LINEAR MATHEMATICS IN INFINITE DIMENSIONS
Signals Boundary Value Problems and Special Functions

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2nd Beta Edition

In its "2nd Beta Edition" this document is meant for private or class room consumption. As such I (gerlach.1@osu.edu) would appreciate feedback about the strengths as well as weaknesses (e.g. lack of clarity, insufficient treatment/explanation, topics that you would like to have seen developed, etc.) of this text book.

Preface

Mathematics is the science of measurement, of establishing quantitative relationships between the properties of entities. The entities being measured occupy the whole spectrum of abstractness, from first-level concepts, which are based on perceptual data obtained by direct observation, to high-level concepts, which are further up in the edifice of knowledge. Furthermore, being the science of measurement, mathematics provides the logical glue that cements and cross-connects the structural components of this edifice.

1. The effectiveness and the power of mathematics (and more generally of logic) in this regard arises from the most basic fact of nature: *to be is to be something*, i.e. *to be is to be a thing with certain distinct properties*, or: *to exist means to have specific properties*. Stated negatively: *a thing cannot have and lack a property at the same time*, or: *in nature contradictions do not exist*, a fact already identified by the father of logic[1] some twenty-four centuries ago.

Mathematics is based on this fact, and on the existence of a consciousness (a physicist, an engineer, a mathematician, a philosopher, etc.) capable of identifying it. Thus mathematics is neither intrinsic to nature (reality), apart from any relation to man's mind, nor is it based on a subjective creation of a man's consciousness detached from reality. Instead, mathematics furnishes us with the means by which our consciousness grasps reality in a quantitative manner. It allows our consciousness to grasp, in numerical terms, the micro-cosmic world of subatomic particles, the macro-cosmic world of the universe and everything in between[2]. In fact, this is what mathematicians are sup-

[1] Aristotle, the Greek philosopher, 384-322 B.C.

[2] The objectivity of mathematics and its relation to physics is explicated in "The Role of Mathematics and Philosophy", Chapter 7, in *THE LOGICAL LEAP: Induction in Physics* by David Harriman (New York: Penguin Group, 2010).

posed to do, to develop general methods for formulating and solving physical problems of a given type.

In brief, mathematics highlights the potency of the mind, its cognitive efficacy in grasping the nature of the world. This potency arises from the mind ability to form concepts, a process which is made most explicit by the science of mathematics.[3]

2. Mathematics is an inductive discipline first and a deductive discipline second. This is because, more generally, induction preceeds deduction[4]. Without the former, the latter is impossible. Thus, the validity of the archetypical deductive reasoning process
"Socrates is a man. All men are mortal. Hence, Socrates is a mortal."
depends on the major premise "All men are mortal." It constitutes an identification of the nature of things. It is arrived at by a process of induction, which, in essence, consists of observing the facts of reality, of identifying their essential properties, and of integrating them with what is already known into new knowledge – here, a relationship between "man" and "mortal". In mathematics, inductively formed conclusions, analogous to this one, are based on motivating examples and illustrated by applications.

Mathematics thrives on examples and applications. In fact, it owes its birth and growth to them. This is manifestly evidenced by the thinkers of Ancient Greece who "measured the earth", as well as by those of the Enlightenment, who "calculated the motion of bodies". It has been rightfully observed that both logical rigor and applications are crucial to mathematics. Without the first, one cannot be certain that one's statements are true. Without the second it does not matter one way or the other[5]. These lecture notes cultivate both. As a consequence they can also be viewed as an attempt to make up for an error committed by mathematicians through most

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[3]Being a philosopher, Leonard Peikoff in his *Objectivism: The Philosophy of Ayn Rand* (New York: Penguin books, 1993, p. 90) describes the role of mathematics this way:

> "The mathematician is the exemplar of conceptual integration. He does professionally and in numerical terms what the rest of us do implicitly and have done ever since childhood, to the extent that we exercise our distinctive human capacity".

[4] "The Structure of Inductive Reasoning", Section 1.5, in *THE LOGICAL LEAP: Induction in Physics* by David Harriman (New York: Penguin Group, 2010, p. 29-35).

[5]David Harriman, "Enlightenment Science and Its Fall", *The Objective Standard;* 1(1): 83-117, Spring 2006;

of history – the Platonic error of denigrating applications[6].

This Platonic error, which arises from placing mathematical ideas prior to their physical origin, has metastasized into the invalid notion '"pure" mathematics'. It is a post-Enlightenment (Kantian) fig leaf[7] for the failure of theoretical mathematicians to justify the rigor and the abstractness of the concepts they have been developing. The roots of this failure are expressed in the inadvertent confession of the chairman of a major mathematics department: "We are all Platonists in this department." Plato and his descendants declared that reality, the physical world, is an imperfect reflection of a purer and higher mystical dimension with a gulf separating the two. That being the case, they aver that "pure" mathematics – and more generally the " a priori" – deals only with this higher dimension, and not with the physical world, which they denigrate as gross and imperfect, and dismiss as mere appearances.

With the acceptance – explicit or implicit – of such a belief system, "pure" mathematics has served as a license to misrepresent theoretical mathematics as a set of floating abstractions cognitively disconnected from the real world. The modifier "pure" has served to intimidate the unwary engineer, physicist or mathematician into accepting that this disconnect is the price that mathematics must pay if it is to be rigorous and abstract.

Ridding a culture's mind from impediments to epistemic progress is a non-trivial task. However, a good first step is to banish detrimental terminology, such as "pure" mathematics, from discourses on mathematics and replace it with an appropriate term such as *theoretical mathematics*. Such a replacement is not only dictated by its nature, but it also tends to reinstate the intellectual responsibility among those who need to live up to their task of justifying rigor (i.e. precision) and abstractness.

3. Mathematics is both complex and beautiful. The complexity of mathematics is a reflection of the complexity of the relationships that exist in the universe. The beauty of mathematics is a reflection of the ability of the human mind to identify them in a unit-economical way[8] : the more eco-

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[6]ibid.

[7]more precisely, a rationalization, i.e. a cover-up, namely a process of providing one's emotions with spurious justifications. ("Philosophic Detection" in Ayn Rand, *Philosophy: Who Needs It*, New American Library, Penguin Group Inc., New York, 1984.)

[8]The *principle of unit-economy* (also known informally as the "crow epistemology") according to which perceptual and conceptual data get compressed into fewer units, was first identified explicitly in "The Cognitive role of Concepts", Chapter 7 in Ayn Rand,

nomical the identification of a constellation of relationships, the more man's mind admires it. Beauty is not in the eyes of the beholder. Instead it is giving credit where credit is due – according to an objective standard. In mathematics that standard is the principle of unit economy. Its purpose is the condensation of knowledge, from the perceptual level all the way to the conceptual at the highest level.

