Identities in Differential Geometry

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1 Second Bianchi Identity

Identity 1. *Let R be the $(4,0)$-curvature tensor for a Riemannian manifold M. Then the second Bianchi identity reads*

$$\nabla R(X,Y,Z,W,T) + \nabla R(X,Y,T,Z,W) + \nabla R(X,Y,W,T,Z) = 0$$

Proof. Since ∇R is a tensor, its enough to show this identity pointwise. Let p be an arbitrary point in M. To simplify the calculations we consider a parallel frame $\{e_1,\dots,e_n\}$ centered at p, so that $\nabla_{e_i} e_j(p) = 0$ for all $1 \le i,j \le n$. Then

$$\begin{aligned}
\nabla R(e_i,e_j,e_k,e_l,e_h) &= e_h(R(e_i,e_j,e_k,e_l)) - R(\nabla_{e_h} e_i, e_j, e_k, e_l) \\
&\quad - R(e_i, \nabla_{e_h} e_j, e_k, e_l) - R(e_i, e_j, \nabla_{e_h} e_k, e_l) - R(e_i, e_j, e_k, \nabla_{e_h} e_l) \\
&= e_h(R(e_i,e_j,e_k,e_l)) = e_h(R(e_k,e_l,e_i,e_j)) \\
&= \langle \nabla_{e_h}\nabla_{e_l}\nabla_{e_k} e_i - \nabla_{e_h}\nabla_{e_k}\nabla_{e_l} e_i + \nabla_{e_h}\nabla_{[e_k,e_l]} e_i, e_j \rangle
\end{aligned}$$

Expanding the other two tensors in a similar way and adding up gives

$$\begin{aligned}
&\nabla R(e_i,e_j,e_k,e_l,e_h) + \nabla R(e_i,e_j,e_l,e_h,e_k) + \nabla R(e_i,e_j,e_h,e_k,e_l) \\
&= \langle \nabla_{e_h}\nabla_{e_l}\nabla_{e_k} e_i - \nabla_{e_h}\nabla_{e_k}\nabla_{e_l} e_i + \nabla_{e_h}\nabla_{[e_k,e_l]} e_i, e_j \rangle \\
&+ \langle \nabla_{e_k}\nabla_{e_h}\nabla_{e_l} e_i - \nabla_{e_k}\nabla_{e_l}\nabla_{e_h} e_i + \nabla_{e_k}\nabla_{[e_l,e_h]} e_i, e_j \rangle \\
&+ \langle \nabla_{e_l}\nabla_{e_k}\nabla_{e_h} e_i - \nabla_{e_l}\nabla_{e_h}\nabla_{e_k} e_i + \nabla_{e_l}\nabla_{[e_h,e_k]} e_i, e_j \rangle \\
&= \langle R(e_l,e_h)\nabla_{e_k} e_i - \nabla_{[e_l,e_h]}\nabla_{e_k} e_i + \nabla_{e_k}\nabla_{[e_l,e_h]} e_i, e_j \rangle \\
&+ \langle R(e_h,e_k)\nabla_{e_l} e_i - \nabla_{[e_h,e_k]}\nabla_{e_l} e_i + \nabla_{e_l}\nabla_{[e_h,e_k]} e_i, e_j \rangle \\
&+ \langle R(e_k,e_l)\nabla_{e_h} e_i - \nabla_{[e_k,e_l]}\nabla_{e_h} e_i + \nabla_{e_h}\nabla_{[e_k,e_l]} e_i, e_j \rangle \\
&= R(e_l,e_h,\nabla_{e_k} e_i, e_j) + R(e_h,e_k,\nabla_{e_l} e_i, e_j) + R(e_k,e_l,\nabla_{e_h} e_i, e_j) \\
&+ R([e_k,e_l],e_h,e_i,e_j) - \langle \nabla_{[[e_k,e_l],e_h]} e_i, e_j \rangle \\
&+ R([e_l,e_h],e_k,e_i,e_j) - \langle \nabla_{[[e_l,e_h],e_k]} e_i, e_j \rangle \\
&+ R([e_h,e_k],e_l,e_i,e_j) - \langle \nabla_{[[e_h,e_k],e_l]} e_i, e_j \rangle \\
&= -\langle \nabla_{[[e_k,e_l],e_h]+[[e_l,e_h],e_k]+[[e_h,e_k],e_l]} e_i, e_j \rangle = 0
\end{aligned}$$

