Candidacy Report

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Contents

1 Introduction

At the intersection of complex geometry and PDE we find a set of important and interesting problems. These are questions in complex geometry that can be transformed into partial differential equations in complex variables. A famous problem of this type would be the Calabi conjecture, which was solved by S.T. Yau in 1978. There are other questions of similar flavour, such as finding Kähler-Einstein metrics on Kähler manifolds. These problems share a feature that they can be transformed into a PDE called the complex Monge-Ampere equation. This is because of a special form that the Ricci curvature takes in the complex setting.

We could also ask some variants of these questions. For example, we can look for metrics that are not necessarily Kähler, but satisfies some weaker assumptions and solves the Calabi-Yau equation. An example would be the balanced metric, which is defined by $d\omega^{n-1} = 0$ instead of the condition $d\omega = 0$ in the Kähler case. In this article we pose some of the same questions for balanced metrics.

A recent breakthrough in this area is solving the Calabi-Yau equation for another non-Kähler metric, called the Gauduchon metric [See Sec 5]. We draw inspiration from some of the ideas in this paper.

2 Hermitian metrics

Let (X, g) be a compact $2n$ - dimensional Riemannian manifold and let J be a complex structure on X. We say that g is a hermitian metric on (X, J) if in addition, it satisfies $g(JX, JY) = g(X, Y)$ for all $X, Y \in TM$. The $(1, 1)$ form $\omega(X, Y) := g(JX, Y)$ is the hermitian form associated to g. We will use the word metric for both ω and g. The most extensively studied class of hermitian manifolds are the Kähler manifolds which are those with closed hermitian forms ($d\omega = 0$). An important problem in Kähler geometry is showing the existence of Kähler-Einstein metrics (those having Ricci curvature proportional to the metric itself) on a given Kähler manifold. This will be discussed in detail in section 3, where we will also find that there are cases that do not admit Kähler-Einstein metrics. Here is a list some other natural metrics on hermitian manifolds.

1. **Balanced metric**: $d\omega^{n-1} = 0$.

2. **Gauduchon metric**: $\partial\bar{\partial}\omega^{n-1} = 0$.

3. **Strongly Gauduchon metric**: $\bar{\partial}\omega^{n-1}$ is ∂-exact.

4. **Astheno-Kähler metric**: $\partial\bar{\partial}\omega^{n-2} = 0$.

5. **Pluriclosed metric or strong KT structure**: $\partial\bar{\partial}\omega = 0$.

6. **k-th Gauduchon metric**: $\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0$.

7. **(l|k)-strong condition**: $\partial\bar{\partial}\omega^k \wedge \omega^l = 0$ for $1 \leq k + l \leq n - 1$.

Remarks:

1. It can be shown that $d\omega^k = 0$ for $2 \leq k \leq n-2$ automatically yields $d\omega = 0$ [2].

2. It is easily verified that if $\partial\bar{\partial}\omega^k = 0$ for $k = 1$ and $k = 2$, then $\partial\bar{\partial}\omega^k = 0$ for any $1 \leq k \leq n - 1$.

There are specific reasons for considering each of these metrics. For example, one can use a Gauduchon metric to define the degree and then make sense of the stability of holomorphic vector bundles over compact non-Kähler manifolds. For more applications of the different cases above see [2]. It is natural to try to extend the major theorems in Kähler geometry to the non-Kähler settings described above. In these notes, we will only consider balanced and Gauduchon metrics. The Calabi-Yau theorem, which shows the existence of a Kähler metric with prescribed Ricci curvature, has been proven for the Gauduchon case in [12]. We will come back to this in section 5. Section 6 explores balanced metrics in detail. Especially we will look at a form of Calabi-Yau theorem for balanced metrics which is one of the main focus of our study.