4. Linearity is as fundamental to mathematics as it is to our mind in forming concepts. The transition from recognizing that $x + y = x + y$ to the act of grasping that $a + a = 2a$ is the explicit starting point of a conceptual consciousness grasping nature in mathematical terms with linearity at the center. Thus it is not an accident that linear mathematics plays its pervasive role in our comprehending the nature of nature around us. In fact, it would not be an exaggeration to say that "Linearity is the exemplary method – simple and primitive – for our grasping of nature in conceptual terms". The appreciation of this fact is found in that nowadays virtually every college and university offers a course in linear algebra, with which we assume the reader is familiar.

5. Twentieth century mathematics is characterized by an inflationary version of Moore's Law. Moore's Law expresses the observation that the number of transistors that fit onto a microchip doubles every two years. This achievement has been a boon to everybody. It put a computer into nearly every household.

The mathematical version of Moore's Law expresses the observation that, up to the Age of Enlightenment, all of Man's mathematical achievements fit into a four-volume book; the achievements up to, say, 1900 fit into a fourteen-volume tome, while the mathematical works generated during the twentieth century take up a whole floor of a typical university library.

Such abundance has its delightful aspects, but it is also characterized by repetitions and non-essentials. This cannot go on for too long. Such an increase ultimately chokes itself off.

One day, confronted with an undifferentiated amorphous juxtaposition of mathematical works, a prospective scientist/engineer/physicist/mathematician might start wondering: "I know that mathematics is very important, but am I learning the *right kind* of mathematics?"

Such a person is looking for orientation as to what is essential, i.e. what is

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Introduction to Objectivist Epistemology, 2nd Edition, edited by H. Binswanger and L. Peikoff. Penguin Books, Inc., New York, 1990

fundamental, and what is not. It has been said that the value of a book[9] , like that of a definition[10], can be gauged by the extent to which it spells out the essential, but omits the nonessential, which is, however, left implied. With that in mind, this text develops from a single perspective six mathematical jewels (in the form of six chapters) which lie at the root twentieth century science.

Another motivation for making the material of this text accessible to a wider audience is that it solves a rather pervasive problem. Most books which the author has tried to use as class texts either lacked the mathematics essential for grasping the nature of waves, signals, and fields, or they misrepresented it as a sequence of disjoint methods. The student runs the danger of being left with the idea that the mathematics consists of a set of *ad hoc* recipes with an overall structure akin to the proverbial village of squat one-room bungalows instead of a few towering skyscrapers well-connected by solid passage ways.

6. We extend and then apply several well-known ideas from finite dimensional linear algebra to infinite dimensions. This allows us to grasp readily not only the overall landscape but it also motivates the key calculations whose purpose is to connect and cross-link the various levels of abstraction in the constructed edifice. Even though the structure of these ideas have been developed in linear algebra, the motivation for doing so and then using them comes from engineering and physics. In particular, the goal is to have at one's disposal the constellation of mathematical tools for a graduate course in electromagnetics and vibrations for engineers or electrodynamics and quantum mechanics for physicists. The benefits to an applied mathematician is the acquisition of nontrivial mathematics from a cross-disciplinary perspective.

All key ideas of linear mathematics in infinite dimensions are already present with waves, signals, and fields whose domains are one-dimensional. The transition to higher dimensional domains is very smooth once these ideas have been digested. This transition does, however, have a few pleasant surprises. They come in the form of special functions, whose existence and properties are a simple consequence of the symmetry properties of the Euclidean plane (or Euclidean three-dimensional space). These properties consist of the

[9]Question and answer period following "You and Your Research" by Richard Hamming. `http://www.cs.virginia.edu/~robins/YouAndYourResearch.html`
Also see the Appendix, page 477.

[10]"Definitions", Chapter 5 in Ayn Rand, *Introduction to Objectivist Epistemology, 2nd Edition*, edited by H. Binswanger and L. Peikoff. Penguin Books, Inc., New York, 1990

invariance under translations and rotations of distance measurements and of the shapes of propagating waves.

7. What is the status of the concept "infinite" appearing in the title of this text? Quite generally the concept "infinite" is invalid metaphysically but valid mathematically.

In the sense of metaphysics (i.e. pertaining to reality, to the nature of things, to existence) infinity falls into the category of invalid concepts, namely attempts to integrate errors, contradictions, or false propositions into something meaningful. Infinity as a metaphysical idea is an invalid concept because metaphysically it is only concretes that exist, and concretes are finite, i.e. have definable properties. An attempt to impart metaphysical significance to infinity is an attempt to rewrite the nature of reality.

However, in mathematics infinity is a well defined concept. It has a definite purpose in mathematical calculations. It is a mathematical method which is made precise by means of the familiar δ-ε process of going to the limit. This text develops only concepts which by their nature are valid. Included is the concept "infinity", which, properly speaking, refers to a mathematical method.

8. The best way to learn something is to teach it. In order to facilitate this motto of John A. Wheeler, the material of this book has been divided into fifty lecture sessions. This means that there is one or two key ideas between "Lecture n" and "Lecture $n+1$", where $n = 1, \cdots 50$. Often the distance between n and $n+1$ extends over more pages than can be digested in a forty-eight minute session. However, the *essentials* of the nth Lecture are developed in a small enough time frame. Thus the first four or five pages following the heading "Lecture n" set the direction of the development, which is completed before the start of lecture "Lecture $n+1$".

Such a division can be of help in planning the schedule necessary to learn everything.

9. It is not necessary to digest the chapters in sequential order. A desirable alternative is to start with Sturm-Liouville theory (chapter 3) before proceeding systematically with the other chapters. Moreover, there is obviously nothing wrong with diving in and exploring each chapter according to one's background and predilections. The opening remarks of each one point out how linear algebra relates it to the others.

10. **Acknowledgments:** The author would like to thank Danai Torrungrueng for valuable comments and Wayne King from the Speech and Hearing Science Department for many fruitful conversations on wavelet theory and

multiresolution analysis.

Ulrich H. Gerlach

Columbus, Ohio, March 24, 2009

Foreword to the Second Edition (*tentative*)

TBD

Mathematics is the language of Physics. Why?

Mathematics is beautiful. Why? Is its beauty in "the eyes of the beholder"? Is beauty an attribute intrinsic to mathematics? Or is it an objective attribute?

