Yau's proof of the Calabi Conjecture

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1 Introduction

Calabi conjecture can be stated as follows.

Let M *be a compact Kähler manifold with Kähler metric* $g_{\alpha\overline{\beta}}dz^{\alpha} \otimes d\bar{z}^{\beta}$. *Let* $\tilde{R}_{\alpha\overline{\beta}}dz^{\alpha} \otimes d\bar{z}^{\beta}$ *be a tensor whose associated* $(1,1)$*-form* $\frac{i}{2\pi}\tilde{R}_{\alpha\overline{\beta}}dz^{\alpha} \wedge d\bar{z}^{\beta}$ *represents the first Chern class of* M. *Then we can find a Kähler metric* $\tilde{g}_{\alpha\overline{\beta}}dz^{\alpha} \otimes d\bar{z}^{\beta}$ *which is cohomologous to the original metric and whose Ricci tensor is given by* $\tilde{R}_{\alpha\overline{\beta}}dz^{\alpha} \otimes d\bar{z}^{\beta}$.

This was proposed by Eugenio Calabi in 1954 and a proof was published in 1978 by S.T. Yau. One direct consequence of this theorem is the existence of Ricci flat Kähler manifolds, now also called as Calabi-Yau manifolds. These are Kähler manifolds with trivial canonical line bundle. These special surfaces find applications in String theory as well. In these notes we go through Yau's proof in detail.

2 Proof of the Calabi conjecture

The proof can be divided into four steps:

1. Reformulating the statement in terms of a PDE called the complex Monge-Ampere equation.
2. Finding apriori estimates for all derivatives of the solution upto the second order.
3. Finding apriori estimates for the third order derivatives of the solution.
4. Using the method of continuity to solve the PDE.

The complex Monge-Ampere equation can be written as

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right) = \det(g_{i\bar{j}}) \exp(F)$$

where the function $F \in C^k(M)$ with $k \geqslant 3$ is given and satisfies

$$\int_M \exp(F) = \mathrm{Vol}(M)$$

It can be seen by integrating both sides of the previous equation that this condition on F is necessary. We will be showing that this indeed is sufficient for the existence of a solution ϕ. Since the proof as a whole can be rather disorienting to read all at once, let's break down the main ideas of each section.

1. **Reformulating the statement in terms of a PDE**

   Observe that the first Chern class of M, by definition, represents the real cohomology class containing $\frac{i}{2\pi}$ times the Ricci form. So the conjecture now implies that the given tensor differs from the Ricci tensor of the given metric by an exact form (say $\partial\bar{\partial}F$). For Kahler manifolds there is a simple identity for Ricci tensor given by $-\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log(\det(g_{i\bar{j}}))$. Since the Kähler metrics mentioned in the theorem are cohomologous to each other, we can relate them by an exact form $\partial\bar{\partial}\phi$, for some scalar-valued function ϕ. Note that any exact form can be globally written in this form as a consequence of the $\partial\bar{\partial}$-lemma. So if we were to have the above tensor $\tilde{R}$ as the Ricci tensor of a the new Kähler metric (say g'), then it is clear that the following has to be satisfied.

   $$\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left[\log(\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1}\exp(-F))\right] = 0$$

   This implies that

   $$\log(\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1}\exp(-F)) = C$$

   by maximum principle on a compact manifold. Now the above condition $\int_M \exp(F) = \mathrm{Vol}(M)$ would imply that $C = 0$, and the resulting expression is the complex Monge-Ampere equation. We can now see that if there is a function ϕ satisfying this equation, then the metric g' defined as $g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}$ has all the required properties.

The difficult part of the proof is finding apriori estimates for the derivatives. So we focus on that first.

2.1 Estimates upto the second order

We will be studying the equation

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1} = \exp(F) \tag{1}$$

where F is assumed to be $C^3(M)$. We want solutions ϕ of (1) such that the matrix $\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right)_{ij}$ is positive definite Hermitian. Then this would define another Kähler metric on M. Denote

$$g'_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}$$

We assume that $\phi \in C^5(M)$. The aim of this section is to obtain estimates on all derivatives of ϕ upto the second order. To guarantee that the solutions are unique, we impose the normalization

$$\int_M \phi = 0 \tag{2}$$

Differentiating (1) wrt z^k we have

$$g'^{i\bar{j}}\left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3 \phi}{\partial z^i \partial \bar{z}^j \partial z^k}\right) - g^{i\bar{j}}\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial F}{\partial z^k} \tag{3}$$

This differentiation is done by first taking log on both sides and then using the formula for the derivative of log of determinant of a matrix $\star$. Differentiating (3) again wrt $\bar{z}^l$

$$\begin{aligned}\frac{\partial^2 F}{\partial z^k \partial \bar{z}^l} =& -g'^{t\bar{j}}g'^{i\bar{n}}\left(\frac{\partial g_{t\bar{j}}}{\partial \bar{z}^l} + \frac{\partial^3 \phi}{\partial z^t \partial \bar{z}^j \partial \bar{z}^l}\right)\left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3 \phi}{\partial z^i \partial \bar{z}^n \partial z^k}\right) + g'^{i\bar{j}}\left(\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \frac{\partial^4 \phi}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l}\right) \\ &+ g^{t\bar{j}}g^{i\bar{j}}\frac{\partial g_{t\bar{j}}}{\partial \bar{z}^l}\frac{\partial g_{i\bar{j}}}{\partial z^k} - g^{i\bar{j}}\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l}\end{aligned} \tag{4}$$

Here we used the formula $\dfrac{\partial g'^{i\bar{j}}}{\partial \bar{z}^l} = -g'^{t\bar{j}}g'^{i\bar{j}}\dfrac{\partial g'_{t\bar{j}}}{\partial \bar{z}^l}$ for the derivative of the inverse of a matrix.
Let Δ' be the Laplacian associated with the metric g'. Then

$$\begin{aligned}\Delta'(\Delta\phi) &= g'^{k\bar{l}}\frac{\partial^2}{\partial z^k \partial \bar{z}^l}\left(g^{i\bar{j}}\frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right) \\ &= g'^{k\bar{l}}g^{i\bar{j}}\frac{\partial^4 \phi}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l} + g'^{k\bar{l}}\frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^l}\frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j} \\ &+ g'^{k\bar{l}}\frac{\partial g^{i\bar{j}}}{\partial z^k}\frac{\partial^3 \phi}{\partial z^i \partial \bar{z}^j \partial \bar{z}^l} + g'^{k\bar{l}}\frac{\partial g_{i\bar{j}}}{\partial \bar{z}^l}\frac{\partial^3 \phi}{\partial z^i \partial \bar{z}^j \partial z^k}\end{aligned} \tag{5}$$

Since LHS is defined independent of the co-ordinate system, we compute the RHS by picking normal co-ordinates at a point so that $g_{i\bar{j}} = \delta_{ij}$, $\dfrac{\partial g_{i\bar{j}}}{\partial z^k} = \dfrac{\partial g_{i\bar{j}}}{\partial \bar{z}^l} = 0$.
Multiplying (4) by $g^{k\bar{l}}$ and adding over k, l gives

$$\Delta F + g'^{t\bar{j}}g'^{i\bar{n}}g^{k\bar{l}}\phi_{t\bar{n}\bar{l}}\phi_{i\bar{j}k} = g^{k\bar{l}}g^{i\bar{j}}\phi_{i\bar{j}k\bar{l}} + g^{k\bar{l}}g'^{i\bar{j}}\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} - g^{i\bar{j}}g^{k\bar{l}}\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} \tag{6}$$

Plugging this in (5) gives

$$\begin{aligned}\Delta'(\Delta\phi) &= \Delta F + g'^{k\bar{j}}g'^{i\bar{n}}\phi_{k\bar{n}\bar{l}}\phi_{i\bar{j}l} + g'^{k\bar{l}}\frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^l}\frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j} + g^{k\bar{l}}g'^{i\bar{j}}\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} - g^{i\bar{j}}g^{k\bar{l}}\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} \\ &= \Delta F + g'^{k\bar{j}}g'^{i\bar{n}}\phi_{k\bar{n}\bar{l}}\phi_{i\bar{j}l} + \sum_{i,j} g'^{k\bar{l}}R_{i\bar{j}k\bar{l}}\phi_{i\bar{j}} + \sum_{l} g'^{i\bar{j}}R_{i\bar{j}l\bar{l}} - \sum_{i,l} R_{i\bar{i}l\bar{l}}\end{aligned} \tag{7}$$

In the last step we used the fact that $g^{i\bar{j}} = \delta_{ij}$ and $R_{i\bar{j}k\bar{l}} = \dfrac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}$.

