# Hodge Theorem

Mathew George

Hodge theorem gives a decomposition of the space of $p$-forms into harmonic, exact and co-exact forms. This is an important theorem in geometry that finds applications frequently. We discuss a proof of this theorem that uses mostly analytic methods. More specifically, we will use the regularity theory of elliptic operators to prove this.

## 1 Preliminaries

### 1.1 Elliptic Operators

Let's start with an example.

$$Lu = -\sum_{i,j} a_{ij} u_{x_i x_j} + \sum_i b_i u_{x_i} + c$$

is said to be elliptic in a domain $\Omega$ if $\sum\limits_{i,j} a_{ij}\xi_i\xi_j > 0$ for any vector $\xi \in \mathbb{R}^n$ for all $x \in \Omega$.

In general, let $L = P_l(D) + \ldots + P_0(D)$, where $P_j(D)$ is an $m \times m$ matrix with each entry being a differential operator $\sum\limits_{|\alpha|=j} a_\alpha D^\alpha$ homogeneous of order $j$. Then $L$ is elliptic if $P_l(\xi)$ is non-singular for each $\xi \in \mathbb{R}^n$. $P_l(\xi)$ is the polynomial obtained by transforming $D^\alpha \mapsto \xi^\alpha$. Clearly ellipticity depends only on the coefficients of the highest order derivatives.

### 1.2 Regularity theory

**Definition 1.1.** *The linear functional* $l$ *is a weak solution of* $\Delta w = \alpha$ *if* $l$ *satisfies*

$$l(L^*\phi) = \langle \alpha, \phi \rangle$$

*for all* $\phi \in \Lambda^p M$ *where* $L^*$ *is the adjoint of* $L$.

Let $L$ be some elliptic operator of order $l$ and $u$ be a weak solution of the problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

in some Sobolev space. Then regularity theory roughly says that

1. If $f \in H_t(\Omega)$ then $u \in H_{t+l}(\Omega)$.
2. If $f \in C^{t,\alpha}(\Omega)$ then $u \in C^{t+l,\alpha}(\Omega)$.
3. If $f \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.

The solutions are as regular as the PDE allows it to be. There might be additional constraints on the coefficients for this to be true. But these are often satisfied. The main technique involved in proving this theorem is to approximate differential operators $D^\alpha u$ by difference quotients $D_h^\alpha u$,

$$\left(\frac{\partial}{\partial x_i}\right)_h u = \frac{u(x + he_i) - u(x)}{h}$$

In the case of L as in the first example, this proof is given in Chapter 6 of Evans' PDE. The general case is given in Warner.

## 1.3 Differential geometry

Let $M$ be a compact, connected, orientable, Riemannian manifold without boundary. We define $\Lambda^p M$ to be the set of $p$-forms on $M$ equipped with the wedge product. Fix an orthonormal basis $\{e_1, \dots, e_n\}$ for $T_x M$ and define an inner product $\langle ., . \rangle$ on $\Lambda^p M$ at $x$ by declaring that $\{e_{i_1}^* \wedge \dots \wedge e_{i_m}^*\}_{1 \leqslant i_1 < \dots < i_m \leqslant n}$ forms an orthonormal basis. That is,

$$\langle e_I^*, e_J^* \rangle = \delta_J^I$$

for multi-indices $I, J$, and then extending it by linearity to $\Lambda^p M$ at $x$.

**Definition 1.2.** *The Hodge star operator is a linear transformation* $\star : \Lambda^p M \to \Lambda^{n-p} M$ *such that*

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle dV$$

*for any* $\alpha, \beta \in \Lambda^p M$, *where* $dV$ *is the volume form on* $M$.

*Example:* If $dV = dx_1 \wedge \dots \wedge dx_n$, then $\star(dx_1 \wedge \dots \wedge dx_3) = dx_4 \wedge \dots \wedge dx_n$.