Does this book address these issues? If so, how?

Columbus, Ohio, January 14, 2016

Contents

Chapter 0

Introduction

Lecture 1

The main focus of the next several chapters is on the mathematical framework that underlies *linear* systems arising in physics, engineering and applied mathematics. Roughly speaking, we are making a generalization from the theory of linear transformation on *finite* dimensional vector space to the theory of linear operators on *infinite* dimensional vector spaces as they occur in the context of homogeneous and inhomogeneous boundary value and initial value problems.

The key idea is linearity. Its *geometrical imagery* in terms of vector space, linear transformation, and so on, is a key ingredient for an efficient comprehension and appreciation of the ideas of linear analysis to be developed. Thus it is very profitable to repeatedly ask the question: *What does this correspond to in the case of a finite dimensional vector space*?

Here are some examples of what we shall generalize to the infinite dimensional vector case:

I. Solve each of the following linear algebra problems

1. $A\vec{u} = 0$ "Homogeneous problem"
2. $A\vec{u} = \vec{b}$ "Inhomogeneous problem"
3. $AG = I$ "Inverse of A"

These are archetypical problems of linear algebra. (If 1. has a non-trivial solution, then 2. has infinitely many solutions or none at all, depending on $\vec{b}$, and 3. has none.)

More generally we ask: For what values of λ do the following have a solution (and for what values do they not):

1. $(A - \lambda B)\vec{u} = 0$
2. $(A - \lambda B)\vec{u} = \vec{b}$
3. $(A - \lambda B)G = I$

Of greatest interest to us is the generalization in which A is (part of) a differential operator with in general non-constant coefficients.

As we know from linear algebra, these three types of problems are closely related, and consequently this must also be the case for our generalization to linear differential equations, ordinary as well as partial. In fact, these three types are called

1. Homogeneous boundary or initial value problems;
2. Inhomogeneous problems;
3. Green's function problems.

II. There is another idea which we shall extend from the finite to the infinite dimensional case. Consider the eigenvalue equation

$$Au = \lambda Iu\,.$$

Let us suppose that there are enough eigenvectors to span the whole vector space, but that at least one eigenvalue is degenerate, i.e., it has more than one eigenvector. In that case, the vector space has an eigenbasis, but it is not unique. Eigenvectors, including those used for a basis, derive their physical and geometrical significance from eigenvalues. Indeed, eigenvalues serve as labels for eigenvectors. Consequently, the lack of enough eigenvalues to distinguish between different eigenvectors in a particular eigensubspace introduces an intolerable ambiguity in our physical and geometrical picture of the inhabitants of this subspace.

In order to remedy this deficiency one introduces another matrix, say T, whose eigenvectors are also eigenvectors of A, but whose eigenvalues are non-degenerate. The virtue of this introduction is that the matrix T recognizes explicitly and highlights, by means of its eigenvalues, a fundamental physical and geometrical property of the linear system characterized by the matrix A.

This explicit recognition is stated mathematically by the fact that T commutes with A

$$AT - TA = 0\,.$$

In general, the matrix T is not unique. Suppose there are two of them, say T_1, which highlights property 1 and T_2, which highlights a different property of the system. Thus

$$AT_1 - T_1A = 0$$

and

$$AT_2 - T_2A = 0\,,$$

but

$$T_1T_2 - T_2T_1 \neq 0\,.$$

Consequently, for hermitian matrices, the matrix A is characterized by two alternative orthonormal eigenbases, one due to T_1, the other due to T_2, and there is a unitary transformation which relates the two bases.

The matrix A does not determine the choice of eigenbasis. Instead, this choice is determined by which of the two physical properties we are told to (or choose to) examine, that of T_1 or that of T_2.

In the extension of these ideas to differential equations, we shall find that

$$\begin{aligned} A &= \text{Laplace operator} \\ T_1 &= \text{translation operator} \\ T_2 &= \text{rotation operator} \end{aligned}$$

and that the T_1-induced eigenbasis consists of plane wave solutions, the T_2-induced eigenbasis consists of the cylinder wave solutions, and the unitary transformation between them is a Fourier transform.

III. A further idea which these notes extend to infinite dimensions is that of an inhomogeneous four-dimensional system,

$$A\vec{u} = \vec{b}\,,$$

which is overdetermined: the matrix A is 4×4, but singular with a one-dimensional null space.

The extension consists of the statement that (a) this equation is a vectorial wave equation which is equivalent to Maxwell's field equation, (b) the four-dimensional vectors $\vec{u}$ and $\vec{b}$ are 4-d vector fields, and (c) the matrix A has entries which are second order partial derivatives.

One solves this system using the method of eigenvectors and eigenvalues. The eigenvectors have entries which are first order derivatives. The nonzero eigenvalues are scalar (D'Alembertian) wave operators acting on scalar wave

functions. For Maxwell's equations there are exactly three of them, and they are the scalars from which one obtains the three respective types of Maxweell fields,

- *transverse electric* (TE),
- *transverse magnetic* (TM),
- *transverse electric magnetic* (TEM).

The power of this linear algebra method is that it yields success in a variety of curvilinear coordinate systems, including cartesian, cylindrical, and spherical.

Chapter 1

Sturm-Liouville Theory

Lecture 2

1.1 Three Archetypical Linear Problems

We shall now take our newly gained geometrical familiarity with infinite dimensional vector spaces and apply it to each of three fundamental problems which, in linear algebra, have the form

1. $(A - \lambda B)\vec{u} = 0$
2. $(A - \lambda B)G = I$
3. $(A - \lambda B)\vec{u} = \vec{b}$,

i.e., the eigenvalue problem, the problem of inverting the matrix $A - \lambda B$, and the inhomogeneous problem.

The most important of these three is the eigenvalue problem because once it has been solved, the solutions to the others follow directly.

Indeed, assume that we found for the vector space a basis of eigenvectors, say

$$\{\vec{u}_1, \ldots, \vec{u}_N\}$$

as determined by

$$A\vec{u} = \lambda B\vec{u}$$

(We are assuming that the matrices A and B are such that an eigenbasis does indeed exist.) In that case, the solutions to problems 2 and 3 are given by

$$G = \sum_{i=1}^{N} \frac{\vec{u}_i \vec{u}_i^H}{\lambda_i - \lambda}$$

and

$$\vec{u} = \sum_{i=1}^{N} \frac{\vec{u}_i \langle \vec{u}_i, \vec{b} \rangle}{\lambda_i - \lambda}$$

respectively, as one can readily verify. Here $\vec{u}_i^H$ refers to the Hermitian adjoint of the vector $\vec{u}_i$.

