Ehresmann's Theorem

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Ehresmann's Theorem states that every proper submersion is a locally-trivial fibration. In these notes we go through the proof of the theorem. First we discuss some standard facts from differential geometry that are required for the proof.

1 Preliminaries

THEOREM 1.1 *(Rank Theorem)* [1] Suppose M and N are smooth manifolds of dimension m and n respectively, and $F : M \to N$ is a smooth map with a constant rank k. Then given a point $p \in M$ there exists charts $(x^1, \ldots, x^m)$ centered at p and $(v^1, \ldots, v^n)$ centered at $F(p)$ in which F has the following coordinate representation:

$$F(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

DEFINITION 1.1 Let X and Y be vector fields on M and N respectively and $F : M \to N$ a smooth map. Then we say that X and Y are F-related if $DF(X|_p) = Y|_{F(p)}$ for all $p \in M$ (or $DF(X) = Y \circ F$).

DEFINITION 1.2 Let X be a vector field on M. Then a curve $c : [0, 1] \to M$ is called an **integral curve** for X if $\dot{c}(t) = X|_{c(t)}$, *i.e.* X at $c(t)$ forms the tangent vector of the curve c at t.

THEOREM 1.2 *(Fundamental Theorem on Flows)* [2] Let X be a vector field on M. For each $p \in M$ there is a unique integral curve $c_p \colon I_p \to M$ with $c_p(0) = p$ and $\dot{c}_p(t) = X_{c_p(t)}$.

Notation: We denote the integral curve for X starting at p at time t by $\phi^t_X(p)$.

PROPOSITION 1.1 X and Y are F-related iff $F \circ \phi^t_X = \phi^t_Y \circ F$ whenever both sides are defined.

Proof. $(\Leftarrow)$

[1]Theorem 5.13 in [2]
[2]Theorem 12.9 & 12.10 in [2]

$$\begin{aligned}
DF(X\big|_p) &= DF\left(\frac{d}{dt}\bigg|_{t=0}\phi_X^t(p)\right)\\
&= \frac{d}{dt}\bigg|_{t=0} F\circ\phi_X^t(p)\\
&= \frac{d}{dt}\bigg|_{t=0} \phi_Y^t\circ F(p)\\
&= Y\big|_{F(p)}
\end{aligned}$$

Thus X and Y are F-related.

$(\Rightarrow)$

From $DF(X) = Y\circ F$ we get that

$$\begin{aligned}
\frac{d}{dt}F\circ\phi_X^t &= DF\left(\frac{d}{dt}\phi_X^t\right)\\
&= DF\left(X\big|_{\phi_X^t}\right)\\
&= Y\big|_{F\circ\phi_X^t}
\end{aligned}$$

This shows that $F\circ\phi_X^t$ is an integral curve of Y. At $t=0$ we have

$$F\circ\phi_X^t\big|_{t=0} = \phi_Y^t\circ F\big|_{t=0}$$

So by uniqueness of integral curves, $F\circ\phi_X^t = \phi_Y^t\circ F$ whenever both sides are defined.

2 Proper Submersions

Definition 2.1 Let $F : M\to N$ be a smooth map. Then F is called a submersion if $\mathrm{rank}_p(DF) = \dim(N)$ for all p in M.

Definition 2.2 A mapping $F : M\to N$ is called proper if $F^{-1}(K)$ is compact for any compact set K in N.

Theorem 2.1 Let $F : M\to N$ be a submersion. Then given any vector field Y in N, there are vector fields X in M that are F-related to Y.

Proof. Let $\dim(M) = m$ and $\dim(N) = n$. Then by rank theorem, given any $p\in M$ there exists local charts $x : U\to\mathbb{R}^m$ and $y : V\to\mathbb{R}^n$ such that $p\in U$ and $F(p)\in V$ with,

$$y\circ F\circ x^{-1}(x^1,\dots,x^m) = (x^1,\dots,x^n)$$

Note that $m\geqslant n$ for a submersion.

We claim that $\frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial x^i}$ are F-related vector fields for $i = 1,2,\dots,n$.

