Calabi-Futaki Invariants

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Calabi-Futaki invariants have been found as an obstruction to the existence of Kähler-Einstein metrics on a given Kähler manifold. In these notes, we define these invariants and show some of their properties. Most of the material here is originally due to Gang Tian and Akito Futaki.

1 Definition

Definition 1.1. *Given a compact Kähler manifold* M*, we define its Kähler cone, denoted by* $\mathrm{Ka}(M)$*, to be the set of all cohomology classes* $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$ *that can be represented by a Kähler metric.*

Definition 1.2. *We say that a vector field* X *is holomorphic if it has a local expression of the form*

$$X = X^i \frac{\partial}{\partial z^i}$$

such that X^i *are holomorphic functions i.e.* $\dfrac{\partial X^i}{\partial \bar{z}^j} = 0$ *for all* i, j.

Let $\eta(M)$ be the set of all holomorphic vector fields on M. Then we define the Calabi-Futaki invariant of M to be the function

$$f_M : \mathrm{Ka}(M) \times \eta(M) \to \mathbb{C}$$

given by

$$f_M([\omega], X) = \int_M X(h_g)\omega_g^n \tag{1}$$

where we picked some metric g from $[\omega]$ and defined h_g as follows. Let $s(g)$ be the scalar curvature of M. Then its easy to see that $s(g) - \dfrac{1}{V}\int_M s(g)\omega_g^n$ has integral 0. (Here $V = \int_M \omega_g^n$ is the volume of M). Then it's a well-known fact that we can construct a function h_g such that

$$s(g) - \frac{1}{V}\int_M s(g)\omega_g^n = \Delta_g h_g$$

This defines f_M. But we need to make sure that f_M defined this way does not depend on the choice of the metric g in $[\omega]$. We will show that the RHS in (1) does not depend on the metric used. This fact was originally proved by Calabi and Futaki, but we give another proof that is not as simple, but is more general.
We show that $\bar{\partial}(i_X\omega_g) = 0$.

$$\begin{aligned}
\bar{\partial}(i_X\omega_g) &= \bar{\partial}\left(\frac{i}{2}g_{i\bar{j}}X^i d\bar{z}^j\right) \\
&= \frac{i}{2}\left(\frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k}X^i + g_{i\bar{j}}\frac{\partial X^i}{\partial \bar{z}^k}\right) dz\bar{z}^k \wedge d\bar{z}^j \\
&= 0
\end{aligned}$$

where we used the fact that X is holomorphic and ω_g is closed. This is because $d\omega_g = 0$ in local co-ordinates looks like $\dfrac{\partial g_{i\bar{j}}}{\partial \bar{z}^k} = \dfrac{\partial g_{i\bar{k}}}{\partial \bar{z}^j}$ and then the anti-symmetry of the wedge product takes care of the rest.
Now it follows from the $\bar{\partial}$ version of the Hodge theorem [see these notes] that we can find a smooth function θ_X and a harmonic[1] 1-form α such that this $\bar{\partial}$-exact form can be decomposed as

$$i_X\omega_g = \alpha - \bar{\partial}\theta_X \tag{2}$$

In local co-ordinates this can be written as $\frac{i}{2}X^i g_{i\bar{j}} d\bar{z}^j = \alpha_{\bar{j}} d\bar{z}^j - \dfrac{\partial\theta_X}{\partial\bar{z}^j}d\bar{z}^j$ and therefore

$$X^i = g^{i\bar{j}}\alpha_{\bar{j}} - g^{i\bar{j}}\frac{\partial\theta_X}{\partial\bar{z}^j} \tag{3}$$

We also have $\bar{\partial}\alpha = 0$ and $\bar{\partial}^*\alpha = 0$. This follows from

$$0 = \int_M \langle\Delta\alpha, \alpha\rangle = \int_M \langle\bar{\partial}\bar{\partial}^*\alpha + \bar{\partial}^*\bar{\partial}\alpha, \alpha\rangle = \int_M ||\bar{\partial}\alpha||^2 + ||\bar{\partial}^*\alpha||^2$$

So we get that

[1] $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$ for Kähler manifolds.