On the other hand, suppose we somehow solved problem 2 and found its solution to be

$$G = G_\lambda \, .$$

Then it turns out that the complex contour integral of that solution, namely

$$\frac{1}{2\pi i} \oint G_\lambda d\lambda = -\sum_{i=1}^{N} \vec{u}_i \vec{u}_i^H \, ,$$

yields the sum of the products

$$-\sum_{i=1}^{N} \vec{u}_i \vec{u}_i^H$$

of the eigenvectors $\vec{u}_i$ $(i = 1, \ldots, N)$ of the eigenvalue problem 1. Thus solving problem 2 yields the solution to problem 1. It also, of course, yields the solution to problem 3, namely

$$\vec{u} = G\vec{b} \, .$$

Thus, in a sense, problem 1 and problem 2 are equally important.

We shall extend our considerations of problems 1-3 from finite dimensional to infinite dimensional vector spaces. We shall do this by letting A be a second order differential operator and B a positive definite function. In this case, problem 1 becomes a homogeneous boundary value problem, most often a so-called Sturm-Liouville problem, which we shall formulate and solve. Problem 2 becomes the problem of finding the so-called Green's function.

This will be done in the next chapter. There we shall also formulate and solve the inhomogeneous boundary value problem corresponding to problem 3.

We extend problems 1-3 to infinite dimensions by focussing on second order linear ordinary differential equations and their solutions. They are the most important and they illustrate most of the key ideas.

It is difficult to overstate the importance of Sturm-Liouville theory. Not only does it provide a practical means for dealing with those phenomena (namely wave propagation and vibrations) that underly twentieth century science and technology, but it also provides a very powerful way of reasoning which deals with the qualitative essentials, and not only with the quantitative details.

A Sturm-Liouville eigenvalue problem gives rise to eigenfunctions. It is extremely beneficial to view them as basis vectors which span an inner product space. Doing so places the theory of linear d.e.'s into the framework of Linear Algebra, thus yielding an easy panoramic view of the field. In particular, it allows us to apply our geometrical mode of reasoning to the Sturm-Liouville problem.

1.2 The Homogeneous Problem

The most basic linear problem consists of finding the null space of

$$A\vec{u} = 0\,.$$

The simplest nontrivial extension to differential equations consists of the homogeneous boundary value problem based on the second order differential equation

$$\left[\frac{d^2}{dx^2} + Q(x,\lambda)\frac{d}{dx} + R(x,\lambda)\right] u(x) = 0$$

where $a < x < b$ and λ is a parameter, with one of the following *end point* conditions:

1. $u(a) = 0$ Dirichlet conditions
   $u(b) = 0$

2. $u'(a) = 0$ Neumann conditions
   $u'(b) = 0$

3. $\left.\begin{array}{l}\alpha u(a)+\alpha' u'(a)=0\\ \beta u(b)+\beta' u'(b)=0\end{array}\right\}$ Mixed D. and N. conditions

4. $\left.\begin{array}{l}u(a)-u(b)=0\\ u'(a)-u'(b)=0\end{array}\right\}$ Periodic boundary conditions

More generally one has

$$\begin{aligned} B_1(u) &\equiv \alpha_1 u(a)+\alpha_1' u'(a)+\beta_1 u(b)+\beta_1' u'(b)=0\\ B_2(u) &\equiv \alpha_2 u(a)+\alpha_2' u'(a)+\beta_2 u(b)+\beta_2' u'(b)=0\,, \end{aligned}$$

which are the most general *end point conditions* as determined by the *given* α's, α''s, β's, and β''s, which are *constants*. These two boundary conditions B_1 and B_2 are supposed to be *independent*, i.e., there do not exist any non-zero numbers c_1 and c_2 such that

$$c_1 B_1(u)+c_2 B_2(u)=0 \qquad \forall\, u(x)\,.$$

By contrast, if there does exist a non-zero solution c_1 and c_2 to this equation, then B_1 and B_2 are *dependent*.
Question: Can one give a clear vector space formulation of

$$\begin{aligned} B_1(u) &= 0\\ B_2(u) &= 0 \end{aligned}$$

in terms of subspaces?
Question: What geometrical circumstance is expressed by "independence"?
Answer: The vector 4-tuples $\{\alpha_1,\alpha_1',\beta_1,\beta_1'\}$ and $\{\alpha_2,\alpha_2',\beta_2,\beta_2'\}$ point into different directions.
Question: What, if any, is the (or a) solution to the homogeneous boundary value problem?
Answer: The general solution to the d.e. is

$$u(x)=eu_1(x,\lambda)+fu_2(x,\lambda)$$

where e and f are integration constants. Let us consider the circumstance where $u(x)$ satisfies the *mixed D.-N. boundary conditions* (3.) at each end point. These conditions imply

$$0=e[\alpha u_1(a,\lambda)+\alpha' u_1'(a,\lambda)]+f[\alpha u_2(a,\lambda)+\alpha' u_2'(a,\lambda)]$$

and

$$0 = e[\beta u_1(b,\lambda) + \beta' u_1'(b,\lambda)] + f[\beta u_2(b,\lambda) + \beta' u_2'(b,\lambda)]\,.$$

The content of the square brackets is known because $u_i(x,\lambda)$, α, α', and β, β' are known or given. The *unknowns* are e and f, or rather their ratio. Note that the *trivial* solution

$$e = f = 0 \Leftrightarrow u(x) = 0 \qquad\qquad \forall\, x$$

is *always* a solution to the homogeneous system. Our interest lies in a *non-trivial* solution. For certain values of λ this is possible. This happens when

$$0 = D(\lambda) = \begin{vmatrix} [\alpha u_1(a,\lambda) + \alpha' u_1'(a,\lambda)] & [\alpha u_2(a,\lambda) + \alpha' u_2'(a,\lambda)] \\ [\beta u_1(b,\lambda) + \beta' u_1'(b,\lambda)] & [\beta u_2(b,\lambda) + \beta' u_2'(b,\lambda)] \end{vmatrix}\,.$$