We compute DF in these coordinates:

$$DF = \begin{pmatrix} \frac{\partial F_1}{\partial x^1} & \cdots & \frac{\partial F_1}{\partial x^m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x^1} & \cdots & \frac{\partial F_n}{\partial x^m} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} I_{n\times n} & \mathbf{O}_{(m-n)\times n} \end{pmatrix}$$

So,

$$DF\left(\frac{\partial}{\partial x^i}\right) = \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \longleftarrow \quad i^{th}\text{position}$$

$$= \frac{\partial}{\partial y^i}$$

for $1 \leqslant i \leqslant n$.

Thus, if $Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial y^i}$ is a vector field on N, then $X = \sum_{i=1}^{n} Y^i \circ F \frac{\partial}{\partial x^i}$ is a vector field on M that is F-related to Y. This gives the local construction. For a global construction, we use partition of unity $\{\lambda_\alpha\}$ subordinate to the covering by charts $\{U_\alpha\}$. Let X_α be F-related to Y in each U_α. Then we get the required vector field by defining $X = \sum_\alpha \lambda_\alpha X_\alpha$.

$$\begin{aligned} DF(X) &= DF\left(\sum_\alpha \lambda_\alpha X_\alpha\right) \\ &= \sum_\alpha \lambda_\alpha DF(X_\alpha) \\ &= \sum_\alpha \lambda_\alpha Y \circ F \\ &= Y \circ F \end{aligned}$$

Proposition 2.1 Let F be proper, X and Y be F-related vector fields. If $F(p) = q$ and $\phi_Y^t(q)$ is defined on $[0, b)$, then $\phi_X^t(p)$ is also defined on $[0, b)$. That is, $F \circ \phi_X^t = \phi_Y^t \circ F$ holds for as long as right hand side is defined.

Proof. We show this by contradiction. Assume ϕ_X^t is defined on $[0, a)$ for some $a < b$. Let $K = F^{-1}\{\phi_Y^t(q) : t \in [0, a]\}$. Then K is compact since F is proper. The integral curve $t \mapsto \phi_X^t(p)$ is contained in K for all $t \in [0, a]$ because of Proposition 1.1. This is not possible if a is finite because the solutions to a linear ODE cannot be contained in a compact set.

3 Ehresmann's Theorem

Definition 3.1 A locally-trivial fibration $F : M \to N$ is a smooth map such that for every $p \in N$ there is a neighborhood U of p that satisfies the following two conditions:

(i) There is a diffeomorphism $\Phi : F^{-1}(U) \to U \times F^{-1}(p)$.

(ii) The following diagram commutes,

$$\begin{tikzcd} F^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "F"'] & & U \times F^{-1}(p) \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

Example 3.1 Projection $\pi : S^1 \times S^1 \to S^1$ gives a locally-trivial fibration.

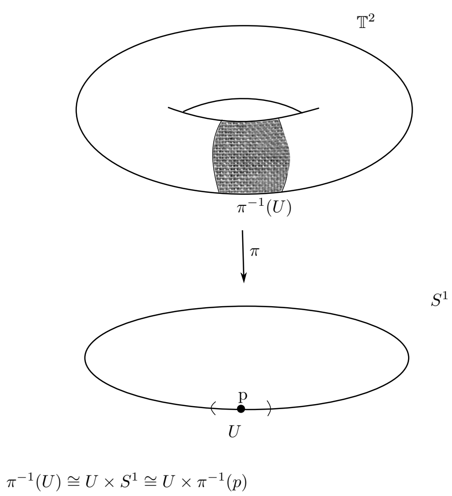

Figure 1: Fibration of S^1 by $\mathbb{T}^2$

Theorem 3.1 *(Ehresmann)* If $F : M \to N$ is a proper submersion, then it is a locally-trivial fibration.

Remark: Consider the special case when the locally-trivial fibration $F : M \to N$ is a vector bundle. Then it is not too difficult to show that, for $F : M \to N$, the existence of local-trivializations is equivalent to the existence of locally-trivializing sections [3]. So it is enough to construct such sections to show that the bundle is locally trivial. For example, we can show that the tangent bundle of a manifold is locally trivial by using the local sections $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ on charts[4]. But this is not possible for surjective submersions, as there is no additional structure on the fibers.

