GAUSS-BONNET THEOREM

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1. Introduction

In this project we will prove one of the most fundamental theorems in differential geometry, namely the Gauss-Bonnet Theorem. The theorem is stated as follows. Let M be an oriented, connected, smoothly triangulated, Riemannian 2-manifold. Then,

$$\int_M K dV = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic and K is the curvature of M.
This is one of the early theorems in differential geometry that relates the geometry of a surface to its topology. The left-hand side of the equation seem to depend on the Riemannian metric given to the surface. But this theorem shows that it does not, since the right-hand side is independent of the metric defined. Similarly, it shows that the Euler characteristic, which appears to depend on the triangulation given, is in-fact, independent of the triangulation given to the surface.

2. Background

A *smoothly triangulated manifold* is a triple (M, K, h), where X is a C^∞ manifold, K is a simplicial complex, and $h : [K] \to M$ is a homeomorphism such that for each simplex s of K, the map $h|_{[s]} \to X$ has an extension h_s to a neighbourhood U of $[s]$ in the plane of $[s]$, such that $h : U \to M$ is a smooth submanifold.

Remark 2.1. For a triangulated 2-manifold M,

$$\chi(M) = n_0 - n_1 + n_2$$

where n_i is the number of i-simplices in the triangulation of M.

Let M be a smoothly triangulated 2-manifold and ω be a 2-form on M. Let $h : [K] \to M$ be the smooth triangulation. Define the *integral* of ω over M as,

$$\int_M \omega = \sum_s \int_{\langle s \rangle} {h_s}^*(\omega)$$

where the sum runs over all the 2-simplices of K.

A *Riemannian manifold* (M, g) is a smooth manifold M, equipped with an inner product g_p on the tangent space T_pM at each point p, that varies smoothly on M, in the sense that if X and Y are smooth vector fields on M, then the map $p \mapsto g_p(X_p, Y_p)$ is a smooth function.

Let M be an oriented Riemannian 2-manifold. Let U be an open set in M. An *oriented orthonormal frame* is a choice of a set of smooth vector fields $\{e_1, e_2\}$ on U such that $\{e_1(p), e_2(p)\}$ forms an oriented orthonormal basis at each point p of U.

Remark 2.2. Oriented orthonormal frames can be constructed on any co-ordinate chart U of M. For let $\{x_1, x_2\}$ be co-ordinate functions on U. We can use Gram-Schmidt method on the co-ordinate vector fields $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ to get an orthonormal frame $\{e_1, e_2\}$ on U.

Define the *dual frame* of $\{e_1, e_2\}$ in U to be the set of 1-forms $\{\omega_1, \omega_2\}$ in U such that $\{\omega_1(p), \omega_2(p)\}$ is the dual basis of $\{e_1(p), e_2(p)\}$ at each point p of U.

The 2-form $\omega_1 \wedge \omega_2$ is independent of the co-ordinates used and hence defines a global non-zero 2-form on M called the *volume element*(dV) of M.

Theorem 2.3. *Let M be a smooth oriented Riemannian 2-manifold. Let $\{e_1, e_2\}$ be an orthonormal frame in a co-ordinate neighbourhood U of M. Let $\{\omega_1, \omega_2\}$ be the corresponding dual frame. Then there exists a unique 1-form ϕ on U such that the following equations hold,*

(1) $d\omega_1 = \phi \wedge \omega_2$
(2) $d\omega_2 = -\phi \wedge \omega_1$
(3) $d\phi = -K\omega_1 \wedge \omega_2$

where K is a smooth function on U which is independent of the co-ordinates used and hence is a smooth function on all of M. K is called as the curvature and ϕ the connection 1-form.

Proof. $\omega_1 \wedge \omega_2$ spans the space of all two forms at each point of U. So we get

$$d\omega_1 = f\omega_1 \wedge \omega_2$$
$$d\omega_2 = h\omega_1 \wedge \omega_2$$

for some smooth functions f and h in U. Taking $\phi = f\omega_1 + h\omega_2$ gives the result. □

Remark 2.4. These equations are called Cartan's structural equations.

Let $\alpha : [a, b] \to M$ be a smooth curve parametrized by arc length. Let $\alpha'(t) = (\alpha(t), \dot{\alpha}(t))$. Then the *geodesic curvature* $k_\alpha(t)$ of α at $t \in [a, b]$ is $\phi(d\alpha'(d/dt))$, where ϕ is the connection 1-form. Define $\tau(\alpha) = \int_a^b k_\alpha(t)dt$.

Geodesic curvature measures how far a curve is from being 'straight' on a manifold.