Values of λ, if any, satisfying $D(\lambda) = 0$ are called *eigenvalues*.
KEY PRINCIPLE: *A differential equation is never solved until* boundary conditions *have been imposed.*
We note that the allowed value(s) of λ, and hence the nature of the solution is determined by these *boundary conditions*.
Example (Simple vibrating string): Solve

$$u'' + \lambda u = 0$$

subject to the boundary conditions

$$\begin{aligned} u(a,\lambda) &= 0 \\ u(b,\lambda) &= 0\,. \end{aligned}$$

Solution: Two independent solutions to the d.e. are

$$\begin{aligned} u_1(x) &= \sin\sqrt{\lambda}x \\ u_2(x) &= \cos\sqrt{\lambda}x\,. \end{aligned}$$

Consequently, the solution in its most general form is

$$u(x) = e\sin\sqrt{\lambda}x + f\cos\sqrt{\lambda}x\,.$$

The boundary conditions yield two equations in two unknowns:

$$\begin{aligned} e\sin a\sqrt{\lambda} + f\cos a\sqrt{\lambda} &= 0 \\ e\sin b\sqrt{\lambda} + f\cos b\sqrt{\lambda} &= 0\,. \end{aligned}$$

In order to obtain a *nontrivial* solution, it is necessary that

$$0 = \begin{vmatrix} \sin a\sqrt{\lambda} & \cos a\sqrt{\lambda} \\ \sin b\sqrt{\lambda} & \cos b\sqrt{\lambda} \end{vmatrix}$$

or

$$\sin(a-b)\sqrt{\lambda} = 0$$

which implies

$$\lambda_n = \left(\frac{\pi n}{a-b}\right)^2 \qquad n = 1, 2, \ldots .$$

Note that $n = 0$ yields a trivial solution only. Why? Negative integers yield nothing different, as seen below.
What are the solutions corresponding to each λ_n? The boundary conditions demand that e be related to f, namely,

$$e \sin a\sqrt{\lambda_n} + f \cos a\sqrt{\lambda_n} = 0 \quad ,$$

or

$$f = -e\frac{\sin a\sqrt{\lambda_n}}{\cos a\sqrt{\lambda_n}} .$$

Thus

$$u(x) = \frac{e}{\cos a\sqrt{\lambda_n}}(\cos a\sqrt{\lambda_n} \sin \sqrt{\lambda_n}x - \sin a\sqrt{\lambda_n} \cos \sqrt{\lambda_n}x)$$

or

$$u_n(x) = c_n \sin \sqrt{\lambda_n}(x-a) \quad .$$

Here we have introduced subscript n to distinguish the solutions associated with the different allowed values

$$\lambda_n = \left(\frac{n\pi}{a-b}\right)^2 \qquad n = 1, 2, \ldots .$$

The negative integers give nothing new. (Why?)
Comment: For $n = 0$, i.e. $\lambda = 0$, there does not exist a non-trivial solution. Why? Because the application of the boundary conditions to the $n = 0$ solution,

$$u(x) = ex + f$$

yields only $e = f = 0$.

Lecture 3

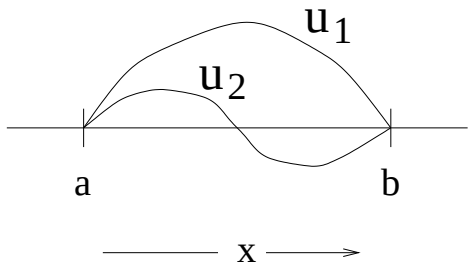


Figure 1.1: First two eigenfunctions of an eigenvalue problem based on Dirichlet boundary conditions.

1.3 Sturm-Liouville Systems

One of the most important and best understood eigenvalue problems in linear algebra is

$$(A - \lambda B)u = 0\,,$$

where A is a symmetric matrix and B is a symmetric positive definite matrix. For this problem we know that

1. its eigenvalues form a finite sequence of *real* numbers
2. the eigenvectors form a B*-orthogonal* basis for the vector space; in other words,
$$U_i^T B U_j = \delta_{ij}\,.$$

A *Sturm-Liouville system* extends this eigenvalue problem to the framework of 2^{nd} order linear ordinary differential equations (o.d.e.'s) where the vector space is infinite dimensional, as we shall see.

1.3.1 Sturm-Liouville Differential Equation

One of the original purposes of the Sturm-Liouville differential equation is the mathematical formulation of the vibration frequency and the amplitude profile of a vibrating string. Such a string has generally a space dependent tension and mass density:

$$T(x) = \text{tension} \quad [\text{force}]$$
$$\rho(x) = \text{density} \quad \left[\frac{\text{mass}}{\text{length}}\right]$$

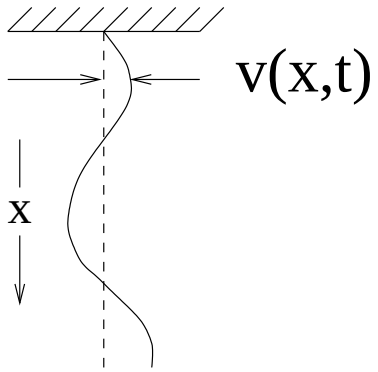


Figure 1.2: Instantaneous amplitude profile of a suspended cable with variable tension and variable mass density.

A cable of variable mass density, say $\rho(x)$, suspended vertically from a fixed support is a good example. Because of its weight, this cable is under variable tension, say $T(x)$, along its length. Let $v(x,t)$ be the instantaneous transverse displacement of the string.

Application of Newton's law of motion, mass×acceleration=force, to the mass $\rho(x)\Delta x$ of each segment Δx leads to the wave equation for the transverse amplitude $v(x,t)$,

$$\rho(x)\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial}{\partial x}T(x)\frac{\partial v(x,t)}{\partial x}\,.$$

The force (per unit length) on the right hand side is due to the bending of the cable. Suppose the cable is imbedded in an elastic medium. The presence of such a medium is taken into account by augmenting the force density on the right-hand side. Being linear in the amplitude $v(x,t)$, the additional restoring force density [force/length] is

$$-k(x)v(x,t) \quad .$$

Here $k(x)\Delta x$ is the position dependent Hooke's constant experienced by the cable segment Δx. Consequently, the augmented wave equation is

$$\boxed{\rho(x)\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial}{\partial x}T(x)\frac{\partial v(x,t)}{\partial x} - k(x)v(x,t)\,.} \tag{1.1}$$

This is the equation of motion for a string imbedded in an elastic medium. Being linear and time-independent, the system has normal modes. They

have vibrational frequencies ω and amplitudes

$$v(x,t) = u(x)\cos\omega(t-t_0)\,.$$

Thus the spatial amplitude profile $u(x)$ of such a mode satisfies

$$\left[\frac{d}{dx}T(x)\frac{d}{dx} + \lambda\rho(x) - k(x)\right]u(x) = 0\,, \qquad \lambda = \omega^2\,. \tag{1.2}$$

