PROBLEMS IN REPRESENTATION THEORY

Representation of finite groups

Problem 1. (10 points) Let G be a finite group acting on a finite set X via group homomorphism $\alpha : G \to S_X$ (here, S_X is the group of permutations of elements of X). Let $V = \mathbb{C}[X]$ be the vector space (over $\mathbb{C}$) of all complex–valued functions on X. We view V as a G–representation, $\rho : G \to \mathrm{GL}(V)$, by:

$$(\rho(g)(f))(x) := f(\alpha(g)^{-1}(x)), \text{ for every } g \in G, f \in V, x \in X.$$

(1) Prove that $\dim(V^G) = |X/G|$ (number of G–orbits in X).
(2) For $g \in G$, show that $\mathrm{Tr}(\rho(g)) = |X^g|$. Here, $X^g := \{x \in X : \alpha(g)(x) = x\}$.
(3) Consider the averaging operator $P = \frac{1}{|G|}\sum_{g\in G} \rho(g)$, acting on V. Show that $\mathrm{Tr}(P) = \dim(V^G)$. Hence we obtain *Burnside's orbit counting formula*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

Problem 2. (15 points) Let D_n be the dihedral group (symmetries of regular n–gon, thus $|D_n| = 2n$). In the problems below, the representations of D_n are over $\mathbb{C}$.

(1) Count the number of conjugacy classes in D_n.
(2) Show that D_n has exactly two 1–dimensional representations, if n is odd; and exactly four 1–dimensional representations, if n is even.
(3) Let V be an irreducible representation of D_n. Show that $\dim(V) \le 2$.
(4) Give explicit description of all irreducible D_n–representations.

Problem 3. (10 points) Consider the dihedral group D_3. It has 3 irreducible representations: two 1–dimensional and one 2–dimensional. Let $V = \mathbb{C}[D_3]$ be the (left) regular representation of D_3. Work out (directly, i.e, without using Theorem §3 of Lecture 2) the decomposition of V into direct sum of irreducible representations.

Problem 4. (10 points) Let S_n be the group of permutations of n letters. Let $\mathbb{C}^n$ be the standard n–dimensional S_n–representation (i.e., in a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$: $\sigma \cdot e_i = e_{\sigma(i)}$, for every $\sigma \in S_n$ and $1 \le i \le n$). Show that $\mathbb{C}^n$ is a direct sum of two irreducible S_n–representations. Give a basis of each of these subrepresentations of $\mathbb{C}^n$.

Problem 5. (10 points) Let G be a finite group and let V_1, V_2, V_3 be three finite–dimensional G–representations over $\mathbb{C}$. Show that the following is an isomorphism of G–representations.

$$\alpha : \mathrm{Hom}_{\mathbb{C}}(V_1, V_2) \otimes V_3 \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_2 \otimes V_3)$$

Here, α is given as follows. For every $X \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, $v_3 \in V_3$ and $v_1 \in V_1$:

$$\alpha(X \otimes v_3) : v_1 \longmapsto X(v_1) \otimes v_3.$$

Problem 6. (10 points) Let G be a finite group and V a finite–dimensional G–representation over $\mathbb{C}$. Assume that $\chi_V(g) = 0$ for every $g \in G \setminus \{e\}$. Prove that $\dim(V)$ is divisible by $|G|$.

Notations: Let G be a finite group, $\mathbb{C}[G]$ its group algebra over $\mathbb{C}$. Recall the convolution product, for $f_1, f_2 \in \mathbb{C}[G]$:

$$(f_1 * f_2)(g) = \sum_{\sigma \in G} f_1(\sigma) f_2(\sigma^{-1} g).$$

We defined a symmetric, bilinear form $B : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$ by:

$$B(f_1, f_2) = \frac{1}{|G|}(f_1 * f_2)(e) = \frac{1}{|G|} \sum_{g \in G} f_1(g^{-1}) f_2(g) \ .$$

Let $\mathbb{C}[G]_{\text{class}}$ denote the (commutative) subalgebra of functions constant on conjugacy classes of G (abbreviated to $\text{Conj}(G)$).