Now choose a co-ordinate system so that in addition to $g_{i\bar{j}} = \delta_{ij}$, we also have $\phi_{i\bar{j}} = \delta_{ij}\phi_{i\bar{i}}$. This is possible because these are symmetric matrices and hence are simultaneously diagonalizable. Then we have

$$g'^{i\bar{j}} = \delta_{ij}\frac{1}{(1+\phi_{i\bar{i}})} \tag{8}$$

and

$$
\begin{aligned}
&\sum_{l} g'^{i\bar{j}} R_{i\bar{j}l\bar{l}} - \sum_{i,l} R_{i\bar{i}l\bar{l}} + \sum_{i,j} g'^{k\bar{l}} R_{i\bar{j}k\bar{l}} \phi_{i\bar{j}} \\
&= -\sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\phi_{i\bar{i}}}{1+\phi_{i\bar{i}}} + \sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\phi_{i\bar{i}}}{1+\phi_{l\bar{l}}} \\
&= -\sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\phi_{i\bar{i}}(\phi_{l\bar{l}}-\phi_{i\bar{i}})}{(1+\phi_{i\bar{i}})(1+\phi_{l\bar{l}})} \qquad (9)
\end{aligned}
$$

Now using the symmetry of this expression wrt i and l,

$$
\begin{aligned}
&= \frac{1}{2}\left[-\sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\phi_{i\bar{i}}(\phi_{l\bar{l}}-\phi_{i\bar{i}})}{(1+\phi_{i\bar{i}})(1+\phi_{l\bar{l}})} - \sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{\phi_{l\bar{l}}(\phi_{i\bar{i}}-\phi_{l\bar{l}})}{(1+\phi_{i\bar{i}})(1+\phi_{l\bar{l}})} \right] \\
&= \frac{1}{2}\sum_{i,l} R_{i\bar{i}l\bar{l}} \frac{(\phi_{l\bar{l}}-\phi_{i\bar{i}})^2}{(1+\phi_{i\bar{i}})(1+\phi_{l\bar{l}})}
\end{aligned}
$$

From (9) and (7) we see that

$$
\Delta'(\Delta\phi) \geqslant \Delta F + \sum_{l} g'^{k\bar{j}} g'^{i\bar{n}} \phi_{k\bar{n}l} \phi_{i\bar{j}l} + (\inf_{i\neq l} R_{i\bar{i}l\bar{l}}) \left[\sum_{i,l} \frac{1+\phi_{i\bar{i}}}{1+\phi_{l\bar{l}}} - m^2 \right] \qquad (10)
$$

This is true because

$$
-\frac{\phi_{i\bar{i}}(\phi_{l\bar{l}}-\phi_{i\bar{i}})}{(1+\phi_{i\bar{i}})(1+\phi_{l\bar{l}})} = \frac{1+\phi_{i\bar{i}}^2+\phi_{l\bar{l}}+\phi_{i\bar{i}}}{(1+\phi_{i\bar{i}})(1+\phi_{l\bar{l}})} - 1
$$

When $\sum_{i,l}$ is taken, we get terms of the form

$$
\frac{\phi_{i\bar{i}}^2+\phi_{l\bar{l}}^2+1+1+\phi_{l\bar{l}}+\phi_{i\bar{i}}+\phi_{i\bar{i}}+\phi_{l\bar{l}}}{(1+\phi_{i\bar{i}})(1+\phi_{l\bar{l}})}
$$

Then we rearrange and factor this as needed.
On the other hand

$$
\begin{aligned}
\Delta'\phi &= \sum_{i} \frac{\phi_{i\bar{i}}}{1+\phi_{i\bar{i}}} \\
&= m - \sum_{i} \frac{1}{1+\phi_{i\bar{i}}} \qquad (11)
\end{aligned}
$$

Let C be a positive constant. Then we have

$$
\begin{aligned}
\Delta'(\exp(-C\phi)(m+\Delta\phi)) = {} & C^2\exp(-C\phi)(g'^{i\bar{j}}\phi_i\phi_{\bar{j}})(m+\Delta\phi) - C\exp(-C\phi)g'^{i\bar{j}}\phi_{i\bar{j}}(m+\Delta\phi) \\
& - C\exp(-C\phi)g'^{i\bar{j}}\phi_{\bar{j}}(\Delta\phi)_i \qquad (12) \\
& - C\exp(-C\phi)g'^{i\bar{j}}\phi_i(\Delta\phi)_{\bar{j}} \qquad (13) \\
& + \exp(-C\phi)\Delta'(\Delta\phi)
\end{aligned}
$$

Now using $g'^{i\bar{j}}\phi_i(\Delta\phi)_{\bar{j}} \leqslant \sqrt{g'^{i\bar{j}}\phi_i\phi_{\bar{j}}}\sqrt{g'^{i\bar{j}}(\Delta\phi)_i(\Delta\phi)_{\bar{j}}}$ on (12) and (13), followed by $ab \leqslant \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ with $\epsilon = C(m+\Delta\phi)$, we get a cancellation with the first term. We are left with

$$
\begin{aligned}
\Delta'(\exp(-C\phi)(m+\Delta\phi)) \geqslant {} & -\frac{\exp(-C\phi)g'^{i\bar{j}}(\Delta\phi)_i(\Delta\phi)_{\bar{j}}}{m+\Delta\phi} - C\exp(-C\phi)\Delta'\phi(m+\Delta\phi) \\
& + \exp(-C\phi)\Delta'(\Delta\phi) \qquad (14)
\end{aligned}
$$

Using $g_{i\bar{j}} = \delta_{ij}$ and $\phi_{i\bar{j}} = \phi_{i\bar{i}}\delta_{ij}$ at a point, (14) can be simplified using (10)

$$\begin{aligned}
-(m+\Delta\phi)^{-1} g'^{i\bar{j}}(\Delta\phi)_i(\Delta\phi)_{\bar{j}} + \Delta'(\Delta\phi) \geqslant & -(m+\Delta\phi)^{-1} \sum_i (1+\phi_{i\bar{i}})^{-1} |\sum_k \phi_{k\bar{k}i}|^2 \\
& + \Delta F + \sum_{i,j,k} (1+\phi_{k\bar{k}})^{-1}(1+\phi_{i\bar{i}})^{-1} \phi_{k\bar{i}\bar{j}}\phi_{i\bar{k}j} \\
& + (\inf_{i\neq l} R_{i\bar{i}l\bar{l}}) \left[\sum_{i,l} \frac{1+\phi_{l\bar{l}}}{1+\phi_{i\bar{i}}} - m^2 \right] \qquad (15)
\end{aligned}$$

