Lecture 33

§1. Proof of Proposition §2 of Lecture 32. -

Recall: $A = (a_{ij})_{i,j\in I}$ is a Cartan matrix ; $D = \mathrm{diag}(d_i : i \in I)$ symmetrizing integers.

$\mathfrak{g}$ = f.d. simple Lie alg. assoc. to A.

$$= \Big(\bigoplus_{\alpha\in Q_+\setminus\{0\}} \mathfrak{g}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in Q_+\setminus\{0\}} \mathfrak{g}_{\alpha}\Big)$$

Claim. - There exists a unique, invariant (symmetric, bilinear) form $K : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ satisfying

(i) $K(h_i, h_j) = a_{ij}/d_j \quad (\forall i,j\in I)$

$K(h_i, e_j) = 0 = K(h_i, f_j)$

(ii) $K(e_i, f_j) = \delta_{ij}\, d_i^{-1} \quad (\forall i,j \in I)$

Proof. - By induction on height $ht : Q_+ \to \mathbb{Z}_{\geq 0}$

$$ht\Big(\sum_{j\in I} k_j\alpha_j\Big) = \sum_{j\in I} k_j$$

Notation. - $\forall\, \ell\in\mathbb{Z}$, let $\mathfrak{g}(\ell) = \bigoplus_{\substack{\beta\in Q_+\\ ht(\beta)=\ell}} \mathfrak{g}_\beta$ if $\ell\geq 0$

$$= \bigoplus_{\substack{\beta\in Q_+\\ ht(\beta)=-\ell}} \mathfrak{g}_{-\beta} \quad \text{if } \ell<0.$$

Remark. - Let $x_0\in\mathfrak{h}$ be s.t. $\alpha_i(x_0)=1\ \forall i\in I$. Then $\mathfrak{g}(\ell)$ is the eigenspace for $\mathrm{ad}(x_0) \circlearrowright \mathfrak{g}$ with eigenvalue $\ell\in\mathbb{Z}$.

By induction, we will prove the existence of K on $\bigoplus_{\ell=-N}^{N} \mathfrak{g}(\ell)$ $(N\in\mathbb{Z}_{\geq 0})$.

Base case $(N=0)$ is covered by $\mathcal{K}(h_i, h_j) = a_{ij}/d_j \quad (\forall\, i,j \in I)$.

$N=1$ case follows from conditions $\mathcal{K}(h_i, e_j) = 0 = \mathcal{K}(h_i, f_j)$

& $\mathcal{K}(e_i, f_j) = \delta_{ij}/d_i$.

Assume that $\mathcal{K}$ has been defined on $\bigoplus_{\ell=-N}^{N} \mathfrak{g}(\ell)$ $(N \geq 1)$ and we
(to satisfy $\mathfrak{g}$-invariance)
carry out the induction step. $\mathfrak{g}$-invariance $\Rightarrow$ $\boxed{\mathcal{K}(\mathfrak{g}(i), \mathfrak{g}(j)) = 0 \text{ if } i+j \neq 0}$.

Note - Since e_i's generate $\bigoplus_{\ell=1}^{\infty} \mathfrak{g}(\ell)$; $\mathfrak{g}(N+1) = \sum_{j=1}^{N} [\mathfrak{g}(j), \mathfrak{g}(N+1-j)]$

f_i's generate $\bigoplus_{\ell \leq -1} \mathfrak{g}(\ell)$

Let $x \in \mathfrak{g}(N+1)$ and write $x = \sum_{j=1}^{N} [x_j', x_{N+1-j}'']$, $x_j' \in \mathfrak{g}(j)$, $x_j'' \in \mathfrak{g}(N+1-j)$

For $y \in \bigoplus_{\ell=-N-1}^{N+1} \mathfrak{g}(\ell)$, define $\mathcal{K}(x,y) = \sum_{j=1}^{N} \mathcal{K}(x_j', [x_{N+1-j}'', y])$ — $(*)$ if $y \in \mathfrak{g}_{-N-1}$

$= 0$ if $y \in \mathfrak{g}(j)$; $j \neq -N-1$.

We need to show that $\mathcal{K} : \mathfrak{g}(-N-1) \times \mathfrak{g}(N+1) \to \mathbb{C}$ defined in $(*)$ above, is independent of the expression $x = \sum_{j=1}^{N} [x_j', x_{N+1-j}'']$ chosen.

