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Minimal Surfaces and Mean curvature flow

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1 Introduction

In this project, we try to construct minimal surfaces using the method of mean curvature flow. In recent years, the theory of manifolds evolving by geometric flows has been developed extensively. Mean curvature flow is one such PDE, that occurs naturally as the gradient flow for the area functional of a surface. It is easy to see that equilibrium points of MCF are minimal surfaces. This motivates us to use evolution of surfaces by MCF as an intricate but effective method for constructing minimal surfaces.

2 Background

2.1 Minimal surfaces

Let Ω be a domain in $\mathbf{R}^2$ and $X(u,v):\Omega\to\mathbf{R}^3$ be a smooth surface embedded in $\mathbf{R}^3$. Let the induced metric on the surface be denoted by the matrix g and the second fundamental form by the matrix h. Then the mean curvature H of the surface at a given point (u,v) is given by the following equation,

$$H(u,v)=tr(hg^{-1})(u,v)$$

where tr stands for the trace of the matrix. We say that $X(u,v)$ is a minimal surface if $H(u,v)=0$ for all $(u,v)\in\Omega$.

2.2 Mean curvature Flow

Definition 2.1. *Let $X_t(u,v)$ be a one-parameter family of smooth surfaces immersed in $\mathbf{R}^3$. Then we say that these surfaces evolve by mean curvature flow if X_t satisfies the following PDE for all u,v in its domain,*

$$\frac{\partial X_t}{\partial t}=H\bar{N} \tag{2.1}$$

where H is the mean curvature and $\bar{N}$ is the outward normal to the surface at a given point.

Example 2.1.1. *Given a sphere $S_R\subset\mathbf{R}^3$ of radius R, we will show that it shrinks to a point through a family of spheres S_r of decreasing radius $r(t)$, under MCF.*

$$S_r.S_r=r^2$$

Differentiating wrt t,

$$\begin{aligned}
&\frac{dS_r}{dt}.S_r=r\frac{dr}{dt}\\
&H\bar{N}.S_r=r\frac{dr}{dt}\\
&-\frac{2}{r}\frac{S_r}{\sqrt{S_r.S_r}}.S_r=r\frac{dr}{dt}\\
&\frac{dr}{dt}=-\frac{2}{r}
\end{aligned}$$

which can easily be solved to get $r(t) = \sqrt{R^2 - 2t}$. Thus S_R shrinks to a point at time $t = \frac{R^2}{2}$.

It is clear that the equilibrium points of the mean curvature flow are minimal surfaces.

3 CONSTRUCTING MINIMAL SURFACES USING MCF

We will analyze how surfaces of revolution evolve under mean curvature flow. We intend to show that initial surfaces can be chosen so that the flow will converge to a minimal surface. Because of the rotational symmetry of MCF, it is clear that the initial surface will preserve its symmetry throughout the evolution. Thus, assuming that a solution exists, we can represent it by the following parametrization,

$$X(u, v, t) = (\phi(u, t)\cos(v), \phi(u, t)\sin(v), \psi(u, t))$$

We assume that ϕ and ψ are smooth functions. For a given time t, this represents a surface obtained by revolving the curve $(\phi(u, t), \psi(u, t))$ in the xz plane around the $y-$axis.

The normal vector and the mean curvature to this surface are given by the following equations,

$$\begin{aligned}
\bar{N} &= \frac{sgn(\phi)}{\sqrt{\phi'^2 + \psi'^2}} \begin{pmatrix} -\psi'\cos(v) \\ -\psi'\sin(v) \\ \phi' \end{pmatrix} \\
H &= \frac{sgn(\phi)}{\sqrt{\phi'^2 + \psi'^2}} \left(\frac{\psi'}{\phi} + \frac{\psi''\phi' - \phi''\psi'}{\phi'^2 + \psi'^2} \right)
\end{aligned}$$