Another interesting direction of study would be to understand the space of Kähler metrics on a given Kähler manifold. In section 7, we will see that the space of all Kähler potentials in a given Kähler class forms an infinite dimensional Riemannian manifold. A surprising fact about the geometry of these spaces is that the geodesic equation for this metric becomes a homogeneous complex Monge-Ampere equation (HCMA) of one dimension higher than the base manifold. This yields some important conclusions about the geometry of the space of Kähler potentials (such as geodesic connectedness) based on the solvability of the corresponding HCMA. This calculation will be shown in section 7. We will also consider some interesting research directions in this topic.

In the last section, we look at a general fully non-linear PDE on hermitian manifolds and discuss how this equation can be used to study various geometric problems, such as the conformal deformations of Chern-Ricci forms.

3 The Calabi Conjecture

Define the first Chern class to be $c_1(M) := [Ric(\omega_g)]$. It can be shown that $c_1(M)$ is independent of the metric g. In the 1950's Calabi conjectured the following:

Theorem 1 (Calabi-Yau)**.** *Given a compact Kähler manifold (M, ω) and a real $(1,1)$-form Ψ in $c_1(M)$, there is another Kähler metric ω', uniquely determined in the Kähler class of ω such that $Ric(\omega') = \Psi$.*

Here $Ric(\omega')$ stands for the Ricci curvature of ω'. This problem was famously solved by S.T. Yau in 1978. Without going into all the details, we present the main ingredients of the proof. This will also serve as a general guideline on how such problems are dealt with.

Step 1: Transform the curvature condition into the complex Monge-Ampere equation as follows. We look for solutions in the Kähler class of ω. So let $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0$ where ϕ is an unknown function. Assuming that this gives the required metric we can write

$$\begin{aligned} Ric(\omega') &= \Psi \\ &= Ric(\omega) - \sqrt{-1}\partial\bar{\partial}F \end{aligned}$$

for some smooth function F.

$$-\sqrt{-1}\partial\bar{\partial}\log(\omega'^n) = -\sqrt{-1}\partial\bar{\partial}\log(\omega^n) - \sqrt{-1}\partial\bar{\partial}F$$

Applying maximum principle we get $\omega'^n = e^F\omega^n$. There should be an additional constant factor which we get rid of by normalizing F. In local co-ordinates this equation becomes

$$\det(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\phi) = e^F \det(g_{i\bar{j}})$$

for $\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge dz_{\bar{j}}$. We immediately note that a necessary condition for this PDE is that $\int_M e^F \omega^n = \int_M \omega^n$. This condition is in fact sufficient to solve it.

Step 2: Solving the equation using the continuity method.

Consider a sequence of equations given by

$$\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \phi_t) = \frac{\int_M \omega^n}{\int_M e^{tF} \omega^n} e^{tF} \det(g_{i\bar{j}})$$
$$\omega_t' = \omega + \sqrt{-1} \partial \bar{\partial} \phi_t > 0$$
$$\int_M e^{tF} \omega^n = \int_M \omega^n$$

for $t \in [0,1]$. At $t = 0$, $\phi_0 \equiv 0$ is a solution. We show that the set $K = \{t \in [0,1] : \phi_t \in C^3(M)$ solves the above equation at $t\}$ is both open and closed. Hence $t = 1$ is solvable (This corresponds to the original equation).

For openness, apply the inverse function theorem to the operator $G(\phi) = \log \dfrac{(\omega + \sqrt{-1} \partial \bar{\partial} \phi)^n}{\omega^n}$. This can be done by showing that the Fréchet derivative of G is invertible at ϕ_t for any $t \in K$.

For closed, we use the above estimates. Let t_k be a sequence in K such that $t_k \to t_0 \in [0,1]$. Then since the equation is solvable for each t_k, we have a uniform $C^{2\alpha}$ estimate $||\phi_{t_k}||_{2,\alpha} \leq C$ (this will be shown in the upcoming steps). Now use Arzela's theorem to extract a uniformly convergent subsequence that converges to a function ϕ in $C^{2,\alpha}(M)$. ϕ satisfies the equation at t_0 since the convergence is in $C^{2,\alpha}(M)$. Thus K is closed.