Here we used

$$\begin{aligned}
(\Delta\phi)_i(\Delta\phi)_{\bar{j}} &= (g^{k\bar{l}}\phi_{k\bar{l}})_i (g^{k\bar{l}}\phi_{k\bar{l}})_j \\
&= (g^{k\bar{l}}\phi_{k\bar{l}i})(g^{k\bar{l}}\phi_{k\bar{l}j}) = \sum_{i,j,k} \phi_{k\bar{k}i}\phi_{k\bar{k}j}
\end{aligned}$$

since $\dfrac{\partial g^{k\bar{l}}}{\partial z^i} = 0$.
But

$$\begin{aligned}
(m+\Delta\phi)^{-1} \sum_i (1+\phi_{i\bar{i}})^{-1} |\sum_k \phi_{k\bar{k}i}|^2 &= (m+\Delta\phi)^{-1} \sum_i (1+\phi_{i\bar{i}})^{-1} \left| \sum_k \frac{\phi_{k\bar{k}i}}{(1+\phi_{k\bar{k}})^{1/2}} (1+\phi_{k\bar{k}})^{1/2} \right| \\
&\leqslant (m+\Delta\phi)^{-1} \left(\sum_{i,k} (1+\phi_{i\bar{i}})^{-1}(1+\phi_{k\bar{k}})^{-1} \phi_{k\bar{k}i}\phi_{\bar{k}k\bar{i}} \right) \left(\sum_k (1+\phi_{k\bar{k}}) \right) \\
&= \sum_{i,k} (1+\phi_{i\bar{i}})^{-1}(1+\phi_{k\bar{k}})^{-1} \phi_{k\bar{k}i}\phi_{\bar{k}k\bar{i}} \\
&= \sum_{i,k} (1+\phi_{i\bar{i}})^{-1}(1+\phi_{k\bar{k}})^{-1} \phi_{i\bar{k}k}\phi_{k\bar{i}\bar{k}} \qquad (16)
\end{aligned}$$

The second line is using the Cauchy-Schwarz inequality and in the third line we just wrote $\sum_k (1+\phi_{k\bar{k}}) = m + \Delta\phi$. The last two lines follow from

$$\phi_{k\bar{k}i} = \phi_{\bar{k}ki} = \phi_{\bar{k}ik} = \phi_{i\bar{k}k}$$

and

$$\phi_{\bar{k}k\bar{i}} = \phi_{k\bar{k}\bar{i}} = \phi_{k\bar{i}\bar{k}}$$

As a side note, the third covariant derivatives of a function commute with the second one only if they are of the same type. When these derivatives are of the opposite type an extra curvature term will appear. This does not happen with first and second covariant derivatives though. Continuing the computation above

$$\leqslant \sum_{i,j,k} (1+\phi_{k\bar{k}})^{-1}(1+\phi_{i\bar{i}})^{-1} \phi_{k\bar{i}\bar{j}}\phi_{i\bar{k}j} \qquad (17)$$

since we are just adding extra positive stuff with $(1+\phi_{i\bar{i}})^{-1} = g'^{i\bar{i}} > 0$ and $\phi_{k\bar{i}\bar{j}} = \overline{\phi_{i\bar{k}j}}$.
So plugging this into (15) gives (after cancellation)

$$-(m+\Delta\phi)^{-1} g'^{i\bar{j}}(\Delta\phi)_i(\Delta\phi)_{\bar{j}} + \Delta'(\Delta\phi) \geqslant \Delta F + \left(\inf_{i\neq l} R_{i\bar{i}l\bar{l}} \right) \left[\sum_{i,l} \frac{1+\phi_{l\bar{l}}}{1+\phi_{i\bar{i}}} - m^2 \right] \qquad (18)$$

Inserting this into (14) we get

$$\Delta'(\exp(-C\phi)(m+\Delta\phi)) \geqslant \exp(-C\phi)\left[\Delta F + \left(\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\left(\sum_{i,l}\frac{1+\phi_{l\bar{l}}}{1+\phi_{i\bar{i}}} - m^2\right)\right] - C\exp(-C\phi)(\Delta'\phi)(m+\Delta\phi) \tag{19}$$

Inserting (11) into this gives

$$\begin{aligned}\Delta'(\exp(-C\phi)(m+\Delta\phi)) &\geqslant \exp(-C\phi)\left(\Delta F - m^2\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right) + \exp(-C\phi)\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\left(\sum_{i,l}\frac{1+\phi_{i\bar{i}}}{1+\phi_{l\bar{l}}}\right)\\ &\quad - C\exp(-C\phi)m(m+\Delta\phi) + C\exp(-C\phi)(m+\Delta\phi)\left(\sum_i \frac{1}{1+\phi_{i\bar{i}}}\right)\\ &= \exp(-C\phi)\left(\Delta F - m^2\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right) - C\exp(-C\phi)m(m+\Delta\phi)\\ &\quad + (C+\inf_{i\neq l} R_{i\bar{i}l\bar{l}})\exp(-C\phi)(m+\Delta\phi)\left(\sum_i\frac{1}{1+\phi_{i\bar{i}}}\right)\end{aligned} \tag{20}$$

where we used $\sum\limits_i 1+\phi_{i\bar{i}} = m+\Delta\phi$.

Notice that the expansion of $\left(\sum\limits_i \frac{1}{1+\phi_{i\bar{i}}}\right)^{m-1}$ would contain $\frac{\sum_i(1+\phi_{i\bar{i}})}{\prod_i(1+\phi_{i\bar{i}})}$ and everything else is positive. This gives the following inequality

$$\sum_i \frac{1}{1+\phi_{i\bar{i}}} \geqslant \left(\frac{\sum_i(1+\phi_{i\bar{i}})}{\prod_i(1+\phi_{i\bar{i}})}\right)^{\frac{1}{m-1}}$$

Therefore

$$\sum_i \frac{1}{1+\phi_{i\bar{i}}} \geqslant (m+\Delta\phi)^{\frac{1}{m-1}}\exp\left(\frac{-F}{m-1}\right) \tag{21}$$

where we have used

$$\begin{aligned}&\det(g'_{i\bar{j}}) = \exp(F)\det(g_{i\bar{j}})\\ &\prod_i(1+\phi_{i\bar{i}}) = \exp(F)\\ &\left(\prod_i(1+\phi_{i\bar{i}})\right)^{-\frac{1}{m-1}} = \exp\left(-\frac{F}{m-1}\right)\end{aligned}$$

Choose C so that

$$C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}} > 1$$

Then it follows from the previous computation that

$$\begin{aligned}\Delta'(exp(-C\phi)(m+\Delta\phi)) &\geqslant \exp(-C\phi)(\Delta F - m^2\inf_{i\neq l} R_{i\bar{i}l\bar{l}}) + (C+\inf_{i\neq l} R_{i\bar{i}l\bar{l}})\exp(-C\phi)\\ &\quad\times\exp\left(-\frac{F}{m-1}\right)(m+\Delta\phi)^{1+\frac{1}{m-1}} - C\exp(-C\phi)m(m+\Delta\phi)\end{aligned} \tag{22}$$

Now we use this to estimate $\exp(-C\phi)(m+\Delta\phi)$. In fact it must achieve its maximum at some point p so that the RHS of (22) is non-positive (since $\Delta'(\dots)$ has to be non-positive. This does no depend on the metric used to compute the Laplacian). At this point

$$0 \geqslant \Delta F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - Cm(m+\Delta\phi) + (C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}) \exp\left(-\frac{F}{m-1}\right)(m+\Delta\phi)^{1+\frac{1}{m-1}}$$

This implies that $(m+\Delta\phi)(p)$ has an upper bound C_1 depending only on $\sup_M\{-\Delta F\}$, $\sup_M |\inf_{i\neq l} R_{i\bar{i}l\bar{l}}|$, Cm and $\sup_M F$. This is because $y^{1+\frac{1}{m-1}} \leqslant ay + b$ would imply that either $y^{1+\frac{1}{m-1}} \leqslant 2ay$ or $y^{1+\frac{1}{m-1}} \leqslant 2b$.
Since $\exp(-C\phi)(m+\Delta\phi)$ achieves its maximum at p, we have the following inequality

$$0 < m + \Delta\phi \leqslant C_1 \exp(C(\phi - \inf_M \phi)) \tag{23}$$

The first inequality is true because $(1+\phi_{i\bar{i}}) > 0$ for all i.
We use this to estimate $\sup_M |\phi|$. Since $\Delta\phi + m = \sum_i 1 + \phi_{i\bar{i}} = g^{i\bar{j}} g'_{i\bar{j}} > 0$, we can estimate $\sup_M \phi$ as follows.
Let $G(p,q)$ be the Green's function of the operator Δ on M. Let K be a constant such that

$$G(p,q) + K \geqslant 0$$

Then

$$\begin{aligned} \phi(p) &= -\int_M G(p,q)\Delta\phi(q)dq \\ &= -\int_M (G(p,q)+K)\Delta\phi(q)dq \end{aligned} \tag{24}$$

since $\int_M \Delta\phi = 0$.