**Remark 1.3.** *Its easy to check that* $\star$ *satisfies* $\star\star = (-1)^{p(n-p)}$.

**Definition 1.4.** *Let* $\mathrm{d} : \Lambda^p M \to \Lambda^{p+1} M$ *be the exterior derivative. Then the coexterior derivative* $\delta : \Lambda^{p+1} M \to \Lambda^p M$ *is given by*

$$\delta = (-1)^{n(p+1)+1} \star \mathrm{d} \star$$

Note that the exterior derivative increases the degree of the form by one whereas the coexterior derivative does the opposite. We use these operators to define the natural generalization of Laplace's operator $\sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$ to a Riemannian manifold.

**Definition 1.5.** *The Laplace-Beltrami operator* $\Delta : \Lambda^p M \to \Lambda^p M$ *is defined as*

$$\Delta = \delta \mathrm{d} + \mathrm{d} \delta$$

Its a good exercise to verify that this becomes the usual Laplacian when applied to smooth functions on $M$.

Define the inner product on $\Lambda^p M$ globally as

$$\langle \alpha, \beta \rangle = \int\limits_M \alpha \wedge \star \beta$$

for any $\alpha, \beta \in \Lambda^p M$. Extend this linearly to $\bigoplus\limits_{p=0}^{n} \Lambda^p M$ with $\Lambda^0 M = C^\infty(M)$.

**Proposition 1.6.**

1. $\delta$ *is the adjoint of* $\mathrm{d}$ *with respect to the above inner product on* $\bigoplus\limits_{p=0}^{n} \Lambda^p M$.
2. $\Delta$ *is self-adjoint.*
3. $\Delta \alpha = 0$ *if and only if* $\mathrm{d}\alpha = \delta\alpha = 0$.

*Proof.*

1. From the Leibniz rule for exterior derivative we know that $\mathrm{d}(\alpha \wedge \star \beta) = \mathrm{d}\alpha \wedge \star \beta + (-1)^{p-1} \alpha \wedge \mathrm{d} \star \beta$ for any $\alpha \in \Lambda^{p-1} M$ and $\beta \in \Lambda^p M$. Now since the manifold does not have a boundary, using Stokes theorem[1] we get

$$\begin{aligned} \int\limits_M \mathrm{d}(\alpha \wedge \star \beta) &= \int\limits_M \mathrm{d}\alpha \wedge \star \beta - \int\limits_M \alpha \wedge \star \delta \beta \\ 0 &= \int\limits_M \mathrm{d}\alpha \wedge \star \beta - \int\limits_M \alpha \wedge \star \delta \beta \\ \langle \mathrm{d}\alpha, \beta \rangle &= \langle \alpha, \delta \beta \rangle \end{aligned}$$

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[1] For an $(n-1)$-form $\omega$ Stoke's theorem says that $\int\limits_M \mathrm{d}\omega = \int\limits_{\partial M} \omega$

2. Follows directly from 1.

3. $\Delta\alpha = 0$ implies that

$$\begin{aligned} 0 &= \langle \Delta\alpha, \alpha \rangle \\ &= \langle \delta d\alpha, \alpha \rangle + \langle d\delta\alpha, \alpha \rangle \\ &= \langle d\alpha, d\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle \\ &= ||d\alpha||^2 + ||\delta\alpha||^2 \end{aligned}$$

Hence $d\alpha = \delta\alpha = 0$. The other implication is straightforward.

□

**Definition 1.7.** $\mathcal{H}^p = \{\omega \in \Lambda^p M : \Delta\omega = 0\}$ *is the set of harmonic* $p$*-forms on* $M$*.*

Regularity theory can be adapted to the setting of $p$-forms. We state two important theorem without proof here.

**Theorem 1.8.**

1. *Let* $\alpha \in \Lambda^p M$ *and let* $l$ *be a weak solution of* $\Delta w = \alpha$*. Then there exists an* $\omega \in \Lambda^p M$ *such that*

$$l(\beta) = \langle \omega, \beta \rangle$$

*for every* $\beta \in \Lambda^p M$

2. *Let* $\{\alpha_n\}$ *be a sequence of smooth* $p$*-forms on* $M$ *such that* $||\alpha_n|| \leqslant C$ *and* $||\Delta\alpha_n|| \leqslant C$ *for all* $n$ *and for some constant* $C > 0$*. Then a subsequence of* $\{\alpha_n\}$ *is Cauchy in* $\Lambda^p M$*.*

## 2 Hodge theorem

The notions developed so far can be used to prove the main theorem of this discussion.