Step 3: Establishing apriori C^0 estimates for the solution.

There are three known ways of doing this [9]. The classical method would be to use the Moser iteration argument. We outline the rough idea here. Multiply the PDE by a power of the solution $\phi^{\alpha+1}$ and integrate by parts. This would yield an L^p estimate for the gradient of the solution for all $p \geq 2$. Now plugging in this estimate into the Sobolev inequality gives the following.

$$||\phi||_{L^{p\beta}} \leq (Cp)^{\frac{1}{p}} ||\phi||_{L^p}$$

for all $p \geq 2$ and some $\beta > 1$. Iterating this inequality for larger and larger p values gives an L^∞ estimate for the solution.

Step 4: Establishing apriori C^1 estimates for the solution. This will follow from the C^0 and C^2 estimates by general interpolation inequalities (see chapter 6 of [5]). Yau originally derived C^2 estimates first that only depends on the C^0 estimates. This was then used to get the gradient estimates. It is now known that one can derive C^1 estimates (depending on the C^0 norm) directly [9].

Step 5: Establishing apriori C^2 estimates for the solution.

Let $\mathrm{tr}_g g' = g^{j\bar{k}} g'_{j\bar{k}}$, and $\mathrm{tr}_{g'} g = g'^{j\bar{k}} g_{j\bar{k}}$ for $g_{i\bar{j}}$, $g'_{i\bar{j}}$ denoting the components of ω and ω' respectively. We can write $g'_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}}$. By direct calculations one can derive the following elliptic inequality.

$$\Delta'(\log \mathrm{tr}_g g' - A\phi) \geq \mathrm{tr}_{g'} g - An$$

Here $\Delta' = g'^{i\bar{j}} \partial_i \partial_{\bar{j}}$ is the Laplacian associated to the metric ω'. Now if $\log \mathrm{tr}_g g' - A\phi$ attains maximum at a point p, then the LHS of the inequality above is non-positive at p. This provides the estimate $\mathrm{tr}_{g'} g(p) \leq An$. Combining this with the original PDE gives a uniform bound

$$\sup_M \log(\mathrm{tr}_g g') \leq C$$

where C depends on $\sup |\phi|$ (which has been estimated). This combined with the original PDE gives upper and lower bounds on all eigenvalues of g'. This is enough since $\mathrm{tr}_g g' = n + \Delta\phi$ in normal coordinates and the latter estimates all the terms in $\sqrt{-1}\partial\bar{\partial}\phi$.

Step 6: Establishing apriori C^3 estimates for the solution.

For the third order estimates we work with the tensor $S^i_{jk} = \Gamma^i_{jk} - \widehat{\Gamma}^i_{jk}$, where Γ^k_{ij} and $\widehat{\Gamma}^k_{ij}$ denote the Christoffel symbols of ω' and ω respectively. As in the case of second order estimates, we derive an elliptic inequality for the tensor S, given by

$$\Delta(|S|^2 + A\mathrm{tr}_g g') \geq |S|^2 - C_2$$

for some C_2. Then by the same method as in step 2 we get an estimate $|S|^2 \leq C_3$, that depends on $\sup |\mathrm{tr}_g g'|$ (which has been estimated). Since the Christoffel symbol Γ^k_{ij} is equal to $g'^{k\bar{l}}\partial_i g'_{j\bar{l}}$, C^3 estimates are obtained from here.

Historically, the third order estimates were obtained by Yau using the 'Calabi computations'. Now we can also derive this by making use of the Evans-Krylov theorem which was proved in the early 1980s. This theorem gives $C^{2,\alpha}$ estimates for fully non-linear uniformly elliptic PDE's of the form $F(x, u, Du, D^2u) = f$, assuming that F is concave and that we have second order estimates.

So far we have shown the existence of solutions with $C^{2,\alpha}$ regularity. For higher regularity, differentiate the PDE and use bootstrapping method. This is done with the standard Schauder estimates for linear elliptic PDEs.