$$
\begin{aligned}
f_M([\omega], X) &= \int_M X(h_g)\omega_g^n \\
&= \int_M X^i \frac{\partial h_g}{\partial z^i}\omega_g^n \\
&= \int_M \left(g^{i\bar{j}}\alpha_{\bar{j}} \frac{\partial h_g}{\partial z^i} - g^{i\bar{j}} \frac{\partial \theta_X}{\partial \bar{z}^j}\frac{\partial h_g}{\partial z^i} \right) \omega_g^n \\
&= \int_M h_g \Delta_g \theta_X \omega_g^n
\end{aligned} \tag{4}
$$

where we used (3) in the third line. In the last line we used the following two equations

$$
\begin{aligned}
\int_M g^{i\bar{j}}\alpha_{\bar{j}} \frac{\partial h_g}{\partial z_i} &= \int_M g(\partial h_g, \alpha) \\
&= \langle \partial h_g, \bar{\alpha} \rangle = \langle h_g, \partial^* \bar{\alpha} \rangle \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\int_M g^{i\bar{j}} \frac{\partial \theta_X}{\partial \bar{z}^j}\frac{\partial h_g}{\partial z^i} &= \int_M g(\partial h_g, \bar{\partial}\theta_X) \\
&= \int_M \partial g(h_g, \bar{\partial}\theta_X) - g(h_g, \partial\bar{\partial}\theta_X)
\end{aligned}
$$

The first term in the last equation is zero by the Stokes theorem. This is just integration by parts written out.

The final formula for f_M given by (4) does not contain α. So we will assume without loss of generality that $\alpha = 0$ and $i_X\omega_g = -\bar{\partial}\theta_X$. This is possible since we can always modify X so that α disappears. Although X might then depend on α, there is exactly one harmonic form α in each cohomology class (as a consequence of the Hodge theorem) and f_M calculated by using the new X would be same as the one calculated using the unmodified X, because of the independence of α in (4).
Define the auxiliary function

$$
F(g, X) = (n+1)2^{n+1} \int_M h_g \Delta_g \theta_X \omega_g^n \tag{5}
$$

We will show that this function does not depend on the choice of g in the class.

Remark 1.3. *Note that F does not change if we replace h_g by $h_g + C$ or θ_X by $\theta_X + C$.*

Lemma 1.4. *Let*

$$\Gamma = -\Delta_g \theta_X + \mathrm{Ric}(g)$$

and

$$\Lambda_j = (n-2j)(\theta_X + \omega_g)$$

where $\mathrm{Ric}(g)$ *is the Ricci form associated with the metric* g*. Then we have the following identity.*

$$F(g, X) = \sum_{j=0}^{n} (-1)^j \frac{1}{j!(n-j)!} \int_M \left[(\Lambda_j + \Gamma)^{n+1} - (\Lambda_j - \Gamma)^{n+1} \right]$$
$$- \mu 2^{n+1} \int_M (\theta_X + \omega_g)^{n+1} \tag{6}$$

where $\mu = \frac{1}{V} \int_M s(g) \omega_g^n$.

Remark 1.5. *This formula makes sense because the integral of a differential form of order less than* $2n$ *is zero by definition.*

Proof. We need the following combinatorial identity to prove this lemma.

$$\sum_{j=0}^{l} (-1)^j \binom{l}{j} (l-2j)^k = \begin{cases} 0 & \text{if } k < l \text{ or } k = l+1 \\ 2^l l! & \text{if } k + l \end{cases} \tag{7}$$

The derivation of this identity will be given at the end of the notes.
Now we expand the first term in the RHS of (6).

$$(\Lambda_j + \Gamma)^{n+1} - (\Lambda_j - \Gamma)^{n+1} = \binom{n+1}{k} \Gamma^n \Lambda_j^{n-k+1} - \binom{n+1}{k} (-1)^{(n+k-1)}$$
$$\times \Gamma^n \Lambda_j^{n-k+1}$$

The combinatorial identity above shows that all the terms in the expansion of

$$\sum_{j=0}^{n} (-1)^j \frac{1}{j!(n-j)!} \left[(\Lambda_j + \Gamma)^{n+1} - (\Lambda_j - \Gamma)^{n+1} \right]$$

except the ones that contain $(n-2j)^n$ are zero.
From these two observations, we get that this sum is equal to

$$\sum_{j=0}^{n} (-1)^j \frac{1}{j!(n-j)!} \left[\binom{n+1}{n} \Gamma^n \Lambda_j - \binom{n+1}{n} (-1) \Gamma^n \Lambda_j \right]$$
$$= \sum_{j=0}^{n} (-1)^j \frac{(n+1)}{j!(n-j)!} 2 \Gamma^n \Lambda_j \tag{8}$$