**Theorem 2.1 (Hodge-Kodaira-Weyl).** *For each* $0 \leqslant p \leqslant n$*,* $\mathcal{H}^p$ *is finite dimensional and we have the following direct sum decomposition of the space* $\Lambda^p M$ *of smooth* $p$*-forms on* $M$*.*

$$\begin{aligned} \Lambda^p M &= \Delta(\Lambda^p M) \oplus \mathcal{H}^p \\ &= d\delta(\Lambda^p M) \oplus \delta d(\Lambda^p M) \oplus \mathcal{H}^p \\ &= d(\Lambda^{p-1} M) \oplus \delta(\Lambda^{p+1} M) \oplus \mathcal{H}^p \end{aligned}$$

*So* $\Delta w = \alpha$ *has a solution in* $\Lambda^p M$ *if and only if* $\alpha \perp \mathcal{H}^p$*.*

*Proof.*

1. If $\mathcal{H}^p$ is finite dimensional then $\mathcal{H}^p$ would contain an infinite orthonormal sequence. But by 2 in theorem 1.8, this orthonormal sequence would contain a Cauchy subsequence. This is not possible since orthonormal elements always maintain a distance of $\sqrt{2}$ between them.

2. Its enough to prove that $\Lambda^p M = \Delta(\Lambda^p M) \oplus \mathcal{H}^p$, since

$$\begin{aligned}\Delta(\Lambda^p M) &= d\delta(\Lambda^p M) \oplus \delta d(\Lambda^p M)\\ &= d(\Lambda^{p-1} M) \oplus \delta(\Lambda^{p+1} M)\end{aligned}$$

Let $\mathcal{H}^{p\perp}$ be the orthogonal complement of $\mathcal{H}^p$ so that

$$\Lambda^p M = \mathcal{H}^{p\perp} \oplus \mathcal{H}^p$$

We will show that $\mathcal{H}^{p\perp} = \Delta(\Lambda^p M)$. Let $H : \Lambda^p M \to \mathcal{H}^p$ be the projection map, so that $H(\alpha)$ is the harmonic part of $\alpha$. Then its not too difficult to show that $\Delta(\Lambda^p M) \subset \mathcal{H}^{p\perp}$, since $\langle \Delta\omega, \alpha\rangle = \langle \omega, \Delta\alpha\rangle = 0$ for all $\alpha \in \mathcal{H}^p$ and $\omega \in \Lambda^p M$. The difficult part is showing that $\mathcal{H}^{p\perp} \subset \Delta(\Lambda^p M)$. We first show that there exists a $c > 0$ such that

$$||\beta|| \leqslant c||\Delta\beta|| \qquad \forall \beta \in \mathcal{H}^{p\perp}$$

Suppose not. Then there is a sequence $\beta_j \in \mathcal{H}^{p\perp}$ such that $||\Delta\beta_j|| \to 0$ and $||\beta_j|| = 1$. By 2 in theorem 1.8, $\{\beta_j\}$ has a Cauchy subsequence, $\{\beta_{j_k}\}$. Thus, $\lim\limits_{k\to\infty} \langle \beta_{j_k}, \psi\rangle$ converges for each $\psi \in \Lambda^p M$.

Define $l(\psi) = \lim\limits_{k\to\infty} \langle \beta_{j_k}, \psi\rangle$ for each $\psi \in \Lambda^p M$.