This technique is called the continuity method. We have skipped many details that are routine in this method and could be found in the references. Yau's original paper [15] does not use this language of tensors and relative endomorphisms. But this approach greatly reduces the calculations required and makes the idea clearer. For more details, see this almost comprehensive survey on complex Monge-Ampere equations [9] and also [11].

4 Kähler-Einstein manifolds

A Kähler manifold is said to be Kähler-Einstein if the Ricci curvature tensor is equal to a constant times the metric.

$$Ric(\omega_g) = \lambda g$$

The problem of finding a Kähler Einstein metric on a given Kähler manifold (M, ω) is very similar to the Calabi-Yau theorem. In fact, the cases when $c_1(M) < 0$ and $c_1(M) = 0$ are completely solved by the same technique. It is worth pointing out that in the case when $c_1(M) < 0$, the C^0 apriori estimate is obtained much more easily. This is because the corresponding complex Monge-Ampere equation has the following form.

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{F+\phi}\omega^n$$

Assume that ϕ attains maximum at a point p. Then since the Hessian matrix of ϕ

is negative definite at that point, it follows that $(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n \leq \omega^n$. Applying this to the above equation gives an upper bound on ϕ. In a similar way we can estimate $\inf(\phi)$ as well. This does not work when $c_1(M) > 0$ since we have $e^{F-\phi}$ in the RHS of that equation.

So the only case left is when $c_1(M) > 0$. As mentioned above the C^0 estimate is not straightforward now because of the $e^{-\phi}$ term, although the higher order estimates can be obtained by same techniques. It turns out that it is not true in general that every Kähler manifold with positive Chern class admits a Kähler-Einstein metric. Some explicit counter examples are known [13].

The next question was to identify and categorize all the obstructions to solving this equation. Here we present one of these. Define

$$\mathfrak{h} := \{\text{holomorphic sections } v \text{ of } T^{1,0}M, \text{ such that } v^j = g^{j\bar{k}}\partial_{\bar{k}} f \text{ for some } f :\ M \to \mathbb{C}\}$$

The function f is called the holomorphy potential for v. These are unique up to addition of constants. A short argument shows that $\mathfrak{h}$ only depends on the Kähler class $[\omega]$. In fact, it has been shown that $\mathfrak{h}$ does not even depend on the choice of Kähler class [7].

Theorem 2. *Let (M, ω) be a compact Kähler manifold. Let us define the functional $F : \mathfrak{h} \to \mathbb{C}$, called the Futaki invariant, by*

$$F(v) = \int_M f(S - \hat{S})\omega^n$$

where f is a holomorphy potential for v, and S, $\hat{S}$ are scalar curvature and its average over the manifold respectively. Then F is independent of the choice of metric in the Kähler class $[\omega]$. In particular, if $[\omega]$ admits a constant scalar curvature metric (cscK), then $F(v) = 0$.

Consequently, since Kähler-Einstein metrics are cscK as well, vanishing of the Futaki invariant is necessary for their existence. Futaki invariants are difficult to compute directly. But some localization formulae have been derived by analyzing the zero set of v. Examples of non-vanishing F are blow up of $\mathbb{C}P^2$ in one or two points. See [13] for the formula and details of this calculation. Futaki invariants are important in the study of cscK metrics in general and relate to other algebraic obstructions such as K-stability [11].