Proof. Since the fibration is defined locally with respect to N, we can assume without loss of generality that $N = \mathbb{R}^n$. Then we need to show that $F^{-1}(\mathbb{R}^n) \cong \mathbb{R}^n \times F^{-1}(0)$.

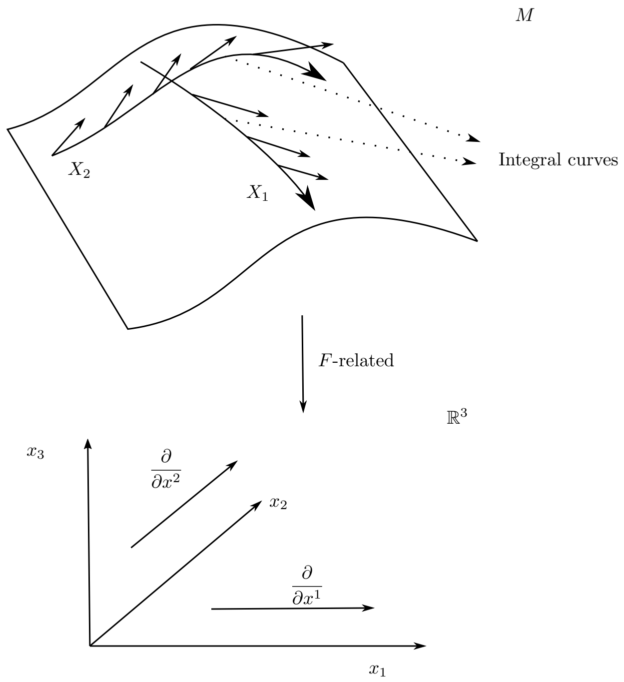

Figure 2: F-related vector fields

We can map the fibers in $F^{-1}(\mathbb{R}^n)$ to the single fiber $F^{-1}(0)$ in a diffeomorphic way. This is done by moving the fibers along integral curves of some set of vector fields, in such a way that the flow takes each fiber to $F^{-1}(0)$. These vector fields are chosen to be F-related to some locally trivial sections of TN, so as to satisfy all the requirements.

[3] set of smooth sections that spans the vector spaces of the bundle point-wise.

[4] The corresponding local trivialization is given by

$$\begin{aligned}\Phi :& \pi^{-1}(U) \to U \times \mathbb{R}^n \\ & \left(x, f^i(x)\frac{\partial}{\partial x^i}\right) \mapsto \left(x, f^1(x), f^2(x), \ldots, f^n(x)\right)\end{aligned}$$

From Theorem 2.1 we know that there exists vector fields X_i that are F-related to $\frac{\partial}{\partial x^i}$ for each $1 \leqslant i \leqslant n$. Let $\phi^t_{X_1}, \phi^t_{X_2}, \ldots, \phi^t_{X_n}$ denote the integral curves of $X_1, \ldots, X_n$. From *Proposition 2.1*, we know that these integral curves are defined for all time $t \in [0, \infty)$.

Define $G : F^{-1}(\mathbb{R}^n) \to \mathbb{R}^n \times F^{-1}(0)$ by,

$$G(z) = \left(F(z), \phi^{-t_n}_{X_n} \circ \cdots \circ \phi^{-t_1}_{X_1}(z)\right)$$

where $F(z) = (t_1, \ldots, t_n)$.

We now show that G is a diffeomorphism:

- *Bijective:* Since G has a well-defined inverse given by $G^{-1}(t_1, \ldots, t_n, x) = \phi^{t_1}_{X_1} \circ \cdots \circ \phi^{t_n}_{X_n}(x)$.
- *Smooth:* Since $\phi^t_X(z)$ depends smoothly on z and since F is smooth.[5]

So G is a diffeomorphism. The commutativity of the diagram is immediate from the way G is defined. Thus F is a locally-trivial fibration.

4 Applications and Examples

Following are some immediate consequences of the Ehresmann's theorem.

Corollary 4.1 Any two fibers of a proper submersion are diffeomorphic.

Corollary 4.2 *(Basic Lemma in Morse Theory)* Let $F : M \to \mathbb{R}$ be a proper map. If F is regular on $(a, b) \subset \mathbb{R}$, then $F^{-1}((a, b)) \cong (a, b) \times F^{-1}(c)$ for all $c \in (a, b)$.