Let $\{V_\lambda : \lambda \in P(G)\}$ be the set of isomorphism classes of finite–dimensional, irreducible G–representations. Let ρ_λ denote the group homomorphism $G \to \text{GL}(V_\lambda)$, or the corresponding algebra homomorphism $\mathbb{C}[G] \to \text{End}(V_\lambda)$. Also, let χ_λ denote the character of the representation V_λ.

Problem 7. (20 points) Prove the following formulae:

- *Fourier inversion formula:* Given $X = (X_\lambda)_{\lambda \in P(G)} \in \oplus_{\lambda \in P(G)} \text{End}_{\mathbb{C}}(V_\lambda)$, show that

$$\psi^{-1}(X)(g) = \frac{1}{|G|} \sum_{\lambda \in P(G)} \dim(V_\lambda) \text{Tr}(\rho_\lambda(g^{-1}) X_\lambda) \ .$$

- *Plancheral formula:* Given $f_1, f_2 \in \mathbb{C}[G]$, show that

$$B(f_1, f_2) = \frac{1}{|G|^2} \sum_{\lambda \in P(G)} \dim(V_\lambda) \text{Tr}(\rho_\lambda(f_1 * f_2)) \ .$$

(Optional: why are these formulae called as such?)

Problem 8. (15 points) Fix $\lambda \in P(G)$ and let $\varphi : V_\lambda \otimes V_\lambda^* \to \text{End}(V_\lambda)$ be the Tensor–Hom adjointness isomorphism. Show that, for $v \in V_\lambda$ and $\xi \in V_\lambda^*$, we have

$$\psi^{-1}(v \otimes \xi) = \sum_{g \in G} \frac{\dim(V_\lambda)}{|G|} \xi(\rho_\lambda(g^{-1}) v) \delta_g \ .$$

For a G–representation V, and $v \otimes \xi \in V \otimes V^*$, the function $c_{v,\xi} : G \to \mathbb{C}$, defined by $c_{v,\xi}(g) = \xi(g^{-1} v)$, is called a matrix coefficient of V.

Problem 9. (10 points) Let $R_1, \ldots, R_N$ be N finite–dimensional representations of G (say over $\mathbb{C}$). Show that $R_1, \ldots, R_N$ are pairwise non–isomorphic, irreducible representations of G if, and only if, their matrix coefficients are linearly independent.

Problem 10. (10 points) (Dedekind) Let G be a finite group. Show that the set of all group homomorphisms $\mathrm{Hom}_{\mathrm{gp}}(G, \mathbb{C}^\times)$ is linearly independent (as a subset of the vector space of all $\mathbb{C}$–valued functions on G).

Problem 11. (15 points) Let Q_8 be the quaternionic group defined as follows. $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ has 8 elements, with the group operation:

- ± 1 commute with every other element and $(-1)^2 = +1$,
- $(-1)x = -x$, for $x = i, j, k$, and
- $ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik,\ i^2 = j^2 = k^2 = -1$

Show that Q_8 and dihedral group D_4 have the same character table, but these groups are not isomorphic.

Problem 12. (25 points) Prove Frobenius' Theorem 2 (slide 18 from Lecture 6) using the fundamental theorem of representation theory of finite groups (Theorem §3 of Lecture 2).

Symmetric group

Problem 13. (10 points) Let $p_1, p_2, \ldots$ be pairwise commuting variables, and for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0)$, let $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_\ell}$. Prove the following identity, in $(\mathbb{Q}[p_1, p_2, \ldots])[\![t]\!]$

$$1 + \sum_\lambda \frac{p_\lambda}{z_\lambda} t^{|\lambda|} = \exp\left(\sum_{r=1}^\infty \frac{p_r t^r}{r}\right)$$

where, if λ has r_j parts equal to j, then $z_\lambda = \prod_{j \geq 1} j^{r_j} r_j!$.