5 Gauduchon metrics

Gauduchon metrics are those hermitian metrics for which $\partial\bar{\partial}\omega^{n-1} = 0$. It was shown by Gauduchon that such a metric always exist in the conformal class of any hermitian metric and is unique up to constants [4]. In addition for Gauduchon metrics, it still holds that

$$\int \Delta u \omega^n = 0.$$

In fact,

$$\begin{aligned}
\int \Delta u \omega^n &= \int \sqrt{-1}\partial\bar{\partial}u \wedge \omega^{n-1} \\
&= \int \sqrt{-1}\partial(\bar{\partial}u \wedge \omega^{n-1}) + \int \sqrt{-1}\bar{\partial}u \wedge \partial\omega^{n-1} \\
&= \int \sqrt{-1}\bar{\partial}(u\partial\omega^{n-1}) - \int \sqrt{-1}u\bar{\partial}\partial\omega^{n-1} \\
&= -\int \sqrt{-1}u\bar{\partial}\partial\omega^{n-1} = 0
\end{aligned}$$

provided that $\bar{\partial}\partial\omega^{n-1} = 0$

Motivated by Yau's theorem, in 1984 Gauduchon posed the following question. Given a compact Kähler manifold M and a real $(1,1)$ form Ψ in the first Bott-Chern class $c_1^{BC}[M]$, does there exist a Gauduchon metric ω on M such that

$$Ric(\omega) = \Psi$$

This conjecture was solved fully only recently by Szekelyhidi, Tosatti and Weinkove [12]. The exact theorem is stated as follows.

Theorem 3. *Let M be a compact complex manifold with a Gauduchon metric α_0, and Ψ a closed real $(1,1)$ form on M with $[\Psi] = c_1^{BC}(M) \in H^{1,1}_{BC}(M,\mathbb{R})$. Then there exists a Gauduchon metric ω satisfying $[\omega^{n-1}] = [\alpha_0^{n-1}]$ in $H_A^{n-1,n-1}(M,\mathbb{R})$ and*

$$Ric(\omega) = \Psi$$

where $H_A^{n-1,n-1}(M,\mathbb{R}) = \dfrac{\{\partial\bar{\partial}\text{-closed real } (n-1,n-1) \text{ forms}\}}{\{\partial\gamma + \overline{\partial\gamma} \mid \gamma \in \Lambda^{n-2,n-1}(M)\}}$ is the *Aeppli* cohomology group and $H^{1,1}_{BC}(M,\mathbb{R}) = \dfrac{\{d\text{-closed real } (1,1) \text{ forms}\}}{\{\sqrt{-1}\partial\bar{\partial}\psi \mid \psi \in C^\infty(M,\mathbb{R})\}}$ is the *Bott-Chern* cohomology group.

Some elements of this proof are as follows. Let α be a background Gauduchon metric. Then by choosing $\omega^{n-1} = \alpha_0^{n-1} + \partial\gamma + \overline{\partial\gamma}$ where $\gamma = \frac{\sqrt{-1}}{2}\bar{\partial}u \wedge \alpha^{n-2}$, we can transform the above theorem to answering the following question. Given a smooth function F, does there exist a function u and a constant b such that there is a hermitian metric ω defined by

$$\omega^{n-1} = \alpha_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \alpha^{n-2} + \mathrm{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}\alpha^{n-2}) > 0$$

that satisfies $\omega^n = e^{F+b}\alpha^n$?

By a previous work of two of the authors [14], this question can be reduced to showing the following apriori C^2 estimate for the solution u.

$$\sup_M |\sqrt{-1}\partial\bar{\partial}u|_\alpha \leq C(1 + \sup_M |\nabla u|_\alpha^2)$$

This is precisely the estimate proved in the paper. We find this paper interesting because the equation in this case is very similar to the one that we will encounter in the next section. There is a possibility that the C^2 estimates for that equation can be obtained by modifying the proof from above.

6 Balanced metrics

Balanced metrics were first studied extensively by M. L. Michelsohn in the 80's [8]. Here balanced metrics are defined as having $\tau_\omega = 0$, where τ_ω denotes the trace of the torsion endomorphism of the metric ω. This is weaker than the Kähler case where torsion is identically zero. For $n = 2$, the Kähler and balanced condition are the same. For all dimensions $n \geq 3$, there are compact balanced manifolds which carry no Kähler metrics. See Michelsohn's paper for details of this construction. Not every complex manifold X admits a balanced metric. This is because the balanced condition ($d\omega^{n-1} = 0$) would imply that the homology group $H_{2n-2}(M, \mathbb{R})$ is non-trivial. Complex manifolds which violate this condition, say for example the Calabi-Eckmann manifolds $S^{2p+1} \times S^{2q+1}$, $p + q > 0$, are not balanced.