Corollary 4.3 *(Reeb's Sphere theorem)* Let M be a closed[6] manifold that admits a map with two non-degenerate critical points. Then M is homeomorphic to a sphere.

Sketch of the proof. Let $\dim(M) = n$.
Let p_1 and p_2 be the critical points where the mapping $f : M \to [a, b]$ attains its maximum and minimum respectively. Then by Morse theorem, $f(x) = x_1^2 + \cdots + x_n^2$ and $f(y) = -y_1^2 - \cdots - y_n^2$ in some charts x and y near p_2 and p_1 respectively.

[5]solutions of linear ODE depends smoothly on the initial data
[6]compact manifold with no boundary

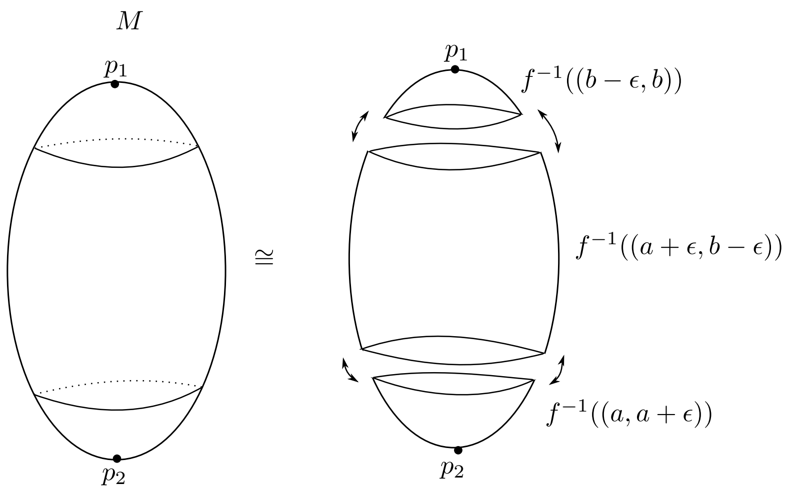

Figure 3: Top and bottom sections are disks with boundaries S^{n-1}

We can see that $f^{-1}((b-\epsilon, b))$ and $f^{-1}((a, a+\epsilon))$ has to look like disks. The function f has to be regular on $(a+\epsilon, b-\epsilon)$, as there are only two critical points on M. So, by Cor 4.2 $f^{-1}((a+\epsilon, b-\epsilon)) \cong (a+\epsilon, b-\epsilon) \times f^{-1}(a+\epsilon)$. The right hand side here is homeomorphic to an n-dim cylinder, since $f^{-1}(a+\epsilon)$ is an $(n-1)$-sphere (being the boundary of an n-disk). So combining all these, we get M as two disks attached to the boundary of a cylinder. Hence M is homeomorphic to a sphere.

Ehresmann's theorem can be used to show that projection of spheres onto projective spaces are fibrations:

Example 4.1 Consider the projection map $p : S^3 \to \mathbb{CP}^1$. p is proper since both S^3 and $\mathbb{CP}^1$ are compact, Hausdorff spaces. Hence, by Ehresmann's theorem, $p : S^3 \to \mathbb{CP}^1$ is a locally-trivial fibration.

We see an example showing that F being proper is necessary:

Example 4.2 Consider the projection map to the first co-ordinate $p : \mathbb{R}^n - \{0\} \to \mathbb{R}$. This is a surjective submersion. But it is not locally trivial because $p^{-1}(0) \times U$ is not simply connected for any open ball U around zero. This map is not proper.

Vector bundles are locally-trivial fibrations that are not proper. So the converse of Ehresmann's theorem is not true. One last remark is that locally-trivial fibrations satisfy the homotopy lifting property. This allows us to compute topological invariants of the manifolds involved.

References

[1] Peter Petersen. *Manifold Theory*, `math.ucla.edu/~petersen/manifolds.pdf`

[2] John Lee. *Introduction to Smooth Manifolds*, Springer, New York, 2003

[3] Bjørn Dundas. *Differential Topology*, JHU, 2002