3. Proof of the theorem

The theorem is first proved for a single 2-simplex, which will then be used to prove the general case. This proof involves two steps. Intuitively the curvature K of M measures the rotation obtained by parallel translation of vectors around small closed curves in M. More precisely, this angle of rotation can be shown to be equal to $\int_{\langle s\rangle} h^*(KdV)$, which is exactly the left-hand side of Gauss-Bonnet equation for a 2-simplex. The same angle can be measured in a different way using properties of the connection 1-form on M and the geodesic curvature of the smooth curve over which we translate the vector. Equating these two angles gives the result for 2-simplices.

Let M be an oriented Riemannian manifold of dimension 2. Let $T(M)$ denote the tangent bundle of M. Let

$$S(M) = \{(m, v) \in T(m) :< v, v >= 1\}$$

$S(M)$ is called the *sphere bundle* of M.

A *connection* on $S(M)$ is a choice of a two dimensional subspace $\mathcal{H}(m,v)$ of $T(S(M)) = \cup_{x\in s(M)} T(S(M),x)$ at each point $(m,v) \in S(M)$ such that the following hold.

(1) $T(S(M),(m,v)) = \mathcal{H}(m,v) \oplus d\pi^{-1}(0)$ where $\pi : S(M) \to M$ is the projection map; that is, the subspace $\mathcal{H}(m,v)$ is complementary to the vertical space($\pi^{-1}(0)$) at (m,v).
(2) $dg(\mathcal{H}(m,v)) = \mathcal{H}(m,gv)$ for each $g \in S^1$.
(3) The choice of $\mathcal{H}$ is smooth; that is at each point $(m,v) \in S(M)$, there exists an open set U about (m,v) and smooth vector fields X and Y defined on U such that X,Y spans $\mathcal{H}$ at each point of U.

Connection provides a way for lifting smooth curves in M to $S(M)$ as given by the next theorem.

Theorem 3.1. *Let $\mathcal{K}$ be a connection on $S(M)$. Let $\alpha : [a,b] \to M$ be a broken C^∞ curve in M. Let $v \in T(M,\alpha(a))$ with $||v|| = 1$. Then there exists a unique broken C^∞ curve $\tilde{\alpha} : [a,b] \to S(M)$, called horizontal lift of α, through $(\alpha(a),v)$ such that*

(1) $\pi \circ \tilde{\alpha} = \alpha$
(2) $\dot{\tilde{\alpha}} \in \mathcal{K}(\tilde{\alpha}(t))$
(3) $\tilde{\alpha}(a) = (\alpha(a),v)$

The vector $\tilde{\alpha}(b) \in T(M,\alpha(b))$ is the parallel translate of v along α to $\alpha(b)$.

We skip the proof here since it is highly technical.

As mentioned before, the rotation of a vector during parallel translation along a curve α can be measured using the properties of the connection 1-form. In-order to do this we introduce the following lemma.

Lemma 3.2. *Let $\alpha : [a,b] \to M$ be a smooth curve in M. Let $\tilde{\alpha} : [a,b] \to S(M)$ and $\tilde{\beta} : [a,b] \to S(M)$ be smooth curves such that $\pi \circ \tilde{\alpha} = \alpha$ and $\pi \circ \tilde{\beta} = \alpha$. Suppose $\tilde{\alpha}$ is horizontal relative to some connection $\mathcal{K}$ on $S(M)$, that is $\dot{\tilde{\alpha}} \in \mathcal{K}(\tilde{\alpha}(t))$ for all $t \in [a,b]$. Let the connection 1-form be ϕ. Then there exists a smooth function $\theta : [a,b] \to \mathbb{R}$ such that*

(1) $\tilde{\beta} = e^{i\theta(t)}\tilde{\alpha}(t)$
(2) $\phi(\dot{\tilde{\beta}}(t)) = \frac{d\theta}{dt}(t)$

Furthermore, if $\tilde{\alpha}(a) = \tilde{\beta}(a)$, then θ can be chosen such that $\theta(a) = 0$.

Proof. Since unit vectors on a tangent space only differ by a rotation, we get

$$\tilde{\beta}(t) = g(t)\tilde{\alpha}(t)$$

where $g : [a,b] \to S^1$ can be verified to be a smooth map. $\mathbb{R}$ is a covering space of S^1 with covering map $p = e^{ir}$. Now since $[a,b]$ is simply connected, we can lift g to a map $\theta : [a,b] \to \mathbb{R}$ such that $g(t) = e^{i\theta(t)}$. This proves (1). When $\tilde{\alpha}(a) = \tilde{\beta}(a)$, we can choose θ to be the lift starting at the origin so that $\theta(a) = 0$.
Proof of (2) involves some manipulations of differential forms which is skipped here. □

Now we give the result which relates curvature K on M to the rotation obtained by parallel translation.