Apart from being a natural metric to consider, balanced metrics have become significant because of their applications in mathematical physics. For example, in superstring theory the internal space X^3 is a complex three-dimensional manifold with a non-vanishing holomorphic three-form Ω. The supersymmetry condition requires

$$d(||\Omega||_\omega \omega^2) = 0$$

for some hermitian metric ω. The above equation in mathematics says that the metric ω is conformally balanced. Thus it is interesting to look for balanced metrics ω such that $||\Omega||_\omega = C$ for some constant C. This problem has been solved on complex torus [3]. In the case of a general Kähler manifold, the openness part of the continuity method is also shown in the same paper.

We wish to consider other interesting questions for balanced metrics such as the Calabi-Yau theorem. Given a compact complex manifold M with a hermitian metric α_0 one could ask the following. Does there exist a hermitian form ω defined by

$$\omega^{n-1} = \alpha_0^{n-1} + \sqrt{-1}\partial\bar{\partial}(\phi\alpha^{n-2}) \tag{1}$$

such that $\omega^n = e^{F+b}\alpha^n$? Here α is a fixed background hermitian metric and ϕ is the unknown function. F and b are a known function and a constant respectively. Again this equation is equivalent to prescribing the Ricci curvature of the metric ω. By choosing α_0 to be balanced (i.e. $d\alpha_0^{n-1} = 0$), we get that ω is balanced as well. If the metric α is chosen to be astheno-Kähler (i.e. $\partial\bar{\partial}\alpha^{n-2} = 0$), then this problem is fully solved. This is because it reduces to the same equation (upto a constant) as in finding Gauduchon metrics with a prescribed volume form [12].

One of the main aims in our study is to consider the same problem after only assuming that α is balanced. The main difference in this case is an extra $\sqrt{-1}\phi\partial\bar{\partial}\alpha^{n-2}$ term in the expansion of (1). This will be difficult to solve since the resulting PDE in this case resembles the equation for finding Kähler-Einstein metrics when the Chern class is positive. But there are some additional good terms here that might make it solvable. As mentioned before, we would like to approach this problem using some of the ideas discussed in [12].

7 The space of balanced metrics and geodesic equations

Let (X, ω_0) be a compact Kähler manifold without boundary of dimension n. Then define the space of Kähler potentials.

$$\mathcal{K} = \{\phi \in C^\infty(X) : \omega_\phi > 0\}$$

where $\omega_\phi := \omega_0 + \frac{i}{2}\partial\bar{\partial}\phi$. Then $\mathcal{K}$ is an infinite dimensional Riemannian manifold with tangent space at ϕ given by $T_\phi\mathcal{K} \cong \{\delta\phi \in C^\infty(X)\}$. The Riemannian metric is given by

$$||\delta\phi||^2 = \int_X |\delta\phi|^2 \omega_\phi^n$$

Our aim is to find the geodesic equation for $(\mathcal{K}, ||.||)$ and show that this is a complex Monge-Ampere equation. For this purpose, we find the variation of the following energy functional.

$$\mathcal{E} = \int_0^T \int_X \dot{\phi_t}^2\ \omega_\phi^n dt$$

for a path $\phi_t : [0, T) \to \mathcal{K}$. Let $\phi(t, s)$ be an end-point fixing variation of paths such that $\phi := \phi(t, 0)$ and $\psi = \frac{\partial\phi}{\partial s}\Big|_{s=0}$.