For the purpose of mathematical analysis one writes this 2^{nd} order linear o.d.e. in terms of the standard mathematical notation

$$\begin{aligned} p(x) &= T(x) \\ \rho(x) &= \rho(x) \\ q(x) &= k(x)\ , \end{aligned}$$

and thereby obtains what is known as the Sturm-Liouville (S-L) equation[1],

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + [\lambda\rho(x) - q(x)]u = 0\,.$$

However, it is appropriate to point out that actually *any* 2^{nd} order linear o.d.e. can be brought into this "Sturm-Liouville" form. Indeed, consider the typical 2^{nd} order homogeneous differential equation

$$P(x)u'' + Q(x)u' + (R(x) + \lambda)u = 0\,.$$

We wish to make the first two terms into a total derivative of something. In that case, the d.e. will have its S-L form. To achieve this, divide by $P(x)$ and then multiply by

$$e^{\int^x \frac{Q}{P}dt} = p(x)\,.$$

[1]The minus sign in front of $q(x)$ is a reflection of the fact that the S-L equation had its origin in the mathematization of vibrating systems. There

$$\left[\frac{1}{2}p(u')^2 + \frac{1}{2}qu^2\right]dx$$

is the total potential energy stored in a string element of width dx: $\frac{1}{2}pu^2$ is the bending energy/length and $\frac{1}{2}qu^2$ is the energy/length stored in the string due to having pushed by an amount u against the elastic medium in which it is imbedded.

The result is

$$e^{\int^x \frac{Q}{P}dt}u'' + \frac{Q}{P}e^{\int^x \frac{Q}{P}dt}u' + \left(\frac{R}{P}e^{\int^x \frac{Q}{P}dt} + \frac{\lambda}{P}e^{\int^x \frac{Q}{P}dt}\right)u = 0$$

or

$$p(x)u'' + p'(x)u' + (\lambda\rho(x) - q(x))u = 0$$

in terms of newly defined coefficients. Combining the first two terms one has

$$\boxed{\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + [\lambda\rho(x) - q(x)]u = 0\,.} \tag{1.3}$$

This is known as the *Sturm-Liouville equation*. In considering this equation, we shall make two assumptions about its coefficients.

The first one is

$$\begin{aligned} \rho(x) &> 0 \\ p(x) &> 0 \end{aligned}$$

in the domain of definition, $a < x < b$. We make this assumption because nature demands it in the problems that arise in engineering and physics.

The second assumption we shall make is that the coefficients $q(x)$, $\rho(x)$, $p(x)$ and $p'(x)$ are continuous on $a < x < b$. We make this assumption because it entails less work. It does happen, though, that $p'(x)$, $q(x)$, or $\rho(x)$ are discontinuous. This usually expresses an abrupt change in the propagation medium of a wave, for example, the tension or the mass density of string, or the refractive index in a wave propagation medium. This discontinuity can be handled by applying "junction conditions" for $u(x)$ across the discontinuity.

1.3.2 Homogeneous Boundary Conditions

We can now state the S-L problem. If the endpoint conditions are of the mixed Dirichlet-Neumann type,

$$\begin{aligned} \alpha u(a) + \alpha' u'(a) &= 0 \\ \beta u(b) + \beta' u'(b) &= 0\,, \end{aligned} \tag{1.4}$$

with the α's and β's independent of λ, then the boxed Eq. (1.3) together with Eq.(1.4) constitute a *regular* Sturm-Liouville system.

If, by contrast,

$$\begin{aligned} u(a) - u(b) &= 0 \quad \text{and} \quad p(a) = p(b) \\ u'(a) - u'(b) &= 0 \end{aligned} \tag{1.5}$$

then Eqs.(1.3) and (1.5) constitute a *periodic* Sturm-Liouville system.
If $p(a) = 0$ and the 1^{st} b.c. in Eq.(1.4) is dropped, then we have a *singular* S-L system. We shall consider the properties of these S-L systems in a subsequent section.

It is difficult to overstate the pervasiveness of these S-L systems in nature. Indeed, natural phenomena subsumed under the regular S-L problem, for example, are quite diverse. Heat conduction and the vibration of bodies illustrate the point.
A. Heat conduction in one dimension.
Consider the temperature profile $u(x)$ of a conducting bar of unit length which is

1. insulated at $x = 0$ (no temperature gradient), and satisfies

2. radiative boundary condition at $x = 1$

Separation of variables applied to the heat equation yields the following S-L system:

$$u'' + \lambda u = 0 \tag{1.6}$$

with

$$u'(0) = 0 \tag{1.7}$$

$$-u'(1) = hu(1). \tag{1.8}$$

Here h is a non-negative constant. Note that at $x = 1$

$h = 0 \Rightarrow$ no radiation
$h > 0 \Rightarrow$ finite heat loss due to radiation (Newton's law of cooling).

B. Vibrations in one dimension.
Alternatively, consider a vibrating string whose transverse amplitude $u(x)$ satisfies the following homogeneous endpoint conditions:

1. At $x = 0$ there is no transverse force to influence the string's motion. The tension produces only a longitudinal force component. In this circumstance the string is said to be *free* at $x = 0$. This *free boundary condition* is expressed by the statement

$$u'(0) = 0 \quad .$$

2. At $x = 1$ the string is tied to a spring so that the vertical spring displacement coincides with the displacement of the string away from equilibrium. Even though the tail end of the string gets accelerated up and down, the total transverse force on it vanishes because it has no mass. Consequently,

$$-u'(1)T - ku(1) = 0 \quad ,$$

or

$$-u'(1) = hu(1) \quad ,$$

where

$$h = \frac{k}{T} = \frac{\text{spring constant}}{\text{string tension}}$$

The transverse amplitude profile of the string is governed by Eq.(1.2). For constant tension and uniform mass density this equation becomes

$$u'' + \lambda u = 0$$

We see that the S-L system for the heat conduction problem, Eqs.(1.6)- (1.8), coincides with that for the vibration problem.

The task of solving this regular S-L system consists of finding *all* possible values of λ for which the solution $u(x, \lambda)$ is non-trivial. Consequently, there are *four* distinct cases to consider:

1. $\lambda = 0$,
2. $\lambda > 0$,
3. $\lambda < 0$, and
4. $\lambda =$ complex.

We shall have to consider cases 1.-3. only. This is because the next subsection (1.3.3) will furnish us with some very powerful theorems about the nature of the allowed values of λ and the corresponding non-trivial solutions $u(x, \lambda)$.