$l$ is clearly bounded and

$$\begin{aligned}l(\Delta\phi) &= \lim_{k\to\infty} \langle \beta_{j_k}, \Delta\psi\rangle\\ &= \lim_{k\to\infty} \langle \Delta\beta_{j_k}, \psi\rangle\\ &= 0\end{aligned}$$

This implies that $l$ is a weak solution of $\Delta w = 0$. By 1 in theorem 1.8, there is a $\beta \in \Lambda^p M$ such that $l(\psi) = \langle \beta, \psi\rangle$. Then $\beta_{j_k} \to \beta$ because $\beta_{j_k}$ converges strongly and the strong and weak limit have to be the same. Since $||\beta_{j_k}|| = 1$ and $\beta_{j_k} \in \mathcal{H}^{p\perp}$, it follows that $||\beta|| = 1$ and $\beta \in \mathcal{H}^{p\perp}$. But $\Delta\beta = 0$, which is not possible.

Let $\alpha \in \mathcal{H}^{p\perp}$ and define $l$ on $\Delta(\Lambda^p M)$ by

$$l(\Delta\phi) = \langle \alpha, \phi \rangle \qquad \forall\ \phi \in \Lambda^p M$$

$l$ is well-defined since if $\Delta(\phi_1 - \phi_2) = 0$ then $(\phi_1 - \phi_2)$ is in $\mathcal{H}^p$ and hence is orthogonal to $\alpha$. We show that $l$ is a bounded linear functional on $\Delta(\Lambda^p M)$.

$$\begin{aligned} |l(\Delta\phi)| &= |l(\Delta\psi)| \qquad \text{where } \phi = \psi + H(p) \\ &= |\langle \alpha, \psi \rangle| \\ &\leqslant ||\alpha|| ||\psi|| \\ &\leqslant c||\alpha|| ||\Delta\psi|| \\ &= c||\alpha|| ||\Delta\phi|| \end{aligned}$$

By Hahn-Banach extension theorem, $l$ extends to a linear functional on $\Lambda^p M$. Thus we have constructed is a weak solution $l$ of $\Delta\omega = \alpha$.

Invoking the regularity theorem 1.8 again, there exists an $\omega \in \Lambda^p M$ such that $\Delta\omega = \alpha$. Hence we have that $\mathcal{H}^{p\perp} \subset \Delta(\Lambda^p M)$.

☐

This concludes the proof of Hodge theorem. Now we prove some important corollaries of this theorem. For this we need to define the following map.

**Definition 2.2.** *Define the Green's function* $G : \Lambda^p M \to \mathcal{H}^{p\perp}$ *as the projection map.*

Then it can be shown that $G$ commutes with $\Delta, d, \delta$. We use this property of $G$ to prove the following.

**Corollary 2.3.** $H^p_{dR}(M) \cong \mathcal{H}^p(M)$

*Proof.* Let $\alpha$ be any closed $p$-form on $M$.

$$\begin{aligned} \alpha &= d\alpha + \delta dG\alpha + H(\alpha) \\ &= d\alpha + \delta G d\alpha + H(\alpha) \\ &= d\delta G\alpha + H(\alpha) \end{aligned}$$

So $H(\alpha) = \alpha$. If $\alpha_1, \alpha_2 \in \mathcal{H}^p$ and $\alpha_1 - \alpha_2 = d\beta$, then

$$\begin{aligned} \langle d\beta, \alpha_1 - \alpha_2 \rangle &= \beta, \delta\alpha_1 - \delta\alpha_2 \rangle \\ &= \langle \beta, 0 \rangle \\ &= 0 \end{aligned}$$

Hence $\alpha_1 = \alpha_2$ and we get that $H^p_{dR}(M) \cong \mathcal{H}^p(M)$. ☐

This says that the de Rham cohomology groups are the same as the set of harmonic forms. In particular we get the following result.

**Corollary 2.4.** $H^p_{dR}(M)$ *is finite dimensional.*

Another theorem that follows from the Hodge theorem is the *Pöincare duality.*

**Corollary 2.5** (Pöincare Duality). *Define the mapping*

$$\begin{aligned} H^p_{dR}(M) \times H^{n-p}_{dR}(M) &\to \mathbb{R} \\ ([\phi],[\psi]) &\mapsto \int\limits_M \phi \wedge \psi \end{aligned}$$

*where* $\phi$ *and* $\psi$ *are closed forms. Then this is a non-singular pairing and consequently* $H^p_{dR}(M) \cong (H^{n-p}_{dR}(M))^*$.