Differentiating $\mathcal{E}$ wrt s-variable gives,

$$\delta\mathcal{E} = \int_0^T \int_X \left(2\dot{\phi}\dot{\psi} + \dot{\phi}^2\Delta_\phi\psi\right) \omega_\phi^n dt$$

Integrating by parts using $\psi(0) = \psi(T) = 0$ we get,

$$\delta\mathcal{E} = -2\int_0^T \int_X \psi \left(\ddot{\phi} - |\nabla\dot{\phi}|^2_{\omega_\phi}\right) \omega_\phi^n dt$$

Thus we deduce the following geodesic equation.

$$\ddot{\phi} - |\nabla\dot{\phi}|^2_{\omega_\phi} = 0$$

An interesting observation due to Donaldson [1] and Semmes [10] is that this geodesic equation is equivalent to a homogeneous complex Monge-Ampere equation (HCMA) of one dimension higher.

$$(\pi^*\omega_0 + \frac{i}{2}\partial\bar{\partial}\Phi)^{n+1} = 0$$

for $\Phi(z, w) = \phi(z, \log|w|)$. This is defined on the manifold $M = X \times A$ with $A = \{w \in \mathbb{C} : e^{-T} < |w| < 1\}$. Here $\partial\bar{\partial}\Phi$ is taken wrt all the $(n + 1)$ variables (z, w), and $\pi^*\omega_0$ is the pull-back of ω_0 to M under the natural projection map $\pi : X \times A \to X$.

Thus solutions of this HCMA would correspond to geodesics in $\mathcal{K}$ and vice-versa. Consequently, to find a geodesic segment connecting ϕ_0 to ϕ_1 in $\mathcal{K}$, it is enough to solve HCMA with the following boundary data.

$$\begin{cases} \Phi_b(z,w) = \phi_0 & , |w| = 1 \\ \Phi_b(z,w) = \phi_1 & , |w| = e^{-T} \end{cases}$$

This can be solved with $C^{1,\alpha}(\bar{M})$ solutions for any $0 < \alpha < 1$ as a consequence of the following theorem [9].

Theorem 4. *Let (X, ω) be a smooth Kähler manifold with smooth boundary. Then the Dirichlet problem*

$$\begin{cases} (\omega_0 + \frac{i}{2}\partial\bar{\partial}\phi)^n = 0 & , \text{ on } M \\ \phi \equiv 0 & , \text{ on } \partial M \end{cases}$$

admits a unique solution, which is of class $C^{1,\alpha}(\bar{M})$ for each $0 < \alpha < 1$.

As a consequence we have the following.

Theorem 5. *Let ϕ_0, ϕ_1 be two points in $\mathcal{K}$. Then there exists a unique geodesic of class $C^{1,\alpha}$, for any $0 < \alpha < 1$, joining ϕ_0 and ϕ_1.*

Remark: It has been shown that the regularity can be improved up to $C^{1,1}$.

Question: What is the geodesic equation for the space of balanced metrics?

There are two possible approaches one could take here. Assume ω_0 is balanced.

1. Define $\mathcal{B}_1 = \{\Phi \in \Lambda^{(n-2,n-2)}M : \omega_\Phi^{n-1} = \omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}\Phi > 0\}$ and perform the same computations as above. These computations become very complicated and long because we are dealing with $(n-2, n-2)$ forms rather than functions.

2. Define $\mathcal{B}_2 = \{\phi \in C^\infty(M) : \omega_\phi^{n-1} = \omega_0^{n-1} + \sqrt{-1}\partial\bar{\partial}\phi\alpha^{n-2} > 0\}$. This is simpler compared to case 1, but not as general.

It would be interesting to see if we can get some important equation in this case. Relatively less is known about this theory. So more interesting questions could be raised.

8 Second order non-linear PDE on hermitian manifolds

The treatment of equations in sections 5 and 6 motivates us to examine the following general non-linear PDE. Let (M, ω) be a hermitian manifold. Consider the fully non-linear PDE on M of the form

$$F(\omega^{-1}U[u]) = \psi(z, u, \partial u, \bar{\partial} u) \tag{2}$$

where $U[u] = \gamma\omega\Delta u - \beta\partial\bar{\partial}u + \chi(z, u, \partial u, \bar{\partial}u)$. Some important progress has been made in [6] for this equation when $\gamma = 1$ and $\beta < 1 (\neq 0)$. In this paper, the authors assume that F satisfies the following conditions.