[To see this, we will show

$$\sum_{j=1}^{N} \mathcal{K}(x_j', [x_{N+1-j}'', y]) = \sum_{\ell=1}^{N} \mathcal{K}([x, y_\ell'], y_{N+1-\ell}'') \quad (**)$$

where $y = \sum_{\ell=1}^{N} [y_\ell', y_{N+1-\ell}']$; $y_\ell' \in \mathfrak{g}(-\ell)$; $y_{N+1-\ell}'' \in \mathfrak{g}(-N-1+\ell)$]

In (**), L.H.S. is independent of the expression for y, while R.H.S. is independent of the one for x.

<u>Proof of (**)</u>: L.H.S. $= \sum_{j,\ell} \mathcal{K}\left(x'_j, [x''_{N+1-j}, [y'_\ell, y''_{N+1-\ell}]]\right)$

$$= \sum_{j,\ell} \mathcal{K}\left(x'_j, [[x''_{N+1-j}, y'_\ell], y''_{N+1-\ell}]\right) + \mathcal{K}\left(x'_j, [y'_\ell, [x''_{N+1-j}, y''_{N+1-\ell}]]\right)$$ (using Jacobi id.)

$$= \sum_{j,\ell} \mathcal{K}\left([x'_j, [x''_{N+1-j}, y'_\ell]], y''_{N+1-\ell}\right) + \mathcal{K}\left([[x'_j, y'_\ell], x''_{N+1-j}], y''_{N+1-\ell}\right)$$ (using invariance of $\mathcal{K}$ on $\bigoplus_{j=-N}^{N} \mathfrak{g}(j)$ - induction hypothesis)

$$= \sum_{\ell} \mathcal{K}\left([x, y'_\ell], y''_{N+1-\ell}\right)$$ (again - Jacobi id.) $\square$

Remark. - The same proof works to give an invariant form on any quotient $\tilde{\mathfrak{g}}/\mathcal{J}$ where $\mathcal{J} \subset \tilde{\mathfrak{g}}$ is a <u>proper ideal</u>

(In particular, $\mathcal{J} \cap \bigoplus_{\ell=-1}^{1} \tilde{\mathfrak{g}}(\ell) = \{0\}$.)

<u>Note</u>. - Non-degeneracy of $\mathcal{K}$ on $\mathfrak{g}$ follows from its simplicity

($\mathrm{Rad}(\mathcal{K}) \subset \mathfrak{g}$ has to be a proper ideal $\Rightarrow$ it is zero.)

§2. Applications of Casimir element - I.

Recall that we obtained the Casimir operator $C \in \mathcal{U}(\mathfrak{g})$ from the non-deg., invariant form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$.

$$C = \sum_{i \in I} y_i^2 + \sum_{\alpha \in R_+} \phi(\alpha) + 2 x_\alpha^- x_\alpha^+ \qquad \text{(see §4,5 of Lecture 32)}$$

We showed : (i) C is central. That is, $\forall\ \mathfrak{g} \overset{\pi}{\curvearrowright} V$ repn,

$$C_\pi = \sum_{i \in I} \pi(y_i)^2 + \sum_{\alpha \in R_+} \pi(\phi(\alpha)) + 2\phi(x_\alpha^-)\phi(x_\alpha^+) \ : V \to V$$

is a $\mathfrak{g}$-intertwiner (Lemma §1 of Lecture 32)

(ii) If $\mathfrak{g} \overset{\pi}{\curvearrowright} V$ and $u \in V$, $\gamma \in \mathfrak{h}^*$ is such that $e_i \cdot u = 0 \ \forall i \in I$, $h \cdot u = \gamma(h) u \ \forall h \in \mathfrak{h}$

then $C_\pi \cdot u = (\gamma + 2\rho, \gamma) \cdot u$, where

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha \ \in \mathfrak{h}^*.$$

(iii) If $\mathfrak{g} \overset{\pi}{\curvearrowright} V$ is a h.w. repn. of highest weight $\lambda \in \mathfrak{h}^*$, then

$$C_\pi = (\lambda + 2\rho, \rho) \cdot Id_V .$$

(see Prop. §5 of Lecture 32).

Lemma. - $\rho(h_i) = 1 \quad (\forall i \in I)$.