Problem 14. (15 points) Recall the definition of elementary, complete and power sum symmetric polynomials, in N variables. Convention: $e_0 = h_0 = 1$.

$$e_r := \sum_{1 \leq i_1 < \ldots < i_r \leq N} x_{i_1} \cdots x_{i_r}, \qquad p_r := \sum_{i=1}^N x_i^r, \qquad h_r := \sum_{1 \leq i_1 \leq \ldots \leq i_r \leq N} x_{i_1} \cdots x_{i_r}.$$

Consider the generating series:

$$E(t) = \sum_{r=0}^N e_r t^r, \qquad H(t) = \sum_{r=0}^\infty h_r t^r, \qquad P(t) = \sum_{n=1}^\infty p_n \frac{t^n}{n}.$$

(a) Show that $H(t)E(-t) = 1$ and $H(t) = \exp(P(t))$.
(b) Prove *Newton's identities*:

$$nh_n = \sum_{r=1}^n p_r h_{n-r}, \qquad ne_n = \sum_{r=1}^n (-1)^{r-1} p_r e_{n-r}.$$

Problem 15. (10 points) Let G be a finite group and $f : G \to \mathbb{C}$ a class function. Show that $f = \chi_V$ for an irreducible representation V if, and only if the following three conditions hold: (i) f can be written as a linear combination, over $\mathbb{Z}$, of characters of finite-dimensional representations; (ii) $B(f, f) = 1$; and (iii) $f(e) > 0$.

Problem 16. (10 points) Recall the notation, for $\lambda \vdash n$:

$$X_\lambda = \left\{ (I_1, \dots, I_\ell) : \{1, \dots, n\} = \bigsqcup_{i=1}^{\ell} I_i \text{ and } |I_j| = \lambda_j, \forall j \right\}$$

(a) Show that X_λ is in bijection with S_n/S_λ, where $S_\lambda = S_{\lambda_1} \times \cdots S_{\lambda_\ell} < S_n$.
(b) Show that there is a bijection between the following three sets:
  (1) $(X_\lambda \times X_\mu)/S_n$,
  (2) $X_\mu^{S_\lambda} := \{a \in X_\mu : w \cdot a = a, \ \forall w \in S_\lambda\}$,
  (3) $\{A = (a_{ij}) \in \mathrm{M}_{n\times n}(\mathbb{Z}_{\geq 0}) : \sum_j a_{ij} = \lambda_i, \forall i \text{ and } \sum_i a_{ij} = \mu_j, \forall j\}$.

Problem 17. (15 points) Recall the definition of $(\cdot, \cdot)$ on $\Lambda_n = \mathbb{Z}[x_1, \dots, x_N]^{S_N}_{\deg = n}$.

(1) For $\lambda \vdash n$, let $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ and $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, where h_r and e_r are defined above (Problem 14). Show that (h_λ, e_μ) is the number of matrices with $0, 1$ entries whose row sums equal λ and column sums equal μ.

$$(h_\lambda, e_\mu) = \left| \left\{ A = (a_{ij}) : a_{ij} \in \{0,1\} \ \forall i, j \text{ and } \sum_j a_{ij} = \lambda_i, \ \forall i; \sum_i a_{ij} = \mu_j, \ \forall j \right\} \right|$$

(2) Let s_λ be the Schur polynomial, and m_μ be the monomial symmetric polynomial. Let $\rho = (N-1, N-2, \dots, 0)$. Show that:

$$\boxed{(s_\lambda, m_\mu) = \sum_{\substack{w \in S_N : \\ \lambda + \rho - w(\rho) \in S_N \cdot \mu}} \varepsilon(w)}$$

Problem 18. (10 points) Let $\mu \vdash (n-1)$. Prove the following (*Pieri's rule*):

$$s_\mu s_{(1)} = \sum s_\lambda \ ,$$

where the sum is over all $\lambda \vdash n$ which can be obtained from μ by adding exactly one box (i.e., $\exists j : \lambda_i = \mu_i, \ \forall i \neq j$, and $\lambda_j = \mu_j + 1$).

Problem 19. (10 points) Let G be a finite group, $H < G$ a subgroup and let $\mathbf{1}$ denote the one dimensional, trivial representation of H. Prove that:

$$\chi_{\mathrm{Ind}_H^G \mathbf{1}}(g) = |(G/H)^g| = \frac{|C \cap H| \cdot |Z_G(g)|}{|H|} \ ,$$

where $C = \{xgx^{-1} : x \in G\}$ is the conjugacy class containing g, and $Z_G(g) = \{x \in G : xg = gx\}$ is its centralizer.