1. $\lambda = 0$ leads to $u = c_1 + c_2 x$

$$\begin{aligned} h > 0 \quad &\Rightarrow \quad c_1 = c_2 = 0 \ \text{ i.e., } u(x) = 0 \quad \text{for all} \quad 0 < x < 1 \\ h = 0 \quad &\Rightarrow \quad u(x) = c_1 \end{aligned}$$

constant solution. (What physical circumstance is expressed by $u(x) = c_1$?)

2. $\lambda = \alpha^2 > 0$, $\alpha > 0$
The general solution to the differential equation is

$$u(x) = c_1 \cos \alpha x + c_2 \sin \alpha x \quad .$$

Now consider the boundary conditions.

(a) Eq.(1.7) $\Rightarrow c_2 = 0$. Consequently, $u(x) = c_1 \cos \alpha x$.

(b) Eq.(1.8) $\Rightarrow -\alpha c_1 \sin \alpha + h c_1 \cos \alpha = 0$. Consequently,

$$\tan \alpha = \frac{h}{\alpha} . \tag{1.9}$$

This transcendental equation determines the allowed values of α and hence of λ. How do we find them? A very illuminating way is based on graphs. Draw the two graphs (Figure 1.3)

$$y = \tan \alpha$$

and

$$y = \frac{h}{\alpha} .$$

Where they intersect gives the allowed values of α, and hence $\lambda = \alpha^2$, the eigenvalues of the S-L problem. We see that there are an infinite number of intersection points

$$\alpha_1, \ \alpha_2, \ \alpha_3, \ldots, \alpha_n, \ldots .$$

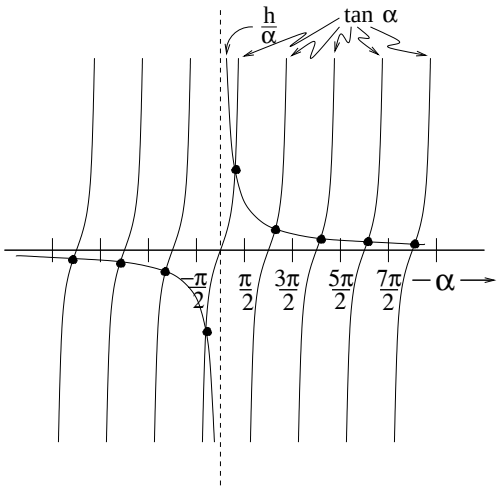


Figure 1.3: There are two graphs in this figure: that of *tan* α and that of the two hyperbolas h/α. The intersection of these two graphs is the solution set to the transcendental eigenvalue Eq.(1.9). The α-values of the heavy dots are the desired solutions. Note that if α is a solution, then $-\alpha$ is another solution, but it yields the same eigenvalue $\lambda = \alpha^2$.

For large n we have $\alpha_n \simeq (n-1)\pi$. The corresponding *eigenvalues* are

$$\lambda_n = \alpha_n^2 \qquad\qquad n = 1, 2, 3, \cdots .$$

Comment: One important question is this: how do the allowed eigenvalues and eigenfunctions depend on boundary conditions? More on that later.

3. $\lambda = -\beta^2 < 0$. This leads to the general solution

$$u(x) = c_1 \cosh \beta x + c_2 \sinh \beta x \ .$$

The boundary conditions yield

$$\tanh \beta = -\frac{h}{\beta} \ .$$

The graph of the hyperbolic tangent does not intersect the graph of the two equilateral hyberbolas. Consequently, the set of solutions for β is the empty set. Thus the S-L system has no solution, except the trivial one $u(x) = 0$.

4. What about complex λ?
   We shall see in the next section that the eigenvalues of a S-L problem are necessarily real.

Lecture 4

1.3.3 Basic Properties of a Sturm-Liouville Eigenvalue Problem

It is surprising how much useful information one can infer about the eigenvalues and eigenfunctions of a S-L problem without actually solving the differential equation explicitly. Thus from very general and simple considerations we shall discover that the eigenvalues *are real, are discrete* if the domain is *finite, have a lowest member, increase without limit*, and that the corresponding *eigenfunctions are orthogonal* to each other, *oscillate, oscillate more rapidly the larger the eigenvalue*, to mention just a few pieces of useful information.

In practice this kind of information is quite often the primary thing of interest. In other words, the philosophy quite often is that one verifies that a certain system is of the S-L types, thus having at one's immediate disposal a concomitant list of properties of the system, properties whose qualitative nature is quite sufficient to answer the questions one had about the system in the first place.

As promised, we shall develop these and other properties by means of a collection of theorems. But before doing so, we remind ourselves about what is meant by a "Sturm-Liouville system", by a "solution", and by "orthogonality". The Sturm-Liouville system we shall consider consists of (i) the *S-L differential equation*

$$\left[\frac{d}{dx}p(x)\frac{d}{dx} + \lambda\rho(x) - q(x)\right]u(x) \;=\; 0\,, \tag{1.10}$$

where q, ρ, p, and p' are continuous and ρ and p are positive definite functions on the open interval (a, b) together with (ii) the *boundary conditions*

$$\begin{aligned} &1.\quad \alpha u(a) + \alpha' u'(a) = 0 \\ &2.\quad \beta u(b) + \beta' u'(b) = 0 \end{aligned} \tag{1.11}$$

where the given constants α, α', β and β' are *independent of the parameter* λ.

Corresponding to an *eigenvalue* of this S-L system, an *eigenfunction* $u_n(x)$ is understood to be that solution which is "regular", i.e.,

$$u_n(x) \text{ and } \frac{du_n(x)}{dx} \text{ are } \textit{continuous}$$

and hence finite, on the closed interval $[a, b]$. In particular, an eigenfunction must not have any finite jump discontinuities anywhere in $[a, b]$.

Orthogonality, Reality, and Uniqueness

Two eigenfunctions $u_m(x)$ and $u_n(x)$ are said to be *orthogonal* relative to the weight function $\rho(x)$ if

$$\int_a^b \overline{u}_m(x) u_n(x) \rho(x) dx = 0 \qquad \text{whenever} \qquad m \neq n\,.$$

They are said to be *orthonormal* with respect to $\rho(x)$ if

$$\int_a^b \overline{u}_m(x) u_n(x) \rho(x) dx = \delta_{mn}\,. \tag{1.12}$$

With these reminders at hand, one can now identify the two most important properties of a S-L system, the orthonormality of its eigenfunctions and the reality of its eigenvalues. The statement and the proof of these properties parallel those of the familiar eigenvalue problem from linear algebra,

$$A\vec{u} = \lambda B\vec{u}$$

where A is a Hermitian and B is a positive definite matrix.