Now

$$\Gamma^n \Lambda_j = (n-2j)^n (\theta_x + \omega_g)^n (-\Delta_g \theta_X + \mathrm{Ric}(g))$$

Its easy to see that the only order $2n$ terms in the expansion of RHS are $n\omega_g^{n-1}\mathrm{Ric}(g)$ and $-\omega_g^n \Delta_g \theta_X$ each multiplied by $(n-2j)^n$. But we know that $\int_M \Delta_g \theta_X \omega_g^n = 0$. Plugging these terms into (8) gives

$$\begin{aligned}
&\sum_{j=0}^{n} (-1)^j \frac{(n+1)}{j!(n-j)!} 2(n-2j)^n n\omega_g^{n-1}\mathrm{Ric}(g) \\
&= 2^{n+1}(n+1)n\omega_g^{n-1}\mathrm{Ric}(g)
\end{aligned} \tag{9}$$

where we used the identity (7) in the last line.
The second integrand in the RHS of (6) contains $\theta_X \omega_g^n$ as the only order $2n$ term. Rest of the terms do not contribute to the integral. So finally we get that the RHS of (6) equals

$$(n+1)2^{n+1} \int_M \left(n\theta_X \mathrm{Ric}(g) \wedge \omega_g^{n-1} - \theta_X \mu \omega_g^n \right)$$

On the other hand we can write (5) as follows

$$\begin{aligned}
F(g, X) &= (n+1)2^{n+1} \int_M \theta_X \Delta_g f_g \omega_g^n \\
&= (n+1)2^{n+1} \int_M \theta_X (s_g - \mu)\omega_g^n \\
&= (n+1)2^{n+1} \int (n\theta_X \mathrm{Ric}(g) \wedge \omega_g^{n-1} - \theta_X \mu \omega_g^n)
\end{aligned}$$

where in the last line we used $s_g \omega_g^n = n\mathrm{Ric}(g) \wedge \omega_g^{n-1}$. This is true because

$$\begin{aligned}
n\mathrm{Ric}(g) \wedge \omega_g^{n-1} &= \frac{i}{2} n(n-1)! R_{i\bar{j}} G_{i\bar{j}} \left(\frac{i}{2}\right)^{n-1} dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n \\
&= \left(\frac{i}{2}\right)^n n(n-1)! R_{i\bar{j}} g^{i\bar{j}} \det(g_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n \\
&= s(g) \left(\frac{i}{2}\right)^n n! \det(g_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n \\
&= s(g)\omega_g^n
\end{aligned}$$

where $G_{i\bar{j}}$ denotes the corresponding cofactors of the matrix $(g_{i\bar{j}})$. □

Lemma 1.6. $\bar{\partial}\Delta_g \theta_X = i_X \mathrm{Ric}(g)$

Proof.

$$
\begin{aligned}
i_X \mathrm{Ric}(g) &= -X^i \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log(\det(g_{k\bar{l}})) d\bar{z}^j \\
&= -\bar{\partial}\left(X^i \frac{\partial}{\partial z^i} \log(\det(g_{k\bar{l}}))\right) && \left(X \text{ holomorphic implies } \frac{\partial X^i}{\partial \bar{z}^j} = 0\right) \\
&= -\bar{\partial}\left(X^i g^{k\bar{l}} \frac{\partial g_{k\bar{l}}}{\partial z^i}\right) \\
&= -\bar{\partial}\left(X^i g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z^k}\right) && \left(d\omega_g = 0 \text{ implies } \frac{\partial g_{k\bar{l}}}{\partial z^i} = \frac{\partial g_{i\bar{l}}}{\partial z^k}\right) \\
&= -\bar{\partial}\left(g^{k\bar{l}} \frac{\partial}{\partial z^k}(X^i g_{i\bar{l}}) - g^{k\bar{l}} g_{i\bar{l}} \frac{\partial X^i}{\partial z^k}\right) \\
&= -\bar{\partial}\left(g^{k\bar{l}} \frac{\partial}{\partial z^k}(X^i g_{i\bar{l}})\right) \\
&= \bar{\partial}\left(g^{k\bar{l}} \frac{\partial}{\partial z^k}(\frac{\partial}{\partial \bar{z}^l} \theta_X)\right) && (\text{since } \bar{\partial}\theta_X = -i_X \omega_g) \\
&= \bar{\partial}\Delta_g \theta_X
\end{aligned}
$$

where we have used

$$
\bar{\partial}\left(g^{k\bar{l}} g_{i\bar{l}} \frac{\partial X^i}{\partial Z^k}\right) = \bar{\partial} \frac{\partial X^i}{\partial z^i} = 0
$$

□

Now we can finally show that f_M is well defined.