Proof. - As s_i preserves $R_+ \setminus \{\alpha_i\}$ and $s_i(\alpha_i) = -\alpha_i$, we have

$$s_i(\rho) = \frac{1}{2} \sum_{\substack{\alpha \in R_+ \\ \alpha \neq \alpha_i}} s_i(\alpha) - \frac{1}{2}\alpha_i = \frac{1}{2} \sum_{\substack{\beta \in R_+ \\ \beta \neq \alpha_i}} \beta - \frac{1}{2}\alpha_i$$

$= \rho - \alpha_i$. On the other hand, $s_i(\rho) = \rho - \rho(h_i)\alpha_i$. □

$\Rightarrow \rho(h_i) = 1.$

Theorem. - Let $\lambda \in P_+$ and V be any finite-dim'l, highest-weight repn., of h.w. λ. Then V is irreducible.

Proof. - As V is highest-weight repn., of h.w. λ, we get

$$C = (\lambda + 2\rho, \lambda) \cdot Id_V.$$

If V is not irreducible, then let $U \subsetneq V$ be a non-zero subrepn of smallest dim. - so that U is irred.

$\Rightarrow$ U has a highest weight vector $0 \neq u \in U[\mu] \subset V[\mu]$.

Note: $\dim U < \infty \Rightarrow \mu \in P_+$.

And - by (ii) of the previous page - $C \cdot u = (\mu + 2\rho, \mu) \cdot u$.

So, $\lambda, \mu \in P_+$; $\lambda - \mu \in Q_+$ and $(\lambda + 2\rho, \lambda) = (\mu + 2\rho, \mu)$.

Claim. - $\lambda = \mu$ (hence $U = V$. contradiction).

Proof of the claim. - $\lambda \in P_+ \Rightarrow (\lambda + 2\rho, \beta) > 0 \quad \forall \beta \in Q_+ \setminus \{0\}$.

Let $\underset{\in Q_+}{\beta = \lambda - \mu}$ and assume $\beta \neq 0$.

We get $(\lambda + 2\rho, \lambda - \mu) > 0 \Rightarrow (\lambda + 2\rho, \lambda) > (\lambda + 2\rho, \mu)$. - (1)

Since $\mu \in P_+$, $(\mu, \beta) \geq 0 \Rightarrow (\mu, (\lambda + 2\rho) - (\mu + 2\rho)) \geq 0$

$\Rightarrow (\lambda + 2\rho, \mu) \geq (\mu + 2\rho, \mu)$ - (2)

Combining (1) & (2) we get $(\lambda + 2\rho, \lambda) > (\mu + 2\rho, \mu)$ contradicting the fact that they are equal. $\square$

<u>Cor.</u> – For $\lambda \in P_+$, the unique f.d. irred. repn. L_λ admits the following – <u>Harish-Chandra presentation</u> –

$$L_\lambda \text{ is generated by } \mathbb{1}_\lambda \text{ subject to } \begin{cases} e_i \cdot \mathbb{1}_\lambda = 0 & \forall\, i \in I \\ h \cdot \mathbb{1}_\lambda = \lambda(h) \cdot \mathbb{1}_\lambda & \forall\, h \in \mathfrak{h} \\ f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda = 0 & \forall\, i \in I. \end{cases}$$

<u>Proof</u>. – We have already proved that the repn described by above presentation is h.w. repn – of h.w. λ ; <u>and</u> finite-dim'l.
(see Pf. of Thm §5 – pages 6, 7 – of Lecture 31.)

Thm $\Rightarrow$ it is irreducible and hence iso. to L_λ. □

§3. <u>Applications of Casimir element II</u>. – Irreducibility of Verma Modules.

Let $\lambda \in \mathfrak{h}^*$ and M_λ be the Verma module. (see Lecture 31 – §2.)

<u>Prop</u>. – If M_λ is <u>not</u> irred., then $\exists\, \beta \in Q_+ \setminus \{0\}$ s.t.

$$(\lambda + \rho, \beta) = \frac{1}{2}(\beta, \beta)$$

<u>Proof</u>. – If M_λ is not irreducible, then there is a proper, non-zero subrepn. $U \subsetneq M_\lambda$. Let $\beta \in Q_+ \setminus \{0\}$ be of smallest height s.t. $U[\lambda-\beta] \neq 0$. Then $U[\lambda - \beta + \alpha_i] = 0 \quad \forall\, i \in I$

$\Rightarrow$ Casimir element on $u \in U[\lambda - \beta] = (\lambda - \beta + 2\rho, \lambda - \beta) \cdot u$

But $C|_{M_\lambda} = (\lambda + 2\rho, \lambda) \cdot Id_{M_\lambda} \Rightarrow (\lambda + 2\rho, \lambda) = (\lambda - \beta + 2\rho, \lambda - \beta)$ and the proposition follows. □