Therefore

$$\begin{aligned} \sup_M(\phi) &\leqslant m \sup_{p\in M} \int_M (G(p,q)+K)dq \\ |\sup_M \phi| &\leqslant m \sup_{p\in M} \int_M |G(p,q)+K|dq \end{aligned} \tag{25}$$

since $\Delta\phi > -m$. So we get

$$\begin{aligned} \int_M |\phi| &\leqslant \int_M |\sup_M(\phi) - \phi| + \int_M |\sup_M(\phi)| \\ &\leqslant (\sup_M \phi)\mathrm{Vol}(M) + |\sup_M(\phi)|\mathrm{Vol}(M) - \int_M \phi \\ &\leqslant 2m\mathrm{Vol}(M) \sup_{p\in M} \int_M (G(p,q)+K)dq \end{aligned} \tag{26}$$

Note that the right hand sides of the above estimate does not depend on ϕ, since the Green's function is prescribed for the manifold independent of ϕ. So far we have estimated $\sup_M(\phi)$ and $\|\phi\|_{L^1(M)}$. To estimate $\inf_M(\phi)$ two different proofs are offered. The first one works for the case when $m = 2$ and the second one works in general. We show both the proofs here.
⋆

For $m = 2$, we renormalize (by adding a constant) ϕ so that $\sup_M(\phi) \leqslant -1$. This is possible because of (25). Let p be any positive number greater than or equal to 1. Then

$$\begin{aligned}\Delta'(-\phi)^p &= p(p-1)\sum_i \frac{|\phi_i|^2}{1+\phi_{i\bar{i}}} - p(-\phi)^{p-1}\Delta'\phi \\ &= p(p-1)(-\phi)^{p-2}\sum_i \frac{|\phi_i|^2}{1+\phi_{i\bar{i}}} - p(-\phi)^{p-1}\left(m - \sum_i \frac{1}{1+\phi_{i\bar{i}}}\right) \end{aligned} \tag{27}$$

When $m = 2$ we find that

$$\begin{aligned}\Delta'(-\phi)^p &= p(p-1)(-\phi)^{p-2}\sum_i \frac{|\phi_i|^2}{1+\phi_{i\bar{i}}} - 2p(-\phi)^{p-1} + p(-\phi)^{p-1}(1+\phi_{1\bar{1}})^{-1}(1+\phi_{2\bar{2}})^{-1}(2+\Delta\phi) \\ &= p(p-1)(-\phi)^{p-2}\sum_i \frac{|\phi_i|^2}{1+\phi_{i\bar{i}}} - 2p(-\phi)^{p-1} + 2p(-\phi)^{p-1} + 2p(\phi)^{p-1}\exp(-F) \\ &+ \exp(-F)(-\Delta(-\phi)^p + p(p-1)(-\phi)^{p-2}|\nabla\phi|^2) \end{aligned} \tag{28}$$

since

$$\begin{aligned} &p(-\phi)^{p-1}\Delta\phi = -\Delta(-\phi)^p + p(p-1)(-\phi)^{p-2}|\nabla\phi|^2 \\ &(1+\phi_{1\bar{1}})(1+\phi_{2\bar{2}}) = \det(g') = \exp(F) \end{aligned}$$

Multiplying (28) by $\exp(F)$ and integrating wrt the volume form of the original metric g we have

$$\begin{aligned} 2p\int_M (-\phi)^{p-1}(1-\exp(F)) &= p(p-1)\int_M (-\phi)^{p-2}\left(\frac{|\phi_i|^2}{1+\phi_{i\bar{i}}}\exp(F)\right) - \int_M \Delta(-\phi)^p + p(p-1)\int_M (-\phi)^{p-2}|\nabla\phi|^2 \\ &= p(p-1)\int_M (-\phi)^{p-2}\left(\frac{|\phi_i|^2}{1+\phi_{i\bar{i}}}\exp(F)\right)\exp(F) + \frac{4(p-1)}{p}\int_M |\nabla(-\phi)^{\frac{p}{2}}|^2 \end{aligned}$$

Since ϕ is negative this implies that for $p \geqslant 2$

$$\begin{aligned}\int_M |\nabla(-\phi)^{\frac{p}{2}}|^2 &\leqslant \frac{p^2}{2(p-1)}\left|\int_M (1-\exp(F))(-\phi)^{p-1}\right| \\ &\leqslant pC_2\int_M |\phi|^{p-1} \end{aligned} \tag{29}$$

where C_2 depends only on $\sup_M(F)$.

Now we use the Sobolev inequality. It states that for any $g \in W^{1,r}$ with $r < \dim(M) = n$, we have the following inequality

$$\|g\|_{L^q(M)} \leqslant C(\|g\|_{L^r(M)} + \|\nabla g\|_{L^r}(M))$$

for some constant C that depend only on M. Here q can be any number in $[p, \frac{nr}{n-r}]$. Observing that the real dimension of M is $2m = 4$ and then applying Sobolev inequality to the function $g = |\phi|^{\frac{p}{2}}$ with $r = 2$ and $q = 4$, we can find a constant C_3 depending only on M such that

$$\begin{aligned}\left(\int_M |\phi|^{2p}\right)^{\frac{1}{2}} &\leqslant C_3\int_M |\phi|^p + C_3\int_M |\nabla(-\phi)^{\frac{p}{2}}|^2 \\ &\leqslant C_3\int_M |\phi|^p + pC_2C_3\int_M |\phi|^{p-1} \end{aligned} \tag{30}$$

The last line follows from

$$\int_M |\nabla(-\phi)^{\frac{p}{2}}|^2 = -\int_M \Delta(-\phi)^{\frac{p}{2}}(-\phi)^{\frac{p}{2}} \qquad \text{(Green's identity)}$$

and

$$\Delta(-\phi)^{\frac{p}{2}}.(-\phi)^{\frac{p}{2}} = \frac{p}{2}\left(\frac{p}{2}-1\right)(-\phi)^{p-2}|\nabla\phi|^2 - \frac{p}{2}(-\phi)^{p-1}\Delta\phi$$

Since $\phi \leqslant -1$, we derive another constant C_4 depending only on M such that

$$\int_M |\phi|^{2p} \leqslant C_4 p^2 \left(\int_M |\phi|^p\right)^2 \tag{31}$$

In order to use (31), we need another estimate of $\int_M |\phi|^2$. But this follows from the estimate of $\int_M |\phi|$ combined with the Poincare inequality and (29) as follows

$$\begin{aligned}\int_M |\phi|^2 &= \int_M (-\phi)^2 \\ &\leqslant C\int_M |\nabla(-\phi)|^2 \\ &\leqslant 2CC_2\int_M |\phi| \end{aligned}\tag{32}$$

We now claim that we can find a constant C_5 depending only on M such that

$$\int_M |\phi|^p \leqslant C_5^p \left(\frac{p}{2}\right)^{\frac{p}{2}} \qquad \text{for all integers } p \geqslant 1 \tag{33}$$

In fact, let p_0 be the first integer such that for $p \geqslant p_0$

$$C_4(1+\mathrm{Vol}(M))\left(\frac{p+2}{4}\right)^{\frac{p+1}{2}} \leqslant \left(\frac{p+1}{2}\right)^{\frac{p-3}{2}} \tag{34}$$

This is possible because $\lim\limits_{p\to\infty} \dfrac{\left(\frac{p+1}{2}\right)^{\frac{p-3}{2}}}{\left(\frac{p+2}{4}\right)^{\frac{p+1}{2}}} = \infty$.