Problem 20. (10 points) Let $\lambda \vdash n$ and let ι_λ is the character of the induced representation U_λ of S_n. Let $\mu \vdash n$ be another partition and let $r_i = |\{j : \mu_j = i\}|$. Use the previous exercise to prove the following formula:

$$\iota_\lambda(\mu) = \sum_{(s_{ij})} \prod_{i \geq 1} \frac{r_i!}{\prod_j s_{ij}!} \ ,$$

where the sum is over all matrices with $\mathbb{Z}_{\geq 0}$ entries such that $\sum_j s_{ij} = r_i$ and $\sum_i i s_{ij} = \lambda_i$.

Problem 21. (15 points) Prove directly that the following are irreducible representations of $\mathrm{GL}_m(\mathbb{C})$:

$$\mathrm{Sym}^\ell(\mathbb{C}^m),\ \ell \geq 0, \qquad \text{and} \qquad \wedge^k(\mathbb{C}^m), 0 \leq k \leq m.$$

Problem 22. (15 points) Recall that, for a graded vector space $U = \oplus_{n=0}^{\infty} U[n]$, such that $\dim(U[n]) < \infty$, for every $n \geq 0$, we define:

$$\text{(Hilbert series)} \qquad P(U;t) = \sum_{n \geq 0} \dim(U[n]) t^n.$$

Let V be a finite-dimensional vector space, and let $T^\bullet(V) = \oplus_{n\geq 0} T^n(V)$, and similarly $\mathrm{Sym}^\bullet(V), \wedge^\bullet(V)$. Prove that:

$$P(T^\bullet(V);t) = \frac{1}{1-\dim(V)t}\ , \qquad P(\mathrm{Sym}^\bullet(V);t) = \frac{1}{(1-t)^{\dim(V)}}\ ,$$
$$P(\wedge^\bullet(V);t) = (1+t)^{\dim(V)}\ .$$

Bonus 10 points: Take $U = \oplus_\lambda \mathbb{S}^\lambda(V)$. What is the corresponding Hilbert series?

Problem 23. (10 points) Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell)$ be a partition, and let $m \geq \ell$. Show that we have an isomorphism of $\mathrm{GL}_m(\mathbb{C})$–representations:

$$L_{\lambda+(1^\ell)} \cong L_\lambda \otimes \wedge^\ell(\mathbb{C}^m).$$

Problem 24. (10 points) *Branching rule for general linear group.* Let λ be a partition of length ℓ, and $m \geq \ell$. Let $L_\lambda^{(m)}$ denote the corresponding representation of $\mathrm{GL}_m(\mathbb{C})$. Consider $\mathrm{GL}_{m-1}(\mathbb{C})$ as the subgroup of $\mathrm{GL}_m(\mathbb{C})$ via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Show that:

$$L_\lambda^{(m)}\Big|_{\mathrm{GL}_{m-1}(\mathbb{C})} \cong \bigoplus_\mu L_\mu^{(m-1)}\ ,$$

where the direct sum is over all $\mu = (\mu_1 \geq \ldots \geq \mu_\ell \geq 0)$ satisfying the interlacing condition:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_\ell \geq \mu_\ell\ .$$

In other words, the Young diagram of μ is contained in the Young diagram of λ and its complement consists entirely of horizontal strips.

Problem 25. (20 points) Solve Problem 2.15.1 of Pasha's book [1] , (a)–(f).

Problem 26. (20 points) Solve Problem 2.15.1, (g)–(k).

Problem 27. (20 points) Solve Problem 2.15.1, (l)–(n).

Problem 28. (30 points) Solve Problem 2.16.5 of Pasha's book.

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Problem 29. (10 points) Prove that V is an irreducible A–representation for (a) $A = \mathrm{Mat}_{n\times n}(K)$ and $V = K^n$, (b) $A = K < x, d > /(dx - xd = 1)$ and $V = K[X]$ where x acts as multiplication by X and d acts as d/dX (assuming characteristic of K is 0).

Problem 30. (15 points) Let $A = \{f : \mathbb{R} \to \mathbb{R}$ continuous $|f(x+1) = f(x)\}$ and let $M = \{g : \mathbb{R} \to \mathbb{R}$ continuous $|g(x+1) = -g(x)\}$. Prove that A and M are non–isomorphic, indecomposable representations of A. Show that $A \oplus A$ is isomorphic to $M \oplus M$.