Theorem 1.7. *The Calabi-Futaki invariant*

$$
f_M([\omega], X) = \int_M X(h_g)\omega_g^n
$$

is independent of the choice of ω_g *in* $[\omega]$.

Proof. Since the space of Kähler metrics is path connected, its enough to show that

$$
\frac{\partial F}{\partial t}(g_t, X)\big|_{t=0} = 0
$$

for any family of metrics $\{g_t\}$ in a particular Kähler class. Now

$$
\begin{aligned}
\bar{\partial}\theta_{X,t} &= -i_X \omega_{g_t} \\
&= -i_X(\omega_g + \partial\bar{\partial}\phi_t) \\
&= \bar{\partial}\theta_{X,0} - \bar{\partial}X(\phi_t)
\end{aligned}
$$

The last line is true because

$$\bar\partial X(\phi_t) = \bar\partial\left(X^i \frac{\partial\phi_t}{\partial z^i}\right) = X^i \frac{\partial^2\phi_t}{\partial z^i\partial\bar z^j} d\bar z^j + \frac{\partial X^i}{\partial\bar z^j}\frac{\partial\phi_t}{\partial z^i} d\bar z^j$$
$$= X^i \frac{\partial^2\phi_t}{\partial z^i\partial\bar z^j} d\bar z^j = i_X(\partial\bar\partial\phi_t)$$

Remember that there are no non-constant holomorphic functions on a compact manifold. Therefore we can write

$$\theta_{X,t} = \theta_{X,0} - X(\phi_t)$$

where we can omit the constant because of Remark 1.3. Now for the ease of computation we introduce the following notations.

$$\psi_{X,t} = -\Delta_{g_t}\theta_{X,t} + (n-2j)\theta_{X,t}$$
$$\tilde\psi_{X,t} = \Delta_{g_t}\theta_{X,t} + (n-2j)\theta_{X,t}$$
$$R(g_t) = \mathrm{Ric}(g_t) + (n-2j)\omega_{g_t}$$

We use this notation and Lemma 1.4 to write

$$F(g_t, X) = \sum_{j=0}^{n} (-1)^j \frac{1}{j!(n-j)!}\int_M (\psi_{X,t} + R(g_t))^{n+1}$$
$$- \sum_{j=0}^{n} (-1)^j \frac{1}{j!(n-j)!}\int_M (\tilde\psi_{X,t} - \mathrm{Ric}(g_t) + (n-2j)\omega_{g_t})^{n+1}$$
$$- \mu 2^{n+1}\int_M (\theta_{X,t} + \omega_{g_t})^{n+1}$$

Now we compute the derivative of the first integral with respect to t and show that its zero. The other two computations are pretty much the same (a slightly different notation would make it look exactly the same as this computation).

$$\frac{\partial F}{\partial t}(g_t, X) = \sum_{j=0}^{n} (-1)^j \frac{n+1}{j!(n-j)!}\int_M (\dot\psi_{X,t} + \dot R(g_t))(\psi_{X,t} + R(g_t))^n + \dots$$

Now using Lemma 1.6 and $-i_x\omega_g = \bar\partial\theta_{X,t}$, we get that

$$\bar\partial\psi_{x,t} = -i_X R(g_t) \tag{10}$$

We know that both $\mathrm{Ric}(g_t)$ and $\dot\omega_{g_t} = -\bar\partial\partial\dot\phi_t$ are $\bar\partial$-exact. So we can define a 1-form α_t by $\bar\partial\alpha_t = \dot R(g_t)$. Then we have

$$\bar\partial\dot\psi_{X,t} = -i_X\dot R(g_t) = -i_X\bar\partial\alpha_t = -\bar\partial(i_X\alpha_t) \tag{11}$$

In the last line we used

$$\begin{aligned}\bar\partial(i_X\alpha_t) &= \left(\frac{\partial X^i}{\partial\bar z^j}(\alpha_t)_i + X^i\frac{\partial(\alpha_t)_i}{\partial\bar z^j}\right)d\bar z^j\\ &= X^i\frac{\partial(\alpha_t)_i}{\partial\bar z^j}d\bar z^j\\ &= i_X\bar\partial\alpha_t\end{aligned}$$