Here the derivatives of ϕ and ψ are taken with respect to the u variable. Substituting these in Eq. 2.1 gives the following set of differential equations,

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= \frac{-\psi'}{\phi'^2 + \psi'^2} \left(\frac{\psi'}{\phi} + \frac{\psi''\phi' - \phi''\psi'}{\phi'^2 + \psi'^2} \right) = F_1(\phi, \phi', \psi', \phi'', \psi'') \\
\frac{\partial \psi}{\partial t} &= \frac{\phi'}{\phi'^2 + \psi'^2} \left(\frac{\psi'}{\phi} + \frac{\psi''\phi' - \phi''\psi'}{\phi'^2 + \psi'^2} \right) = F_2(\phi, \phi', \psi', \phi'', \psi'')
\end{aligned} \tag{3.1}$$

The initial surface at $t = 0$ will be represented by $X_0(u, v) = (\phi_0(u)\cos(v), \phi_0(u)\sin(v), \psi_0(u))$.

From Huisken's work on mean curvature flow, we know that any convex compact surface will deform to a sphere very rapidly under MCF. In particular, he shows that no singularities are formed during this process. As we have seen, a sphere will evolve through a family of spheres of decreasing radius that shrinks to a point in finite time. So taking all these into consideration, for obtaining a minimal surface through MCF, we choose a concave surface of revolution as our initial surface.

3.1 Existence and Uniqueness of Solution

We use Picard's iteration method to show that a solution exists for Eq. 3.1 for a short time.

Theorem 3.1. *(Picard's theorem) Let $y, f \in \mathbf{R}^d$; $f(t, y)$ continuous on a parallelepiped $R = \{(t, y) : t_0 \le t \le t_0 + a, |y - y_0| \le b\}$ and uniformly Lipschitz continuous wrt y. Let M be a bound for $|f(t, y)|$ on R; $\alpha = \min(a, \frac{b}{M})$. Then*

$$y' = f(t, y), \qquad y(t_0) = y_0$$

has a unique solution $y = y(t)$ on $[t_0, t_0 + \alpha]$.

Proof: The proof follows by successive approximations of the solution, by functions of the form

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s)) ds$$

with $y_0(t) = y_0$. It can be shown that these approximations will uniformly converge to a unique solution for the given differential equation.

Theorem 3.2. *Given Eq. 3.1 with $(\phi(u, 0), \psi(u, 0)) = (\phi_0(u), \psi_0(u))$ representing a hyperboloid. Then there exists a unique solution for some time $t \in [0, \epsilon]$.*

Let $y = (\phi, \psi)$, $F = (F_1, F_2,)$ and $y_0 = (\phi_0, \psi_0)$. Then Eq. 3.1 can be written as

$$y' = F(t, y), \qquad y(t_0) = y_0$$

The idea is to show that F is uniformly Lipschitz wrt each of the variables and hence a unique solution exists for a short time by Picard's theorem.

4 Thread kept on a Catenoid

4.1 Introduction

We try to find the shape assumed by a thin thread kept on a catenoidal soap bubble. The thread will assume the shape that minimizes its total bending energy. Assuming that the equation representing the thread as a smooth curve in $\mathbf{R}^3$ is parametrized by arc-length, the energy functional is given by the following equation,

$$E = \int_0^L k^2(t) dt \tag{4.1}$$

where $k(t)$ is the curvature of the curve at t and L is the total length of the thread. We try to minimize this functional subject to the constraint that the curve should always remain on the catenoid. We neglect the effects of gravity in this equation for energy.