Notice there are only finitely many values that p cannot assume. So we can choose a constant C_5 such that (33) is valid for all $1 \leqslant p \leqslant 2p_0$. This can be done because $\|\phi\|_{L^1(M)}$ and $\|\phi\|_{L^2(M)}$ are estimated above and $\|\phi\|_{L^p(M)}$ can be bounded by using (31) repeatedly with Hölder inequality.
We prove (33) for all integers $p \geqslant 1$ by induction. There are two cases. If $p+1$ is divisible by two, then from (31) we have

$$\begin{aligned}\int_M |\phi|^{p+1} &\leqslant C_4\left(\frac{p+1}{2}\right)^2\left(\int_M |\phi|^{\frac{p+1}{2}}\right)^2 \\ &\leqslant C_4\left(\frac{p+1}{2}\right)^2 C_5^{p+1}\left(\frac{p+1}{4}\right)^{\frac{p+1}{2}} \qquad \text{by induction hypothesis}\end{aligned}$$

Now apply (34) to this to get

$$\int_M |\phi|^{p+1} \leqslant C_5^{p+1}\left(\frac{p+1}{2}\right)^{\frac{p+1}{2}}$$

which is (33) for $p+1$.
On the other hand, if $p+1$ is not divisible by two, then by (31)

$$
\begin{aligned}
\int_M |\phi|^{p+1} &\leqslant C_4 \left(\frac{p+1}{2}\right)^2 \left(\int_M |\phi|^{\frac{p+1}{2}}\right)^2 && \text{since (31) is true for non-integers as well} \\
&\leqslant C_4 \left(\frac{p+1}{2}\right)^2 \left(\int_M |\phi|^{\frac{p+2}{2}}\right)^{\frac{2(p+1)}{p+2}} (\mathrm{Vol}(M))^{\frac{2}{p+2}} \\
&\leqslant C_4 \left(\frac{p+1}{2}\right)^2 C_5^{p+1} \left(\frac{p+2}{4}\right)^{\frac{p+1}{2}} (\mathrm{Vol}(M))^{\frac{2}{p+2}} \\
&\leqslant C_5^{p+1} \left(\frac{p+2}{4}\right)^{\frac{p+1}{2}}
\end{aligned}
$$

The second inequality is using Hölder inequality with conjugate exponents $\frac{p+2}{p+1}$ and $p+2$. The second to last line follows by induction hypothesis since $(\mathrm{Vol}(M))^{\frac{2}{p+2}} \leqslant \mathrm{Vol}(M)$ if $\mathrm{Vol}(M) \geqslant 1$. So we get (33) for all p.
Using this we have the following

$$
\begin{aligned}
\int_M \exp(k\phi^2) &\leqslant \sum_{p=0}^{\infty} \frac{k^p}{p!} \int_M |\phi|^{2p} \\
&\leqslant \sum_{p=0}^{\infty} \frac{k^p}{p!} (C_5^2)^p p^p
\end{aligned}
\tag{35}
$$

By Stirling's formula

$$
p! \geqslant \sqrt{2\pi} p^{p+\frac{1}{2}} e^{-p}
$$

for all positive integers p. This implies that

$$
\int_M \exp(k\phi^2) \leqslant \sum_{p=1}^{\infty} (kC_5^2 e)^p p^{-\frac{1}{2}} \tag{36}
$$

When

$$
k < C_5^{-2} e^{-1} \tag{37}
$$

the RHS of (36) is finite and we have thus obtained an estimate of $\int_M \exp(k\phi^2)$ with any k satisfying (37).
We can now estimate $\sup_M |\phi|$. Rewrite

$$
\Delta\phi = f \tag{38}
$$

where

$$
-m \leqslant f \leqslant C_1 \exp(\sup_M(\phi)) \exp(-C \inf_M(\phi)) \tag{39}
$$

We will now use Schauder estimates which essentially gives gradient estimates for the solution of (38) in terms of $C(M)$-norm of f and $\|\phi\|_{L^1(M)}$. So there is a constant C_6 depending only on M and $C_1 \exp(C \sup_M(\phi))$ such that

$$
\sup_M |\nabla\phi| \leqslant C_6 \left(\exp(-C \inf_M \phi) + \int_M |\phi|\right) \tag{40}
$$

Since we have already proved estimates for $\sup_M(\phi)$ and $\int_M |\phi|$, we have a constant C_7 depending only on M and C such that

$$\sup_M |\nabla\phi| \leqslant C_7(\exp(-C\inf_M(\phi))+1) \tag{41}$$

Let q be a point in M where $\phi(q) = \inf_M(\phi)$. Then in the geodesic ball, with center q and radius $-\frac{1}{2}\inf_M(\phi)C_7^{-1}(\exp(-C\inf_M(\phi))+1)^{-1}$, ϕ is not greater that $\frac{1}{2}\inf_M(\phi)$, because of the mean value theorem applied with (41).
Since we may assume that $-\inf_M(\phi)$ to be large, we can take the radius of the geodesic ball to be smaller than the injectivity radius of M. If $-\inf_M(\phi)$ was too small for this, then we have an estimate in terms of the injectivity radius anyway. Therefore the integral of $\exp(k\phi^2)$ in this ball is not less than

$$C_8 \exp(\frac{1}{4}k\inf_M(\phi)^2)\left(-\frac{1}{2}\inf_M(\phi)\right)^{2m} C_7^{-2m}(\inf_M(\phi)+1)^{-2m}$$

where C_8 is a positive constant depending only on M. Here we have used $\exp(k\phi^2) \leqslant \exp\left(\frac{k}{4}(\inf_M(\phi))^2\right)$ and comparing to the volume of the corresponding ball in T_qM. Since we have estimated $\int_M \exp(k\phi^2)$, the last quantity is estimated. This, of course gives an estimate for $|\inf_M \phi|$.
⋆
Now we give an estimate of $|\inf_M(\phi)|$ without assuming $m = 2$. Let N be any positive number. Then (20) shows that

$$\begin{aligned}\Delta'(\exp(-N\phi)(m+\Delta\phi)) \geqslant{}& \exp(-N\phi)(\Delta F - m^2\inf_{i\neq l})R_{i\bar{i}l\bar{l}} - N\exp(-N\phi)m(m+\Delta\phi)\\ &+ (N+\inf_{i\neq l}R_{i\bar{i}l\bar{l}})\exp(-N\phi)(m+\Delta\phi)\left(\sum_i \frac{1}{1+\phi_{i\bar{i}}}\right)\end{aligned} \tag{42}$$

Choose N so that

$$N + \inf_{i\neq l} R_{i\bar{i}l\bar{l}} \geqslant \frac{1}{2}N \tag{43}$$

Then by (21)

$$(N+\inf_{i\neq l}R_{i\bar{i}l\bar{l}})(m+\Delta\phi)\left(\frac{1}{1+\phi_{i\bar{i}}}\right) \geqslant \frac{N}{2}\exp\left(-\frac{F}{m-1}\right)(m+\Delta\phi)^{\frac{m}{m-1}} \tag{44}$$