Problem 31. (10 points) (See Problem 3.9.1 of Pasha's book). Let V, W be two representations of A. Given a linear map $f : A \to \mathrm{Hom}_k(W, V)$, define $\rho_f : A \to \mathrm{End}_k(V \oplus W)$ given by:

$$\rho_f(a) = \begin{pmatrix} \rho_1(a) & f(a) \\ 0 & \rho_2(a) \end{pmatrix}.$$

(a) Show that ρ_f is a representation of A if, and only if $f(ab) = \rho_1(a)f(b) + f(a)\rho_2(b)$, for every $a, b \in A$. Let the space of all the solutions of this equation be denoted by $Z^1(W, V) \subset \mathrm{Hom}_k(A, \mathrm{Hom}_k(W, V))$ (called 1–cocycles).

(b) Given $X \in \mathrm{Hom}_k(W, V)$, define the automorphism of the vector space $g \in \mathrm{Aut}_k(V \oplus W)$ given by:

$$g = \begin{pmatrix} \mathrm{Id}_{V_1} & X \\ 0 & \mathrm{Id}_{V_2} \end{pmatrix}.$$

Let $f : A \to \mathrm{Hom}_k(W, V)$ be an element of $Z^1(W, V)$ as in part (a). Show that g is an isomorphism between ρ_f and ρ_0 if, and only if $f(a) = \rho_1(a)X - X\rho_2(a)$ for every $a \in A$.

Problem 32. (10 points) Prove Weyl's dimension formula: if L_λ is the unique irreducible representation of $\mathrm{GL}_m(\mathbb{C})$, for a partition λ with $\ell(\lambda) \leq m$, then

$$\dim(L_\lambda) = \prod_{1\leq i<j\leq m} \frac{\lambda_i - \lambda_j + j - i}{j - i} .$$

Problem 33. (15 points) For any $\lambda \in \mathbb{C}$, define M_λ to be (infinite–dimensional) $\mathfrak{sl}_2$–representation as follows: M_λ has a basis $\{m_r : r \in \mathbb{Z}_{\geq 0}\}$ with $\mathfrak{sl}_2$–action given by (with the understanding that $m_{-1} = 0$):

$$h \cdot m_\ell = (\lambda - 2\ell)m_\ell, \qquad e \cdot m_\ell = (\lambda - \ell + 1)m_{\ell-1}, \qquad f \cdot m_\ell = (\ell + 1)m_{\ell+1}.$$

Show that M_λ (called Verma module) is irreducible for $\lambda \notin \mathbb{Z}_{\geq 0}$. For $\lambda \in \mathbb{Z}_{\geq 0}$, show that M_λ has a subrepresentation isomorphic to $M_{-\lambda-2}$, so that we have the following non–split short exact sequence of $\mathfrak{sl}_2$–representations:

$$0 \to M_{-\lambda-2} \to M_\lambda \to L_\lambda \to 0 .$$

Here, L_λ is $\lambda + 1$–dimensional, irreducible representation of $\mathfrak{sl}_2$.

Problem 34. (15 points) For a linear operator $X \in \mathrm{End}_{\mathbb{C}}(V)$ acting on a finite–dimensional vector space, let $\exp(X) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} X^n$.

(a) Show that $\det(\exp(X)) = e^{\mathrm{Tr}(X)}$.

(b) For $X \in \mathrm{End}(V)$ and $A \in \mathrm{GL}(V)$, consider the linear operators on $\mathrm{End}_{\mathbb{C}}(V)$:

$$\mathrm{ad}(X) : Y \mapsto XY - YX, \qquad \mathrm{Ad}(A) : Y \mapsto AYA^{-1} \ .$$

Show that $\exp(\mathrm{ad}(X)) = \mathrm{Ad}(\exp(X))$.

(c) Let A be an algebra over $\mathbb{C}$ and $\partial : A \to A$ be a nilpotent derivation. Show that $\exp(\partial) : A \to A$ is an algebra isomorphism.