So again $\dot\psi_{X,t} = -i_X\alpha_t$, again because of the Remark 1.3, adding a constant times t to $\theta_{X,t}$ will not affect $F(g_t, X)$.
Now we have

$$\begin{aligned}&\int_M(\dot\psi_{X,t}+\dot R(g_t))(\psi_{X,t}+R(g_t))^n = \int_M(-i_X\alpha_t+\bar\partial\alpha)(\psi_{X,t}+R(g_t))^n\\ &= \int_M(-i_X\alpha_t)(\psi_{X,t}+R(g_t))^n + (\bar\partial\alpha_t)(\psi_{X,t}+R(g_t))^n\\ &= \int_M(-i_X\alpha_t)(\psi_{X,t}+R(g_t))^n - n\alpha_t(\psi_{X,t}+R(g_t))^{n-1}(\bar\partial\psi_{X,t}+\bar\partial R(g_t))\\ &= \int_M(-i_X\alpha_t)(\psi_{X,t}+R(g_t))^n - n\alpha_t(\psi_{X,t}+R(g_t))^{n-1}(i_X(\psi_{X,t}+R(g_t)))\\ &= -\int_M i_X(\alpha_t(\psi_{X,t}+R(g_t))^n)\end{aligned}\tag{12}$$

In the third line we used integration by parts with the $\bar\partial$ operator. In the fourth line we used that $\bar\partial R_{g_t} = 0$ (since ω_g and $\mathrm{Ric}(g)$ are closed) and that

$$\bar\partial\psi_{X,t} = -i_X R(g_t) = -i_X(\psi_{X,t}+R(g_t))$$

where $i_X\psi_{X,t} = 0$ since i_X always lowers the order. In the last line we used the following Leibniz rule for i_X.

$$i_X(\alpha\wedge\beta) = i_X(\alpha)\wedge\beta - \alpha\wedge i_X(\beta)$$

for one and two forms α and β respectively.
Denote $\eta = \alpha_t(\psi_{X,t}+R(g_t))^n$. Now it follows easily that the integral above is zero, since there are no forms of order $2n$ in $i_X\eta$. For, if there was a form of order $2n$ in the expansion of $i_X\eta$, then that would imply that there are non-zero forms of order $2n+1$ in η.

□

Now we deduce some important corollaries.

Corollary 1.8.

1. f_M *is a holomorphic invariant of M i.e. remains unchanged under any biholomorphism form* M *to itself.*

2. *There is a Kähler metric in* $[\omega]$ *with constant scalar curvature only if* $f_M([\omega], \_) = 0$.

Proof.

1. Follows from the fact that holomorphic automorphisms preserve the Kähler class of a metric and also the set of all holomorphic functions on M.
2. Constant scalar curvature implies that $h_g =$ constant. Hence $f_M([\omega], \_) = 0$.

$\square$

In particular, Calabi-Futaki invariants vanish for Kähler-Einstein metrics. Now we see another application of these invariants to *extremal Kähler metrics*.

Definition 1.9. *A Kähler metric* g *is called extremal if the scalar curvature* $s(g)$ *satisfies*

$$s(g)_{\bar{i}\bar{j}} = 0$$

Extremal metrics naturally arise as the critical points of the Calabi functional, which is defined by

$$Ca(M) = \int_M s(g)^2 \omega_g^n$$

Clearly, all metrics of constant scalar curvature are extremal. But does the converse also hold under certain conditions? The following corollary answers this question.

Corollary 1.10. *If* $f_M([\omega], \_) = 0$*, then any extremal Kähler metric* g *in* $[\omega]$ *has constant scalar curvature.*

Proof. If g is extremal, then $X = g^{i\bar{j}} s(g)_{\bar{j}} \frac{\partial}{\partial z^i}$ is a holomorphic vector field. So we get

$$\begin{aligned} f_M([\omega], X) &= \int_M g^{i\bar{j}} s(g)_{\bar{j}} \frac{\partial h_g}{\partial z^i} \omega_g^n \\ &= -\int_M g^{i\bar{j}} s(g) \frac{\partial^2 h_g}{\partial z^i \partial \bar{z}^j} \omega_g^n \\ &= -\int_M s(g) \Delta_g h_g \omega_g^n \\ &= -\int (s(g) - \mu) \Delta_g h_g \omega_g^n \\ &= -\int (s(g) - \mu)^2 \omega_g^n \end{aligned}$$

Hence $s(g) = \mu$.

$\square$