4.2 Minimizing the Energy

Let the equation of a catenoid embedded in $\mathbf{R}^3$ be given by,

$$x(u,v) = (\cosh(v)\cos(u), \cosh(v)\sin(u), v)$$

Let $\alpha(t) = x(u(t), v(t))$ be the equation of the curve on the catenoid that minimizes the energy functional given by 4.1. We will calculate the curvature of $\alpha(t)$ at a given point.

$$\begin{aligned} \dot{\alpha} &= x_u\dot{u} + x_v\dot{v} \\ \ddot{\alpha} &= x_{uu}\dot{u}^2 + 2x_{uv}\dot{v}\dot{u} + x_{vv}\dot{v}^2 + x_u\ddot{u} + x_v\ddot{v} \end{aligned} \tag{4.2}$$

Let Γ^k_{ij} denote the i,j,k th Christoffel symbol. Then we have the following set of basic equations from differential geometry,

$$\begin{aligned} x_{uu} &= \Gamma^1_{11}x_u + \Gamma^2_{11}x_v + eN \\ x_{uv} &= \Gamma^1_{12}x_u + \Gamma^2_{12}x_v + fN \\ x_{vv} &= \Gamma^1_{22}x_u + \Gamma^2_{22}x_v + gN \end{aligned} \tag{4.3}$$

where N is the normal to the surface and e, f, g are the components of the second fundamental form. Substituting Eq. 4.3 in Eq. 4.2 gives,

$$\begin{aligned} \ddot{\alpha} &= (\Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 + \ddot{u})x_u + (\Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 + \ddot{v})x_v + (e\dot{u}^2 + 2f\dot{u}\dot{v} + g\dot{v}^2)N \\ &= Ax_u + Bx_v + CN \end{aligned} \tag{4.4}$$

For the above parametrization of the catenoid, u, v are isothermal co-ordinates. Therefore the components of the first and second fundamental forms are given by,

$$\begin{aligned} &x_u.x_u = x_v.x_v = E = G = \cosh^2(v) \\ &x_u.x_v = F = 0 \\ &e = -1 \qquad f = 0 \qquad g = 1 \end{aligned}$$

So

$$\begin{aligned} k^2 = |\ddot{\alpha}|^2 &= E(A^2 + B^2) + C^2 \\ &= (2\tanh(v)\dot{u}\dot{v} + \ddot{u})^2\cosh^2(v) + (\tanh(v)(\dot{v}^2 - \dot{u}^2) + \ddot{v})^2\cosh^2(v) + (\dot{v}^2 - \dot{u}^2)^2 \\ &= \cosh^2(v)\dot{u}^4 + (-2 + 2\sinh^2(v))\dot{u}^2\dot{v}^2 + \cosh^2(v)\dot{v}^4 + \cosh^2(v)\ddot{u}^2 + \cosh^2(v)\ddot{v}^2 + \\ &\quad 2\sinh(2v)\dot{u}\ddot{u}\dot{v} - \sinh(2v)\dot{u}^2\ddot{v} + \sin(2v)\dot{v}^2\ddot{v} = F(u,v) \end{aligned} \tag{4.5}$$

The idea is to use variational calculus to minimize the functional $E = \int_0^L k^2(t)dt$. Since $\alpha(t)$ is the energy minimizing curve, we argue that it has to be a critical point of the energy functional. Therefore, the functional derivative of E should vanish at $\alpha(t) = x(u(t), v(t))$. Using this idea we can generate a set of differential equations that represents the solution to the minimization problem.

Let $(u(t), v(t))$ be the smooth curve in the uv-plane whose image under the map x is the curve $\alpha(t)$. Define

$$Q_{h_1} = \lim_{\lambda\to 0}\int_0^L \frac{F(u+\lambda h_1, v) - F(u,v)}{\lambda}\,dt \tag{4.6}$$

and

$$Q_{h_2} = \lim_{\lambda\to 0}\int_0^L \frac{F(u, v+\lambda h_2) - F(u,v)}{\lambda}\,dt \tag{4.7}$$

where h_1 and h_2 are arbitrary smooth functions defined on $[0, L]$. From calculus of variations, we know that $\alpha(t)$ minimizes E if and only if $Q_{h_1} = Q_{h_1} = 0$, for all possible h_1 and h_2 within its range. Expanding the above expression for Q_{h_1} using Eq. 4.5 and taking limits gives