1. F is defined in an open convex cone $\mathcal{A}$ with vertex at 0 which is contained in the space of real $(1, 1)$ forms $\mathcal{H}$, and contains the positive cone $\mathcal{H}^+$ in $\mathcal{H}$.
2. $F(X + Y) \geq F(X), \qquad \forall\, X \in \mathcal{A},\ Y \in \overline{\mathcal{H}}^+.$
3. F is a concave function in $\mathcal{A}$.
4. $\lim\limits_{R\to+\infty} F(R\omega) - \sup\limits_M \psi[u] \geq c_0 > 0$ for all admissible ($U[u] \in \mathcal{A}$) solutions u.

From these four conditions, it is possible to show that the non-linear PDE above is elliptic (i.e. the linearization is elliptic). In [6], the manifold is allowed to have boundary. Both interior and boundary estimates up to second order are obtained after assuming a few additional growth rate conditions on ψ and χ. From here it can be shown by continuity method that the Dirichlet problem given by

$$\begin{cases} F(U[u]) = \psi[u] & \text{in } \overline{M} \\ u = \phi & \text{on } \partial M \end{cases}$$

for $\phi \in C^\infty(\partial M)$, has an admissible solution with a global apriori estimate $|u|_{C^2(\overline{M})} \leq C$.

The authors then go on to use this method to study conformal deformations of the Chern-Ricci curvature of a hermitian metric ω. Indeed, for a hermitian metric ω there are four possible Ricci forms based on the indices we contract on the $(4, 0)$ curvature tensor. These are denoted by

$$R^{(1)}_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}},\ R^{(2)}_{k\bar{l}} = g^{i\bar{j}} R_{i\bar{j}k\bar{l}}$$
$$R^{(3)}_{i\bar{l}} = g^{k\bar{j}} R_{i\bar{j}k\bar{l}},\ R^{(4)}_{k\bar{j}} = g^{i\bar{l}} R_{i\bar{j}k\bar{l}}$$

Let $Rc^{(k)} = \sqrt{-1}R^{(k)}_{i\bar{j}} dz_i \wedge dz_{\bar{j}}$ denotes the k-th Chern-Ricci form. Then the (α, β, γ)-Chern Ricci form is defined as the linear combination

$$Rc^{\langle\alpha,\beta,\gamma\rangle} := \alpha Rc^{(1)} + \beta Rc^{(2)} + \gamma(Rc^{(3)} + Rc^{(4)})$$

Let $\tilde{\omega} = e^{-au}\omega$ be a conformal metric for some function u on M and a is some constant. In connection to (2), we consider the equation

$$F(\tilde{\omega}^{-1} Ric^{\langle\alpha,\beta,\gamma\rangle}_{\tilde{\omega}}) = \psi$$

The Ricci form for the conformal metric $\tilde{\omega}$ can be computed as

$$Ric^{\langle\alpha,\beta,\gamma\rangle}_{\tilde{\omega}} = Ric^{\langle\alpha,\beta,\gamma\rangle}_{\omega} + a\beta(\Delta u)\omega + a(n\alpha + 2\gamma)\sqrt{-1}\partial\bar{\partial}u$$

This shows that the above equation falls into the general category of equation (2) and can be used to study such deformations.

It is also worth noting that in (2), we can choose F to be a symmetric function of the eigenvalues of $U[u]$. This, in particular yields the complex Monge-Ampere equation (letting F to be the product of eigenvalues and $\gamma = 0$). However, the theory developed here does not apply to these cases due to constraints on γ and β.

In our research project, we would like to consider more general fully non-linear equations. Over the recent years, such equations are becoming more significant in different areas of geometry.

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