*Proof.*

Take $\phi \neq 0$ harmonic in $H^p_{dR}(M)$. Then $\star\phi$ is harmonic as well since the Laplacian commutes with $\star$. So $\star\phi$ is in $H^{n-p}_{dR}(M)$ by corollary 2.3. So

$$\int\limits_M \phi \wedge \star\phi = \int\limits_M \langle \phi, \phi \rangle dV = ||\phi||^2 \neq 0$$

This shows that its a non-singular pairing and the duality follows. □

# 3 Hodge theory for complex manifolds

Hodge theorem can also be extended to Hermitian manifolds (complex manifolds with a Hermitian structure $\bar{\partial}$ on its tangent bundle). Observe that for Hermitian manifolds there are two different Laplace's operators corresponding to $\partial$ and $\bar{\partial}$.

$$\begin{aligned} \Delta^{\partial} &= \partial\partial^* + \partial^*\partial \\ \Delta^{\bar{\partial}} &= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \end{aligned}$$

where $\partial^* = - \star \bar{\partial} \star$ and $\bar{\partial}^* = - \star \partial \star$ forms the adjoints of $\partial$ and $\bar{\partial}$ respectively.

The Hodge theorem extends to each of these operators and we get the corresponding decompositions. Note that the cohomology corresponding to $\bar{\partial}$ is the Dolbeault cohomology $H^{p,q}(M)$. Following the same proof as for corollary 2.3, we get an isomorphism between $H^{p,q}(M)$ and $\bar{\partial}$-harmonic $(p,q)$-forms $\mathcal{H}^{p,q}(M)$. In place of Pöincare duality, we prove the *Serre duality* in this case. Let $\Lambda^{p,q}$ be the set of $(p,q)$-forms and $h^{p,q}$ (Hodge numbers) be the dimension of $H^{p,q}(M)$.

**Theorem 3.1.** *(Serre duality)* $H^{p,q}(M) \cong H^{n-p,n-q}(M)$ *and hence* $h^{p,q} = h^{n-p,n-q}$.

*Proof.* Define $\bar{\star} : \Lambda^{p,q} \to \Lambda^{n-p,n-q}$ to be the composition of Hodge star operator with complex conjugation. Then

$$\begin{aligned}
\bar{\star}\Delta^{\bar{\partial}}(\omega) &= \star\overline{(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\omega} \\
&= - \star (\partial \star \bar{\partial} \star + \star \partial \star \bar{\partial})\bar{w} \\
&= \bar{\partial}^*\bar{\partial}\bar{\star}\omega - \bar{\partial} \star \partial \star^2 \bar{\omega} \\
&= \bar{\partial}^*\bar{\partial}\bar{\star}\omega + \bar{\partial}\bar{\partial}^*\bar{\star}\omega \\
&= \Delta^{\bar{\partial}}\bar{\star}\omega
\end{aligned}$$

From this its clear that $\bar{\star}\omega$ is a $\mathbb{C}$-anti-linear isomorphism. □

If in addition, $M$ is a Kähler manifold[2], then $\Delta^{\bar{\partial}} = \Delta^{\partial}$ and the Laplacian of the underlying Riemannian manifold is $\Delta = 2\Delta^{\bar{\partial}}$. This would imply that $H^{p,q}(M) \subset \mathcal{H}^{p+q}(M)$. Note that the opposite inclusion does not hold because there could be harmonic $(p+q)$-forms which are not even $(p,q)$-forms. But since $\Delta^{\bar{\partial}}$ leaves $\Lambda^{p,q}$ invariant, the following decompostion holds.

$$\mathcal{H}^k(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$$

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[2] A Kähler manifold is an even dimensional Riemannian manifold $M$ with an almost complex structure $J^2 = -I$ on its tangent space, such that the metric satisfies $g(JX, JY) = g(X, Y)$, the 2-form (Kähler form) $\omega_g(X, Y) = g(JX, Y)$ is closed, and the almost complex structure $J$ is induced by a complex manifold structure on $M$.