$$Q_{h_1} = \int_0^L A_1(t)\dot{h}_1(t)\,dt + \int_0^L A_2(t)\ddot{h}_1(t)\,dt \tag{4.8}$$

where,

$$\begin{aligned}
A_1(t) &= 2\ddot{u}(t)\dot{v}(t)\sinh(2v(t)) - 2\dot{(u)}(t)\ddot{v}(t)\sinh(2v(t)) + \\
&\quad 2\dot{u}(t)\dot{v}(t)^2\left(2\sinh^2(v(t)) - 2\right) + 4\dot{u}(t)^3\left(\sinh^2(v(t)) + 1\right) \\
A_2(t) &= 2\ddot{u}(t)\cosh^2(v(t)) + 2\dot{u}(t)\dot{v}(t)\sinh(2v(t))
\end{aligned}$$

Integrating the second term in Eq. 4.8 by parts,

$$\begin{aligned}
Q_{h_1} &= \int_0^L \left(A_1(t) - \dot{A}_2(t)\right)\dot{h}_1(t)\,dt + \left(A_2(L)\dot{h}_1(L) - A_2(0)\dot{h}_1(0)\right) \\
&= 0
\end{aligned} \tag{4.9}$$

This equality should hold for all smooth functions h_1 defined on $[0, L]$. It is easy to see that h_1 can be chosen appropriately to show that,

$$\begin{aligned}
A_1 - \dot{A}_2 &\equiv 0 \\
A_2(0) = A_2(L) &= 0
\end{aligned} \tag{4.10}$$

Doing similar calculations for Q_{h_2} yields

$$Q_{h_2} = \int_0^L B_1(t)h_2(t)\,dt + \int_0^L B_2(t)\dot{h}_2(t)\,dt + \int_0^L B_3(t)\ddot{h}_2(t)\,dt$$

where

$$\begin{aligned}
B_1(t) =& 2\ddot{u}(t)^2\sinh(v(t))\cosh(v(t)) - 2\dot{u}(t)^2\ddot{v}(t)\cosh(2v(t)) + 4\dot{u}(t)^2\dot{v}(t)^2\sinh(v(t))\cosh(v(t)) + \\
& 2\dot{u}(t)^4\sinh(v(t))\cosh(v(t)) + 4\dot{u}(t)\ddot{u}(t)\dot{v}(t)\cosh(2v(t)) + 2\ddot{v}(t)^2\sinh(v(t))\cosh(v(t)) + \\
& 2\dot{v}(t)^4\sinh(v(t))\cosh(v(t)) + 2\dot{v}(t)^2\ddot{v}(t)\cosh(2v(t))
\end{aligned} \tag{4.11}$$

$$B_2(t) = 2\left(\dot{u}(t)^2\dot{v}(t)(\cosh(2v(t))-3) + \dot{u}(t)\ddot{u}(t)\sinh(2v(t)) + 2\dot{v}(t)^3\cosh^2(v(t)) + \dot{v}(t)\ddot{v}(t)\sinh(2v(t))\right) \tag{4.12}$$

$$B_3(t) = -\dot{u}(t)^2\sinh(2v(t)) + 2\ddot{v}(t)\cosh^2(v(t)) + \dot{v}(t)^2\sinh(2v(t)) \tag{4.13}$$

Integrating by parts and by using the same argument as before, we obtain the following set of differential equations and initial conditions.

$$\begin{aligned} &B_1(t) - \dot{B}_2(t) + \ddot{B}_3(t) = 0 \\ &B_3(0) = B_3(L) \\ &B_2(L) = \dot{B}_3(L) \\ &B_2(0) = \dot{B}_3(0) \end{aligned} \tag{4.14}$$

This set of differential equations and initial conditions given by Eq. 4.10 and Eq. 4.14, can be solved numerically.