Theorem 3.3. *Let M be an oriented Riemannian 2-manifold. Let $\langle s\rangle$ be an oriented 2-simplex in $\mathbb{R}^2$ and let $h : [s] \to M$ be a map with a smooth extension, mapping a neighbourhood of $[s]$ into M. Let α be a closed broken C^∞ curve in M obtained by restricting h to $\partial\langle s\rangle$. Then the rotation obtained by parallel translation around the closed curve α is*

$$e^{\int_{\langle s\rangle} h^*(KdV)}$$

so that the angle of rotation is $\int_{\langle s\rangle} h^*(KdV)$

Proof. We give a sketch of the proof here.

Let $\langle s\rangle = \langle v_0, v_1, v_2\rangle$. Then $[s]$ can be covered by line segments starting at v_1 to the opposite edge. h maps these line segments to curves in M which cover $h([s])$. Consider the horizontal lifts of these curves in $S(M)$ and let $\tilde{h} : [s] \to S(M)$ be obtained by sending each of these line segments to their corresponding lifts in $S(M)$ through a fixed vector w_1 at $h(v_1)$. w_1 is assumed to be obtained by parallel translation of a vector w_0 at $h(v_0)$ over $\alpha_{\langle v_0,v_1\rangle}$. Let $\tilde{\beta} = \tilde{h}|_{\langle s\rangle}$.

It can be shown using Cartan's structural equation and Stoke's theorem that

$$\int_{\langle s\rangle} h^*(KdV) = -\int_{\partial s} \phi(d\tilde{\beta}(\frac{d}{dt}))dt$$

Let $\tilde{\alpha}$ be the horizontal lift of α through w_0. Then it is clear that $\tilde{\alpha}$ and $\tilde{\beta}$ coincide on the sides $h_{\langle v_1,v_0\rangle}$ and $h_{\langle v_1,v_2\rangle}$ of $h_{\langle s\rangle}$ by construction. Now on the third side $h_{\langle v_2,v_0\rangle}$, using lemma 3.2 we have

(1) $\tilde{\beta} = e^{if(t)}\tilde{\alpha}(t)$
(2) $\phi(d\tilde{\beta}(\frac{d}{dt})) = \frac{df}{dt}$

for $f : [v_2, v_0] \to \mathbb{R}$ with $f(v_2) = 0$.

But $\tilde{\alpha}$ is the horizontal lift of α, so that $\tilde{\alpha}(v_0)$ is the parallel translate of w_0 around α. On the other hand, $\tilde{\beta}(v_0) = w_0$. Hence $e^{if(v_0)}$ is just the rotation mapping the parallel translate of w_0 around α into w_0; that is, $e^{-if(v_0)}$ rotates w_0 into its parallel translate around α. Now combining the two equations above gives,

$$\begin{aligned}\int_{\langle s\rangle} h^*(KdV) &= -f(v_0)\\ &= \textit{the angle of rotaion from } w_0 \textit{ to its parallel translate around } \alpha\end{aligned}$$

□

The machinery developed so far can be used to prove the Gauss-Bonnet theorem for 2-simplices.

Lemma 3.4. *(The Gauss-Bonnet Theorem for 2-simplices) Let M be an oriented Riemannian 2-manifold. Let $\langle s\rangle$ be an oriented 2-simplex in $\mathbb{R}^2$, and let $h : [s] \to M$ be a map which has a smooth non-singular extension mapping a neighbourhood of $[s]$ into M. Let α be the closed broken C^∞ curve in M obtained by restricting h to $\partial\langle s\rangle$. Then*

$$\int_{\langle s\rangle} h^*(KdV) = -\tau(\alpha) + \sum \textit{ interior angles of } h[s] - \pi$$

Proof. Suppose α is broken up into three of its smooth components, $\alpha_0, \alpha_1, \alpha_2$ so that $\tau(\alpha) = \sum \tau(\alpha_i)$ and $\alpha_i : [a_i, a_{i+1}] \to M$ with $a_0 = a$ and $a_3 = b$. We claim that $e^{i\tau(\alpha_i)}$ is the rotation from the parallel translate of $\dot{\alpha}_i(a_i)$ to $\dot{\alpha}_i(a_{i+1})$. If β_i is the horizontal lift of α_i with $\beta_i(a_i) = \dot{\alpha}_i(a_i)$ then by Lemma 3.2, there exists a smooth function $\theta : [a_i, a_{i+1}] \to \mathbb{R}$ with $\theta(a_i) = 0$ and $\phi(\dot{\alpha'}_i(t)) = \frac{d\theta}{dt}(t)$. Then,

$$\int_{a_i}^{a_{i+1}} \phi(d\alpha_i'(\frac{d}{dt}))dt = \int_{a_i}^{a_{i+1}} \phi(\dot{\alpha'}_i(t))dt = \theta(a_{i+1})$$

This verifies the claim using Lemma 3.2.1.