There is a constant C_9 depending only on $\sup_M(F)$ and m such that

$$\frac{1}{2}N\exp\left(-\frac{F}{m-1}\right)(m+\Delta\phi)^{\frac{m}{m-1}} \geqslant 2Nm(m+\Delta\phi) - NC_9 \tag{45}$$

since the function $(any - by^{\frac{n}{n-1}})$ attains a minimum that depends only on n, a and b. Inserting (43), (44) and (45) into (42) we find

$$\Delta'(\exp(-N\phi)(m+\Delta\phi)) \geqslant \exp(-N\phi)(\Delta F - m^2\inf_{i\neq l}R_{i\bar{i}l\bar{l}} - NC_9) + N\exp(-N\phi)m(m+\Delta\phi) \tag{46}$$

Therefore

$$
\begin{aligned}
\exp(F)\Delta'(\exp(-N\phi)(m+\Delta\phi)) &\geqslant \exp(-N\phi)\exp(F)(\Delta F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - NC_9) + N\exp(-N\phi) \\
&\quad \times \exp(\inf_M F) m(m+\Delta\phi) \\
&= \exp(-N\phi)\left[\exp(F)(\Delta F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - NC_9) + m^2 N \exp(\inf_M F)\right] \\
&\quad - mN\exp(\inf_M F)\exp(-N\phi)\Delta\phi \\
&= \exp(-N\phi)\left[\exp(F)(\Delta F - m^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - NC_9) + m^2 N \exp(\inf_M F)\right] \\
&\quad + m\exp(\inf_M F)(-\Delta\exp(-N\phi) + N^2\exp(-N\phi)|\nabla\phi|^2) \\
&\geqslant -C_{10}\exp(-N\phi) + m\exp(\inf_M F)(-\Delta\exp(-N\phi) \\
&\quad + N^2\exp(-N\phi)|\nabla\phi|^2)
\end{aligned}
\tag{47}
$$

where C_{10} depends only on N, F and M. In the second to last line we used

$$\Delta\exp(-N\phi) = \exp(-N\phi)N^2|\nabla\phi|^2 + N\exp(-N\phi)\Delta\phi$$

Integrating we obtain

$$
\begin{aligned}
\int_M |\nabla\exp\left(-\frac{1}{2}N\phi\right)|^2 &= \frac{1}{4}N^2\int_M \exp(-N\phi)|\nabla\phi|^2 \\
&\leqslant \frac{1}{4}C_{10}m^{-1}\exp(-\inf_M F)\int_M \exp(-N\phi)
\end{aligned}
\tag{48}
$$

since $\int_M \Delta' = \int_M \Delta = 0$ and $\exp(F - \inf_M F) \geqslant 1$.
We claim that for each N satisfying (43), the inequalities (48) and (26) furnish an estimate of

$$\int_M \exp(-N\phi)$$

that depends only on N, F and M. We are going to prove this statement by contradiction.
Suppose that there exists a sequence $\{\phi_i\}$ satisfying (26) and (48) such that $\lim_{i\to\infty}\int_M \exp(-N\phi_i) = \infty$.
Then we define

$$\exp(-N\tilde{\phi}_i) = \exp(-N\phi_i)\left[\int_M \exp(-N\phi_i)\right]^{-1} \tag{49}$$

It follows from (48) that the sequence

$$\int_M \left|\nabla\exp\left(-\frac{1}{2}N\tilde{\phi}_i\right)\right|^2$$

is uniformly bounded from above by a constant depending only on N, F and M. Since $\int_M \exp(-N\tilde{\phi}_i) = 1$ for all i, this last fact implies that a subsequence of $\{\exp\left(-\frac{1}{2}N\tilde{\phi}\right)\}$ converges in $L^2(M)$ to some function $f \in L^2(M)$. This is a consequence of $W^{1,2} \subset\subset L^2(M)$. We assume that this subsequence is denoted using the same indices.
On the other hand, we know that, for any $\lambda > 0$

$$\mathrm{Vol}\left\{x\middle|\lambda \leqslant \exp\left(-\frac{1}{2}N\tilde{\phi}_i\right)\right\} = \mathrm{Vol}\left\{x\middle|\frac{2}{N}\log(\lambda) + \frac{1}{N}\log\left(\int_M \exp(-N\phi_i)\right) \leqslant -\phi_i\right\} \tag{50}$$

Since $\lim_{i\to\infty}\int_M \exp\left(-\frac{1}{2}N\tilde{\phi}_i\right) = \infty$ we conclude that for i large enough

$$\mathrm{Vol}\left\{x\middle|\lambda \leqslant \exp\left(-\frac{1}{2}N\tilde{\phi}_i\right)\right\} \leqslant \mathrm{Vol}\left\{x\middle|0 < \frac{2}{N}\log(\lambda) + \frac{1}{N}\log\left(\int_M \exp(-N\phi_i)\right) \leqslant |\phi_i|\right\} \tag{51}$$

By (26) $\{\int_M |\phi_i|\}$ is uniformly bounded and the inequality (51) implies that

$$\lim_{i\to\infty} \mathrm{Vol}\left\{x\middle|\lambda \leqslant \exp\left(-\frac{1}{2}N\tilde{\phi_i}\right)\right\} = 0 \tag{52}$$

for all $\lambda > 0$. Clearly

$$\begin{aligned}\mathrm{Vol}\{x|\lambda \leqslant f\} &\leqslant \mathrm{Vol}\left\{x\middle|\frac{1}{2} \leqslant \left|f - \exp\left(-\frac{1}{2}N\tilde{\phi_i}\right)\right|\right\} + \mathrm{Vol}\left\{x\middle|\frac{1}{2} \leqslant \exp\left(-\frac{1}{2}N\tilde{\phi_i}\right)\right\} \\ &\leqslant \frac{4}{\lambda^2}\int_M \left|f - \exp\left(-\frac{1}{2}N\tilde{\phi_i}\right)\right|^2 + \mathrm{Vol}\left\{x\middle|\frac{1}{2} \leqslant \exp\left(-\frac{1}{2}N\tilde{\phi_i}\right)\right\}\end{aligned} \tag{53}$$

First inequality is true because if $\lambda \leqslant f$, then $\frac{1}{2}\lambda$ can lie either between f and $\exp\left(-\frac{1}{2}N\tilde{\phi_i}\right)$ or be less than $\exp\left(-\frac{1}{2}N\tilde{\phi_i}\right)$. The last line is just the Chebyshev's inequality.
Since $\exp\left(-\frac{1}{2}N\tilde{\phi_i}\right)$ converges to f in $L^2(M)$, (52) and (53) shows that

$$\mathrm{Vol}\{x|\lambda \leqslant f\} = 0 \tag{54}$$

Since f is the L^2-limit of $\exp\left(-\frac{1}{2}N\tilde{\phi_i}\right)$, this implies that $f = 0$ almost everywhere. This is a contradiction because $\int_M f^2 = 1$.
So we get the conclusion that whenever N satisfies (43), $\int_M \exp(-N\phi)$ has an estimate above that depends only on N, F and M.
We can now repeat the previous argument to find an estimate for $|\inf_M(\phi)|$. (41) is valid for any dimension. As before, we find a geodesic ball with radius $-\frac{1}{2}\inf_M(\phi)C_7^{-1}(\exp(-C\inf_M(\phi))+1)^{-1}$ (which is not greater than the injectivity radius), such that ϕ is not greater than $\frac{1}{2}\inf_M(\phi)$ in this ball. Then we choose N so large that (43) is satisfied. Now the integral of $\exp(-N\phi)$ in the above geodesic ball is not less than

$$C_{12}\exp\left(-\frac{1}{2}N\inf_M(\phi)\right)\left(-\frac{1}{2}\inf_M(\phi)\right)^{2m}(C_7)^{-1}(\exp(-C\inf_M(\phi))+1)^{-2m}$$

since $\phi \leqslant -1$ and $\phi \leqslant \frac{1}{2}\inf_M(\phi)$ in this ball. Here C_{12} is a positive constant depending only on M. Now if we make N larger than $4mC$, this gives an estimate for $-\inf_M(\phi)$. Since we had already found an estimate for $\sup_M(\phi)$, we combine this to get an estimate for $\sup_M|\phi|$. (41) and (23) then gives estimates for $\sup_M|\nabla\phi|$ and $\sup_M(m+\Delta\phi)$. on the other hand, since $(\delta_{ij}+\phi_{i\bar{j}})$ is a positive-definite Hermitian matrix, we can find upper estimates for $1+\phi_{i\bar{i}}$ for each i. The equation $\prod_{i=1}^{m}(1+\phi_{i\bar{i}}) = \exp(F)$ then gives a positive lower estimate for $1+\phi_{i\bar{i}}$ for each i. In conclusion, we have proved the following:

