let $\{x_a\}_{a=1}^{n}$ & $\{x^a\}_{a=1}^{n}$ be dual bases of $\mathfrak{g}$ as above.

For $x \in \mathfrak{g}$, let

$$[x, x_a] = \sum_r C_{a,r}(x)\, x_r \qquad (C_{a,r}(x) \in \mathbb{C})$$

$$[x, x^b] = \sum_s D_{b,s}(x)\, x^s \qquad (D_{b,s}(x) \in \mathbb{C})$$

Note: Invariance $\Rightarrow$ $\boxed{D_{r,s}(x) + C_{s,r}(x) = 0 \quad \forall x \in \mathfrak{g}}$.

$$\Bigg(\text{Proof.} \quad C_{s,r}(x) = \text{coeff. of } x_r \text{ in } [x, x_s]$$

$$= B(x^r, [x, x_s])$$

$$= B([x^r, x], x_s) = -\text{Coeff of } x^s \text{ in } [x, x^r]$$

$$= -D_{r,s}(x). \qquad \square\Bigg)$$

(a):

$$\left[x \otimes 1 + 1 \otimes x, \sum_{a=1}^{n} x_a \otimes x^a\right] = \sum_a [x, x_a] \otimes x^a + \sum_b x_b \otimes [x, x^b]$$

$$= \sum_{r,s} \big(C_{s,r}(x) + D_{r,s}(x)\big)\,(x_r \otimes x^s) = 0.$$

(b):

$$\left[x, \sum_{a=1}^{n} x_a x^a\right] = \sum_a [x, x_a]\, x^a + \sum_b x_b\, [x, x^b]$$

$$= \sum_{r,s} \big(C_{s,r}(x) + D_{r,s}(x)\big) x_r x^s = 0. \qquad \square$$

§2. In our case, $\mathfrak{g}$ = f.d. simple Lie alg. assoc. to A.

Prop. - There is a unique, invariant, bilinear form on $\mathfrak{g}$

$$K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \quad \text{satisfying :}$$

(i) $K(h_i, h_j) = a_{ij}\, d_j^{-1} \quad \forall\, i,j \in I.$

(ii) $K(e_i, f_j) = \delta_{ij}\, d_i^{-1}$ (iii) $K(h_i, e_j) = 0 = K(h_i, f_j)$.

Remark.- Such a symmetric, invariant, bilinear form exists on $\tilde{\mathfrak{g}}$ as well - let us denote it by $\tilde{K} : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \mathbb{C}$.

The radical of $\tilde{K}$ is an ideal of $\tilde{\mathfrak{g}}$ (Lemma §1 above)

and $\tilde{K}\big|_{\mathfrak{h}\times\mathfrak{h}}$ is given by non-deg. matrix $A \cdot D^{-1}$

$\Rightarrow$ $\mathrm{Rad}(\tilde{K})$ is a proper ideal of $\tilde{\mathfrak{g}}$, hence is contained in the unique, max'l proper ideal $\mathfrak{r} \subset \tilde{\mathfrak{g}}$.

Hence $\tilde{K}$ descends to a non-deg., symmetric, bilinear, invariant form on $\mathfrak{g}$.

§3. More properties of $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$.

(i) $K(x, y) = 0$ $(\forall\, x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_\beta)$ if $\alpha + \beta \neq 0$.
$(\alpha, \beta \in R \cup \{0\})$

(ii) Let $\alpha \in R_+$. Then $K : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ is non-deg.

Proof of (i): Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ where $\alpha, \beta \in R \cup \{0\}$ and $\alpha+\beta \neq 0$. For every $h \in \mathfrak{h}$, we have

$$\begin{array}{ccccc} \mathcal{K}(h,[x,y]) & = & \mathcal{K}([h,x],y) & = & \alpha(h)\,\mathcal{K}(x,y) \\ \| & & & & \\ -\mathcal{K}(h,[y,x]) & = & -\mathcal{K}([h,y],x) & = & -\beta(h)\,\mathcal{K}(y,x) \end{array}$$

$\Rightarrow$ $(\alpha+\beta)(h)\cdot \mathcal{K}(x,y) = 0$. As $\alpha+\beta \neq 0$, $\exists\ h \in \mathfrak{h}$ s.t. $(\alpha+\beta)(h) \neq 0$, $\Rightarrow$ $\mathcal{K}(x,y) = 0$.