In the second to last equality we use the fact that $[e_i,e_j](p) = \nabla_{e_i} e_j(p) - \nabla_{e_j} e_i(p) = 0$ and that R is tensorial in all its arguments. The last equality follows from the Jacobi identity for Lie brackets. The general case for any set of vector fields follows by linearity. □

2 Contracted Bianchi Identity for Kähler manifolds

Identity 2. *For a Kaḧler manifold we have the following identities*

$$S_{,m} = Ric^k_{m,k}$$

$$S_{,m\bar{p}} = Ric^k_{m,k\bar{p}}$$

where S and Ric are the scalar and Ricci curvatures respectively.

Proof. The second Bianchi identity for Riemannian manifolds is given by

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0$$

where $R_{ijkl} = R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l})$ denotes the coefficients of the $(4,0)$ curvature tensor in local coordinates. We look at the same identity for Kähler manifolds. Note that for a Kähler metric, $R_{ijkl} = 0$ if either of the pairs (i,j) or (k,l) are of the same type. This means that the second Bianchi identity takes the following form

$$R_{i\bar{j}k\bar{l},m} + R_{i\bar{j}\bar{l}m,k} = 0$$

Contracting the indices $(i,\bar{l})$ and $(\bar{j},k)$ followed by a type change, we get

$$\begin{aligned}
&R_{i\bar{j}k\bar{l},m} = R_{i\bar{j}m\bar{l},k}\\
&g^{\bar{a}i}R_{i\bar{j}k\bar{l},m} = g^{\bar{a}i}R_{i\bar{j}m\bar{l},k}\\
&R^{\bar{a}}_{\bar{j}k\bar{l},m} = R^{\bar{a}}_{\bar{j}m\bar{l},k}\\
&Ric_{\bar{j}k,m} = Ric_{\bar{j}m,k}\\
&g^{b\bar{j}}Ric_{\bar{j}k,m} = g^{b\bar{j}}Ric_{\bar{j}m,k}\\
&Ric^b_{k,m} = Ric^b_{m,k}\\
&S_{,m} = Ric^k_{m,k}
\end{aligned}$$

Here S is the scalar curvature. First equality is obtained by the skew symmetry of the curvature tensor in the first two indices. Ric_{ij} denotes the coefficients of the Ricci curvature in local coordinates. The final expression is called the contracted Bianchi identity for Kähler manifolds. It differs from the Riemannian case by a factor of $\frac{1}{2}$. We can take another covariant derivative to get the following identity.

$$S_{,m\bar{p}} = Ric^k_{m,k\bar{p}}$$

□

3 A Kähler identity

Identity 3.
$$\left(\frac{\sqrt{-1}}{2}\right)^2 \partial\phi \wedge \overline{\partial\phi} \wedge \omega_g^{n-2} = \frac{1}{n(n-1)} \left(|\partial\phi|^2 - |\bar{\partial}^*\phi|^2\right) \omega_g^n$$

Proof. Its enough to prove this identity point-wise since both the sides are tensorial. So we consider local co-ordinates $\{z_1, \ldots, z_n\}$ such that $g_{i\bar{j}}(p) = \delta_{ij}$. We prove the identity for the 1-form $\phi = \phi_1 d\bar{z}_1$. The general case can be done in a similar manner.

$$\partial\phi = \frac{\partial\phi_1}{\partial z_i} dz_i \wedge d\bar{z}_1 = -\frac{\partial\phi_1}{\partial z_i} d\bar{z}_1 \wedge dz_i$$
$$\bar{\partial}\overline{\phi} = \bar{\partial}(\overline{\phi_1} dz_1) = -\frac{\partial\overline{\phi_1}}{\partial\overline{z_i}} dz_1 \wedge d\bar{z}_i$$

It follows that

$$\partial\phi \wedge \bar{\partial}\overline{\phi} = \left|\frac{\partial\phi_1}{\partial z_i}\right|^2 dz_1 \wedge d\bar{z}_1 \wedge dz_i \wedge d\bar{z}_i$$