Adding the exterior angles around each vertex (by which $\dot{\alpha}_i(a_{i+1})$ is rotated to $\dot{\alpha}_{i+1}(a_{i+1})$), we get total rotation by parallel translation around α to be equal to $e^{-\tau(\alpha) - \sum \text{ exterior angles}}$. Hence, by taking logarithms, we get

$$\int_{\langle s \rangle} h^*(KdV) = -\tau(\alpha) - \sum \text{ exterior angles} + 2\pi l$$

where l is an integer. Now suppose ρ_0 is a flat Riemannian metric in a neighbourhood of $h[s]$ (say transferred from $\mathbb{R}^2$ via h). Then $K = 0$, $\tau(\alpha) = 0$, and $\sum$ exterior angles is 2π. Hence $l = 1$ for flat metrics. Suppose ρ is the given metric, and let $\rho_t = t\rho_0 + (1-t)\rho$ be a family of metrics, $t \in [0, 1]$. l is a continuous function of t since K_t, $\tau_t(\alpha)$ and exterior angles of ρ_t are continuous. Since l is an integer for all t and equal to 1 for $t = 0$, we obtain $l = 1$.

Since interior angle + exterior angle = π, the lemma is proved. □

Now we give the proof of Gauss-Bonnet theorem using the previous result.

Theorem 3.5. *(Gauss-Bonnet Theorem) Let M be an oriented, connected, smoothly triangulated, Riemannian 2-manifold. Then*

$$\int_M KdV = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M.

Proof. It can be easily verified that each 1-simplex in M is an edge of exactly two 2-simplices in M. Thus the total number number n_1 of 1-simplices in M is given by $n_1 = 3n_2/2$, where n_2 is the number of 2-simplices in M. Hence the Euler characteristic is given by

$$X(M) = n_0 - n_1 + n_2 = n_0 - n_2/2$$

where n_0 is the number of vertices in the triangulation of M.

$$\begin{aligned} \int_M KdV &= \sum_s \int_{\langle s \rangle} h^*(KdV) \\ &= \sum_s (-\tau(\partial\langle s \rangle) + \sum \text{interior angles of } h[s] - \pi) \\ &= -\tau(\sum \partial s) + \sum_s \Big(\sum \text{interior angles of } h[s] - n_2\pi \Big) \end{aligned}$$

But $\sum \partial s$ is zero since each 1-simplex in the boundary lies in exactly two 2-simplices which are coherently oriented and hence has opposite orientations. So they cancel in pairs in the sum. $\sum_s \Big(\sum \text{interior angles of } h[s] \Big)$ equals the sum over all vertices

v in M of the sum of the interior angles at v of all 2-simplices with v as a vertex. Taking a coordinate neighbourhood of v contained in $Star(v)$, we see that the sum of the interior angles at v is exactly 2π. Hence

$$\begin{aligned}\int_M K dV &= 2\pi n_0 - n_2\pi \\ &= 2\pi(n_0 - \frac{n_2}{2}) = \chi(M)\end{aligned}$$

□

4. Some Consequences of Gauss-Bonnet Theorem

(1) S^2 cannot admit a Riemannian metric with curvature $K \leq 0$ everywhere. This follows from the fact that $\chi(S^2) = 2$.

(2) The 2-dimensional torus cannot admit a Riemannian metric with $K < 0$ everywhere or $K > 0$ everywhere. If $K \geq 0$ everywhere or if $K \leq 0$ everywhere, then $K = 0$ identically. This follows from the fact that $\chi(T^2) = 0$.

(3) An orientable surface of genus $g \geq 2$ cannot admit a Riemannian metric with curvature $K \geq 0$ everywhere. This is because $\chi(M) = 2 - 2g$ for an orientable surface of genus g.

(4) Let M be as in the theorem. Suppose on M there exists a smooth vector field which is never zero. Then $\chi(M) = 0$. In particular, there exist no non-vanishing vector field on any even dimensional sphere.

Proof. Let X be a non-vanishing vector field on M. Then we can take $e_1 = X/||X||$ and e_2 to be the unique vector field such that $\{e_1, e_2\}$ forms an orthonormal frame on all of M. So we can define the corresponding dual frame $\{\omega_1, \omega_2\}$ globally on M. Now as in the proof of Cartan's equations we can find a unique 1-form ϕ defined globally on M, such that $d\phi = KdV$. Hence KdV becomes an exact form on M. So we get

$$\int_M KdV = \int_M d\phi = \int_{\partial M} \phi = 0$$

using Stoke's Theorem.

□

References

[1] I.M. Singer and J.A. Thorpe, Lecture Notes on Elementary Topology and Geometry