(ii) follows from non-deg. of $\mathcal{K}$, and (i).

§4. Let $C \in \mathcal{U}(\mathfrak{g})$ be the Casimir operator assoc. to $\mathcal{K}$. In order to write it explicitly, let $x_\alpha \in \mathfrak{g}_\alpha$ $(\alpha \in R_+)$, $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be chosen so that $\mathcal{K}(x_\alpha, x_{-\alpha}) = 1$.

Lemma. – $[x_\alpha, x_{-\alpha}] = \phi(\alpha)$ $\forall\ \alpha \in R_+$

(recall: $\phi : \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ given by $\gamma(h) = \mathcal{K}(\phi(\gamma), h)$ $\forall\ \gamma \in \mathfrak{h}^*$, $h \in \mathfrak{h}$.)

Proof.– $\mathcal{K}(h,[x_\alpha, x_{-\alpha}]) = \mathcal{K}([h,x_\alpha], x_{-\alpha})$

$= \alpha(h)\,\mathcal{K}(x_\alpha, x_{-\alpha}) = \alpha(h)$. □

Let $\{y_i\}_{i\in I}$ be an orthonormal basis of $\mathfrak{h}$. Then

$$C = \sum_{i\in I} y_i^2 + \sum_{\alpha\in R+} \left(x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha\right)$$

$$= \sum_{i\in I} y_i^2 + \sum_{\alpha\in R_+} \left(\phi(\alpha) + 2\, x_{-\alpha} x_\alpha\right)$$

Note: $\forall\, \lambda\in\mathfrak{h}^*$, $\sum_{i\in I} \lambda(y_i)^2 = (\lambda,\lambda)$.

(Proof.- $\phi : \underset{\lambda}{\underset{\rotatebox{90}{$\in$}}{\mathfrak{h}^*}} \to \mathfrak{h}$. Write $\phi(\lambda)$ in the basis $\{y_i\}$ (orthonormal)

$$\phi(\lambda) = \sum_i (\phi(\lambda), y_i)\, y_i = \sum_{i\in I} \lambda(y_i)\, y_i$$

$\Rightarrow (\lambda,\lambda) = \lambda(\phi(\lambda)) = \sum_{i\in I} \lambda(y_i)^2 .\ \square$)

§5. Casimir action on highest weight repns.

Let $\rho\in\mathfrak{h}^*$ be given by $\rho = \frac{1}{2}\sum_{\alpha\in R+} \alpha$.

Prop. (i) If $\mathfrak{g} \circlearrowright V$ is a h.w. repn. of h.w. $\lambda\in\mathfrak{h}^*$, then

$$C\big|_V = (\lambda+2\rho, \lambda)\cdot \mathrm{Id}_V .$$

(ii) If $\mathfrak{g}\circlearrowright U$ is a repn. and $u\in U$ is s.t. $e_i\cdot u = 0\ \forall i$

$h\cdot u = \gamma(h)\cdot u\ \forall h\in\mathfrak{h}$.

$\gamma\in\mathfrak{h}^*$

then $C \cdot u = (\gamma + 2\rho , \gamma) \cdot u$.

<u>Proof of (ii)</u> .- $$C \cdot u = \sum_{i \in I} y_i^2 \cdot u + \sum_{\alpha \in R_+} \phi(\alpha) \cdot u + \cancelto{0}{2 x_\alpha^- x_\alpha^+} \cdot u$$

$$= \left[\sum_{i \in I} \gamma(y_i)^2 + \left(\gamma , \sum_{\alpha \in R_+} \alpha \right) \right] \cdot u$$

$$= \left((\gamma , \gamma) + (\gamma , 2\rho) \right) \cdot u = (\gamma + 2\rho , \gamma) \cdot u$$

(i) follows from (ii) and the fact that V is generated by of $v \in V[\lambda]$ and C commutes with $\mathfrak{g}$- action. □