Since the frame is orthonormal, the Kähler form can be written as

$$\omega_g = \frac{\sqrt{-1}}{2} \sum_i dz_i \wedge d\bar{z}_i$$

Its easy to see that 2-forms commute with each other. So we get

$$\omega_g^{n-2} = \left(\frac{\sqrt{-1}}{2}\right)^{n-2} \left(\sum_i dz_i \wedge d\bar{z}_i\right)^{n-2}$$
$$= \frac{\sqrt{-1}}{2}(n-2)! \sum_{i<j} \left(dz_1 \wedge d\bar{z}_1 \wedge \ldots \widehat{dz_i \wedge d\bar{z}_i} \wedge \ldots \widehat{dz_j \wedge d\bar{z}_j} \wedge \ldots dz_n \wedge d\bar{z}_n\right)$$

where $\widehat{.}$ is used to highlight the terms which are absent.
Therefore

$$\left(\frac{\sqrt{-1}}{2}\right)^2 \partial\phi \wedge \bar{\partial}\overline{\phi} \wedge \omega_g^{n-2} = \left(\frac{\sqrt{-1}}{2}\right)^n (n-2)! \left[\left|\frac{\partial\phi_1}{\partial z_2}\right|^2 + \left|\frac{\partial\phi_1}{\partial z_3}\right|^2 + \ldots + \left|\frac{\partial\phi_1}{\partial z_n}\right|^2\right] .vol$$

We're done with the left hand side. Now we compute the right hand side and show that it matches with the above.

$$
\begin{aligned}
|\partial\phi|^2 &= \left\langle \frac{\partial\phi_1}{\partial\bar{z}_i} d\bar{z}_1 \wedge dz_i, \frac{\partial\phi_1}{\partial z_j} d\bar{z}_1 \wedge dz_j \right\rangle \\
&= \frac{\partial\phi_1}{\partial z_i} . \overline{\frac{\partial\phi_1}{\partial z_i}} \delta_{ij} \\
&= \sum_i \left| \frac{\partial\phi_1}{\partial z_i} \right|^2
\end{aligned}
$$

Using the definition of the $\star$-operator, $\alpha \wedge \star\beta = \langle\alpha, \beta\rangle vol$, we get

$$
\begin{aligned}
\partial^*\phi &= -\star\bar{\partial}\star\phi = -\frac{\partial\phi_1}{\partial z_1} \\
|\partial^*\phi|^2 &= \left| \frac{\partial\phi_1}{\partial z_1} \right|^2
\end{aligned}
$$

Therefore,

$$
|\partial\phi|^2 - |\partial\phi^*|^2 = \left[\left| \frac{\partial\phi_1}{\partial z_2} \right|^2 + \left| \frac{\partial\phi_1}{\partial z_3} \right|^2 + \ldots + \left| \frac{\partial\phi_1}{\partial z_n} \right|^2 \right]
$$

This gives the final result since $w_g^n = \left(\frac{\sqrt{-1}}{2}\right)^n n!.vol$.

□

4 Derivative of $\det(A)$

This is a simple computation in linear algebra. But its useful to know. We come across the derivative of the determinant at many places in geometry. For example, the formula for the Ricci curvature of a Kähler manifold involves finding the derivative of $\log(\det(g_{i\bar{j}}))$.

Identity 4.

$$\frac{1}{det(A)}\frac{d}{dt}\det A = a'_{ij}a^{ij}$$

where $A(t) = (a_{ij}(t))$ *and* $A^{-1}(t) = (a^{ij}(t))$.

Proof. $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$ where A_i's are the row vectors of A. Then it can be shown from the definition of a determinant that

$$\frac{d}{dt}det(A) = \begin{vmatrix} \frac{d}{dt}A_1 \\ A_2 \\ \vdots \\ A_n \end{vmatrix} + \begin{vmatrix} A_1 \\ \frac{d}{dt}A_2 \\ \vdots \\ A_n \end{vmatrix} + \ldots + \begin{vmatrix} A_1 \\ A_2 \\ \vdots \\ \frac{d}{dt}A_n \end{vmatrix}$$

Expanding the i^{th} determinant along the i^{th} row we get

$$\begin{vmatrix} A_1 \\ \vdots \\ \frac{d}{dt}A_i \\ \vdots \\ A_n \end{vmatrix} = \sum_j a'_{ij}A_{ij}$$

where A_{ij}'s are the corresponding cofactors. But $A_{ij} = det(A)a^{ij}$. The final result is obtained by combining these three equations. □