Problem 35. (15 points) Let $m \geq 2$ and let L be the free Lie algebra over a vector space of dimension m. Recall that L is $\mathbb{Z}_{\geq 1}$–graded: $L = \oplus_{n=1}^{\infty} L[n]$. Let $d_n := \dim(L[n])$. Show that the following identity holds:

$$1 - mt = \prod_{n=1}^{\infty} (1 - t^n)^{d_n} \ .$$

Use this identity to prove that $d_n = \displaystyle\frac{1}{n} \sum_{d|n} \mu(d) m^{n/d}$.

Here, $\mu : \mathbb{Z}_{\geq 1} \to \{-1, 0, +1\}$ is the Möbius function defined as follows. $\mu(1) = 1$, $\mu(n) = 0$ if there exists a prime p such that p^2 divides n, and $\mu(p_1 \cdots p_k) = (-1)^k$ for distinct primes $p_1, \dots, p_k$.

Problem 36. (20 points) Recall that for an $\mathfrak{sl}_2$–representation V such that e, f act locally nilpotently on V, we defined:

$$\mathsf{S}_V := \exp(e)\exp(-f)\exp(e) : V \xrightarrow{\sim} V.$$

Take $V = L_n$, irreducible representation of dimension $n+1$ (see Problem 33 above). Thus, we have a basis $\{v_0, \dots, v_n\}$ with $\mathfrak{sl}_2$–action given as in Problem 33.

Below, you will prove that $\mathsf{S}_{L_n}(v_\ell) = (-1)^{n-\ell} v_{n-\ell}$.

(a) Prove that, taking $V = \mathfrak{sl}_2$ via adjoint action, we have

$$\mathsf{S}_{\mathfrak{sl}_2} : \begin{array}{rcl} e & \mapsto & -f \\ f & \mapsto & -e \\ h & \mapsto & -h \end{array}$$

(b) Now, let $V = L_n$ as above. Show that $\mathsf{S}_{L_n} v_0 = (-1)^n v_n$. Use this, and the previous part to conclude that $\mathsf{S}_{L_n} v_\ell = (-1)^{n-\ell} v_{n-\ell}$, for all $0 \leq \ell \leq n$.

Problems 37, 38 are inspired by the following beautiful paper by Etingof and Varchenko: Dynamical Weyl group and applications, Advances in Mathematics, **167** *(2002) pp. 74–127.*

Problem 37. (20 points) Let $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$. Let M_λ be the Verma module (see Problem 33) with basis denoted by $\{m_r^{(\lambda)} : r \in \mathbb{Z}_{\geq 0}\}$. Let L_n be the $n+1$–dimensional irreducible representation of $\mathfrak{sl}_2$, with basis denoted by $\{v_0, \ldots, v_n\}$. Let $0 \leq k \leq n$.

Show that, for all but finitely many $\lambda \in \mathbb{C}$, there is a unique $\mathfrak{sl}_2$–intertwiner:

$$\Psi_\lambda^{(n,k)} : M_\lambda \to M_\mu \otimes L_n$$

where $\mu = \lambda - (n-2k)$, whose value on $m_0^{(\lambda)}$ is of the following form, for uniquely determined $c_r \in \mathbb{C}$:

$$\Psi_\lambda^{(n,k)} : m_0^{(\lambda)} \mapsto m_0^{(\mu)} \otimes v_k + \sum_{r=1}^{k} c_r m_r^{(\mu)} \otimes v_{k-r} \ .$$

Deduce that we have the following isomorphism (generically in λ):

$$\mathrm{Hom}_{\mathfrak{sl}_2}(M_\lambda, M_\mu \otimes V) \to V[\lambda - \mu]$$

sending an intertwiner ψ to the coefficient of $m_0^{(\mu)}$ in $\psi(m_0^{(\lambda)})$.

Problem 38. (20 points) Now assume that $\lambda \in \mathbb{Z}_{\geq 0}$ and $\lambda \gg 0$. Let n, k and $\Psi_\lambda^{(n,k)}$ be as in Problem 37 above. Show that:

$$\Psi_\lambda^{(n,k)} \left(m_{\lambda+1}^{(\lambda)} \right) = m_{\mu+1}^{(\mu)} \otimes (a(\lambda) v_{n-k}) + \sum_{r \geq 1} b_r(\lambda) m_{\mu+1+r}^{(\mu)} \otimes v_{n-k-r}$$

for some $a(\lambda), b_r(\lambda)$. Compute $a(\lambda)$ explicitly.