Proposition 2.1. *Let M be a compact Kähler manifold with the metric $g_{i\bar{j}}dz^i \otimes d\bar{z}^j$. Let ϕ be a real-valued function in $C^4(M)$ such that $\int_M \phi = 0$ and $\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i\partial\bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1}dz^i \otimes d\bar{z}^j$ defines another metric on M. Suppose that ϕ satisfies*

$$\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i\partial\bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1} = \exp(F)$$

Then there are positive constants C_1, C_2, C_3 and C_4 depending only on $\inf_M F$, $\sup_M F$, $\inf_M \Delta F$ and M such that

$$\begin{aligned}&\sup_M|\phi| \leqslant C_1\\ &\sup_M|\nabla\phi| \leqslant C_2\\ &0 < C_3 \leqslant 1+\phi_{i\bar{i}} \leqslant C_4\end{aligned}$$

for all i.

2.2 Third order estimates

In this section we find estimates for third order derivatives, $\phi_{i\bar{j}k}$ of the solution ϕ. We assume that $F \in C^3(M)$. Consider the following function

$$S = g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}} \phi_{i\bar{j}k} \phi_{\bar{r}s\bar{t}} \tag{55}$$

We are going to compute the Laplacian of S. For convenience, we shall introduce the following convention. We say that $A \simeq B$ if $|A - B| \leqslant C_1\sqrt{S} + C_2$ where C_1 and C_2 are constants that can be estimated. We also say that $A \cong B$ if $|A - B| \leqslant C_3 S + C_4\sqrt{S} + C_5$ where C_3, C_4 and C_5 are constants that can be estimated.
As before, we diagonalize the metric and the Hessian $(\phi_{i\bar{j}})$ at a point under consideration. Then by computation $\star$ we have

$$\begin{aligned}\Delta' S \cong & \sum (1+\phi_{i\bar{i}})^{-1}(1+\phi_{j\bar{j}})^{-1}(1+\phi_{k\bar{k}})^{-1}(1+\phi_{\alpha\bar{\alpha}})^{-1} \\ & \times \left[\left| \phi_{\bar{i}j\bar{k}\alpha} - \sum_p \phi_{\bar{i}p\bar{k}} \phi_{\bar{p}j\alpha} (1+\phi_{p\bar{p}})^{-1} \right|^2 + \left| \phi_{\bar{i}jk\alpha} - \sum_p (\phi_{\bar{p}i\alpha}\phi_{p\bar{j}k} + \phi_{\bar{p}ik}\phi_{p\bar{j}\alpha})(1+\phi_{p\bar{p}})^{-1} \right|^2 \right]\end{aligned} \tag{56}$$

where the first summation is over all the indices.
On the other hand by (10)

$$\Delta'(\Delta\phi) \geqslant \sum (1+\phi_{k\bar{k}})^{-1}(1+\phi_{i\bar{i}})^{-1} |\phi_{k\bar{i}\bar{j}}|^2 - C_6 \tag{57}$$

where C_6 is a constant that can be estimated. Therefore, by letting C_7 be a large positive constant we find

$$\Delta'(S + C_7\Delta\phi) \geqslant C_8 S - C_9 \tag{58}$$

where C_7, C_8 and C_9 are positive constants that could be estimated. This is true because the first term in the RHS of (57) is greater than S and $\sqrt{S}$ can be estimated by S itself ($S + b^2 \geqslant 2b\sqrt{S}$ for any positive constant b).
Observe that at the point where $S + C_7\Delta\phi$ achieves its maximum, (58) shows that $C_8 S \leqslant C_9$ and hence

$$C_8(S + C_7\Delta\phi) \leqslant C_9 + C_8 C_7 \Delta\phi \tag{59}$$

Since we already have estimates for $\Delta\phi$, this gives an estimate for the quantity $\sup_M (S + C_7\Delta\phi)$ and hence of $\sup_M (S)$. This in turn gives estimates of $\phi_{i\bar{j}k}$ for all i, j and k. So we have proved the following proposition.

Proposition 2.2. *Let M be a compact Kähler manifold with metric $g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Let ϕ be a real-valued funtion in $C^5(M)$ such that $\int_M \phi = 0$ and $\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right) dz^i \otimes d\bar{z}^j$ defines another metric on M. Suppose that*

$$\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right) (\det(g_{i\bar{j}}))^{-1} = \exp(F)$$

Then there are estimates for the derivatives $\phi_{i\bar{j}k}$ in terms of $g_{i\bar{j}} dz^i \otimes d\bar{z}^j$, $\sup_M |F|$, $\sup_M |\nabla M|$, $\sup_M \sup_i |F_{i\bar{i}}|$ and $\sup_M \sup_{i,j,k} |F_{i\bar{j}k}|$.

Note that the last two factors in the proposition comes from the computation of $\Delta' S$.

2.3 Solution of the equation

With the estimates so far we can now solve the equation

$$\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1} = \exp(F)$$

where $F \in C^3(M)$.
If Ω is the Kähler form of M, then the above equation is equivalent to

$$(\Omega + \partial\bar{\partial}\phi)^m = (\exp(F))\Omega^m \tag{60}$$

Hence integrating (60) we immediately see that

$$\int_M \exp(F) = \mathrm{Vol}(M) \tag{61}$$

Conversely, we shall now prove that if $F \in C^k(M)$ with $k \geqslant 3$ satisfies (61), then we can find a solution ϕ of (60) with $\phi \in C^{k+1,\alpha}(M)$ for any $0 \leqslant \alpha < 1$.
For this we use the continuity method. Let

$$S = \Bigg\{ t \in [0,1] \,\Bigg|\, \det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1} = \mathrm{Vol}(M)\left[\int_M \exp(tF)\right]^{-1} \exp(tF)$$
$$\text{has a solution in } C^{k+1,\alpha}(M) \text{ and } (1 + \phi_{i\bar{i}}) > 0\ \forall i \Bigg\} \tag{62}$$

Note that $0 \in S$, since $\phi \equiv 0$ is a solution in this case. Hence we need to only show that S is both closed and open in $[0,1]$. This will imply that $1 \in S$ and that our original equation has a solution in $C^{k+1,\alpha}$.
To see that S is open, we use the inverse function theorem. Let

$$A = \left\{ \phi \in C^{k+1,\alpha}(M) \,\middle|\, (1 + \phi_{i\bar{i}}) > 0\ \forall i \text{ and } \int_M \phi = 0 \right\} \tag{63}$$

$$B = \left\{ f \in C^{k-1,\alpha}(M) \,\middle|\, \int_M f = \mathrm{Vol}(M) \right\} \tag{64}$$

Then A is open in the Banach space $C^{k+1,\alpha}(M)$ and B is an affine subspace of the Banach space $C^{k-1,\alpha}(M)$. We have a map $G : A \to B$ given by

$$G(\phi) = \det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1} \tag{65}$$

Its clear that G maps to B since all the metrics in the same cohomology class has the same volume.The differential of G at a point ϕ_0 is given by

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi_0}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1}\Delta_{\phi_0} \tag{66}$$

This can be seen as follows. Let $\phi_t = \phi_0 + t\eta$. Then

$$\begin{aligned}
\frac{d}{dt}G(\phi_t)\Big|_{t=0} &= \frac{d}{dt}\det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1} \\
&= \det\left(g_{i\bar{j}} + \frac{\partial^2 \phi_0}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1}\frac{\partial^2 \dot{\phi}_t}{\partial z^i \partial \bar{z}^j} {g'_{\phi_0}}^{i\bar{j}}\Big|_{t=0} \\
&= \Delta_{\phi_0}\eta
\end{aligned}$$

where the subscript ϕ_0 is used to denote quantities defined using the metric $\det\left(g_{i\bar{j}} + \frac{\partial^2\phi_0}{\partial z^i \partial \bar{z}^j}\right)$.
Note that B is the affine space corresponding to the vector space $\left\{ f \in C^{k-1,\alpha}(M) \middle| \int_M f = 0 \right\}$. So the tangent space of B is same as this vector space.
It is well-known (using Fredholm alternative for instance) that the condition for $\Delta_{\phi_0}\phi = g$ to have a weak solution on M is that $\int_M g dV_{\phi_0} = 0$. Hence the condition for

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\phi_0}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1}\Delta_{\phi_0} = f$$

to have a weak solution is that $\int_M f = 0$, since $\Delta_{\phi_0}\phi = (\det(g_{\phi_0}))^{-1} f \det(g_{i\bar{j}})$ is solvable if and only if $0 = \int_M (\det(g_{\phi_0}))^{-1} f \det(g_{i\bar{j}}) dV_{\phi_0} = \int_M f dV_g$.

Schauder theory makes sure that $\phi \in C^{k+1,\alpha}(M)$ when $f \in C^{k-1,\alpha}(M)$. The solution is clearly unique if we require that $\int_M \phi = 0$ (by maximum principle two solutions will always differ by a constant). Hence the differential of G at ϕ_0 is bijective and hence invertible. So G maps an open neighborhood of ϕ_0 to an open neighborhood of $G(\phi_0)$ in B. All solutions ϕ described in the set S that also satisfy $\int_M \phi = 0$ are contained in the set A, since we can write $\mathrm{Vol}(M)\left[\int_M \exp(tF)\right]^{-1} \exp(tF)$ as $f_t \in B$. Let $t_0 \in S$ and ϕ_0 be the corresponding solution with $f_0 = \mathrm{Vol}(M)\left[\int_M \exp(t_0F)\right]^{-1} \exp(t_0F)$. All functions f_t with $|t - t_0| < \epsilon$ denote an open set in B around f_0 and hence is the image of an open set in A around ϕ_0 under G. This shows that $|t - t_0| < \epsilon$ is contained in S and S is open.
It remains to prove that the set S is closed. For this, we will set up a well-known bootstrapping technique in PDE, whereby we successively improve the regularity of the solution. If $\{t_q\}$ is a sequence in S, then we have a sequence $\phi_q \in C^{k+1,\alpha}(M)$ such that

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\phi_q}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1} = \mathrm{Vol}(M)\left[\int_M \exp(t_qF)\right]^{-1} \exp(t_q qF)$$

Normalizing we can assume that $\int_M \phi_q = 0$. Differentiating the above equation we have

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\phi_q}{\partial z^i \partial \bar{z}^j}\right) g_q'^{\,i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\frac{\partial \phi_q}{\partial z^p}\right) = \mathrm{Vol}(M)\left[\int_M \exp(t_qF)\right]^{-1} \frac{\partial}{\partial z^p}\left[\exp(t_qF)(\det(g_{i\bar{j}}))\right] \tag{67}$$

where $(g_q'^{i\bar{j}})$ is the inverse matrix of $\left(g_{i\bar{j}} + \frac{\partial^2\phi_q}{\partial z^i \partial \bar{z}^j}\right)$ for each q.
Proposition 2.1 shows that $(\phi_q)_{i\bar{j}}$'s are uniformly bounded and hence the operator on the LHS of (67) is uniformly elliptic. Proposition 2.2 shows that the coefficients are Hölder continuous with exponent $0 \leqslant \alpha \leqslant 1$. Then the Schauder estimate gives an estimate on the $C^{2,\alpha}$-norm of $\frac{\partial \phi_q}{\partial z^p}$. Similarly we can estimate the $C^{2,\alpha}$-norm of $\frac{\partial \phi_q}{\partial \bar{z}^p}$. From this information, we deduce that the coefficients of the LHS of (67) are $C^{1,\alpha}$. The Schauder estimate again provides better differentiability for $\frac{\partial \phi_q}{\partial z^p}$ and $\frac{\partial \phi_q}{\partial \bar{z}^p}$. Iterating this, one finds $C^{k+1,\alpha}$-estimates of ϕ_q. Therefore, the sequence is uniformly bounded with a uniform bound on its derivatives. So it (a subsequence to be precise) converges in $C^{k+1,\alpha}$-norm to a function ϕ by the Arzela-Ascoli theorem. Now taking limit as $q \to \infty$ on the equation gives

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1} = \mathrm{Vol}(M)\left[\int_M \exp(t_0F)\right]^{-1} \exp(t_0F) \tag{68}$$

where $t_0 = \lim_{q\to\infty} t_q$. This proves that S is closed. So we have proved the following theorem.

Theorem 2.3. *Assume that M is a compact Kähler manifold with metric $g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Let $F \in C^k(M)$ with $k \geqslant 3$ and $\int_M \exp(F) = \mathrm{Vol}(M)$. Then there is a function $\phi \in C^{k+1,\alpha}(M)$ for any $0 \leqslant \alpha < 1$ such that $\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right) dz^i \otimes d\bar{z}^j$ defines a Kähler metric and*

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right) = \exp(F)\det(g_{i\bar{j}})$$

As a consequence of Theorem 2.3, we can prove the Calabi conjecture. We restate it here

Let M be a compact Kähler manifold with Kähler metric $g_{\alpha\overline{\beta}}dz^\alpha \otimes d\bar{z}^\beta$. Let $\tilde{R}_{\alpha\overline{\beta}}dz^\alpha \otimes d\bar{z}^\beta$ be a tensor whose associated $(1,1)$-form $\frac{i}{2\pi}\tilde{R}_{\alpha\overline{\beta}}dz^\alpha \wedge d\bar{z}^\beta$ represents the first Chern class of M. Then we can find a Kähler metric $\tilde{g}_{\alpha\overline{\beta}}dz^\alpha \otimes d\bar{z}^\beta$ which is cohomologous to the original metric and whose Ricci tensor is given by $\tilde{R}_{\alpha\overline{\beta}}dz^\alpha \otimes d\bar{z}^\beta$.

To see this, notice the following well-known formula for Ricci curvature

$$R_{\alpha\bar{\beta}} = -\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}\log(\det(g_{i\bar{j}})) \tag{69}$$

Since the given tensor $\frac{i}{2\pi}\tilde{R}_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$ represents the first Chern class of M, we can conclude that

$$\tilde{R}_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}} - \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}f \tag{70}$$

for some smooth real-valued function f. This is because the first Chern class of M is the real cohomology class represented by the Ricci form of M. Now using Theorem 2.3, we can find a smooth function ϕ so that $\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial z^\alpha \partial \bar{z}^\beta}\right)dz^\alpha \otimes d\bar{z}^\beta$ defines a Kähler metric with

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right)(\det(g_{i\bar{j}}))^{-1} = C\exp(F)$$

where $\int_M C\exp(f) = \mathrm{Vol}(M)$.
Now the Ricci tensor of this metric is given by

$$\begin{aligned}
R'_{\alpha\bar{\beta}} &= -\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}\log(C\exp(f)\det(g_{\alpha\bar{\beta}})) \\
&= R_{\alpha\bar{\beta}} - \frac{\partial^2 f}{\partial z^\alpha \partial \bar{z}^\beta} \\
&= \tilde{R}_{\alpha\bar{\beta}}
\end{aligned}$$

This proves the theorem.