Theorem 1 *(Orthogonality) Let λ_m and λ_n be any two distinct eigenvalues of the S-L problem 1.10 and 1.11 with corresponding eigenfunctions u_m and u_n. Then u_m and u_n are orthogonal with respect to the weight $\rho(x)$.*

Orthogonality also holds in the following cases

1. when $p(a) = 0$ and the first (1) of the boundary conditions 1.11 is *dropped*. This is equivalent to setting $\alpha = \alpha' = 0$

2. when $p(b) = 0$ and the second (2) of the conditions 1.11 is *dropped*. This is equivalent to setting $\beta = \beta' = 0$

3. when $p(a) = p(b)$ and 1.11 are replaced by

$$\begin{aligned} u(a) &= u(b) \\ u'(a) &= u'(b) \ . \end{aligned} \tag{1.13}$$

Remarks:

1. In case (1.) or (2.), the S-L problem is said to be *singular*.

2. The S-L problem with mixed Dirichlet-Neumann conditions at both ends is said to be *regular*.

3. The same terminology, ("singular") is also used when

$$\begin{aligned} \rho(x) &\quad \text{vanishes at an endpoint,} \\ q(x) &\quad \text{is singular at an endpoint,} \\ (a,b) &\quad \text{is unbounded .} \end{aligned}$$

   In other words, we are not interested in the actual value of $u(x)$, just that it stays *finite*. This is sufficient to select (a multiple of the correct) one of two independent solutions to the differential equation.

4. The boundary conditions 1.13 are those of a *periodic* S-L problem, for example, the one where x is the angle φ in cylindrical coordinates. (More on that later.)

5. This theorem is analogous to the orthogonality $\langle u_m, Bu_n \rangle = 0$, of the eigenvectors u_m and u_n of the familiar eigenvalue problem $A\vec{u} = \lambda B\vec{u}$.

6. The physical significance of the orthogonality of the eigenfunctions is exemplified by the energy of a vibrating system governed by, say the wave equation, Eq.(1.1). Its total energy,

$$T.E. = \frac{1}{2}\int_a^b \left[\rho(x)\left(\frac{\partial v}{\partial t}\right)^2 + T(x)\left(\frac{\partial v}{\partial x}\right)^2 + k(x)v^2\right] dx \ , \tag{1.14}$$

   is the sum of its *kinetic energy* and its combined *potential energies* due to the tension in the string and due to the elasticity of the ambient

medium in which the string makes its transverse excursions. Performing an integration by parts on the second term, dropping the endpoint terms due to the imposed homogeneous boundary conditions, and using the governing wave equation, Eq.(1.1), one finds that the total energy is

$$\begin{aligned} T.E. &= \frac{1}{2}\int_a^b \left[\rho(x)\left(\frac{\partial v}{\partial t}\right)^2 + \left(-\frac{\partial}{\partial x}T(x)\frac{\partial v}{\partial x} + k(x)v\right)v\right] dx \\ &= \frac{1}{2}\int_a^b \left[\left(\frac{\partial v}{\partial t}\right)^2 - \frac{\partial^2 v}{\partial t^2}v\right]\rho(x)\,dx \end{aligned}$$

Suppose the total vibrational amplitude is a superposition of the amplitudes associated with with each eigenfrequency ω_n,

$$v(x,t) = \sum_n c_n u_n(x)\cos(\omega_n t + \delta_n)\ .$$

Then the total energy becomes

$$\begin{aligned} T.E. &= \frac{1}{2}\sum_m\sum_n \left[\omega_m\omega_n \sin(\omega_m t + \delta_m)\sin(\omega_n t + \delta_n)\right. \\ &+ \left.\omega_m^2 \cos(\omega_m t + \delta_m)\cos(\omega_n t + \delta_n)\right]\bar{c}_m c_n \int_a^b \bar{u}_m(x)u_n(x)\rho(x)\,dx \end{aligned}$$

The orthonormality, Eq.(1.12), implies that

$$\boxed{T.E. = \frac{1}{2}\sum_n \omega_n^2 |c_n|^2}\ .$$

Thus we see that the orthonormality of the S-L eigenfunctions expresses the fundamental fact that the total energy, a constant independent of time, is composed of the mutually exclusive and constant energies residing in each normal mode ("vibratory degree of freedom").

Proof in 3 Steps: In analogy to $Au_m = \lambda_m B u_m$ and $Au_n = \lambda_n B u_n$ one first considers

Step (1) $\underbrace{-(p\bar{u}_m')' + q\bar{u}_m}_{\|\ L\bar{u}_m} = \lambda_m \rho \bar{u}_m; \qquad \underbrace{-(pu_n')' + qu_n}_{\|\ Lu_n} = \lambda_n \rho u_n.$

Then multiply the equations respectively by u_n and $\overline{u}_m$ and subtract them. The left hand side becomes

$$l.h.s. \;=\; u_n L\overline{u}_m - \overline{u}_m L u_n \equiv \frac{d}{dx}p(\overline{u}_m u_n' - u_n \overline{u}_m') \tag{1.15}$$

We now interrupt the three-step proof to remark that this is an important identity known as *Lagrange's Identity*. We shall meet it and refer to it in several subsequent sections. This identity generalizes to higher dimensions by means of the vector identity $u_n\nabla^2\overline{u}_m - \overline{u}_m\nabla^2 u_n = \nabla\cdot(u_n\vec{\nabla}\overline{u}_m - \overline{u}_m\vec{\nabla}u_n)$. The integral version of Lagrange's Identity is known as *Green's identity*

$$\int_a^b (u_n L\overline{u}_m - \overline{u}_m L u_n)dx = p(x)(\overline{u}_m u_n' - u_n\overline{u}_m')\Big|_a^b \tag{1.16}$$

in 1 dimension. Observe the parallel of this with Green's Identity in three dimensions:

$$\iiint\limits_{\text{volume}} (u_n\nabla^2\overline{u}_m - \overline{u}_m\nabla^2 u_n)d^3x = \iint\limits_{\text{boundary}} (u_n\vec{\nabla}\overline{u}_m - \overline{u}_m\vec{\nabla}u_n)\cdot d\vec{S}\,.$$

We now continue the three-step proof by considering the right hand side of the above subtraction result,

$$r.h.s. = (\lambda_m - \lambda_n)\rho\overline{u}_m u_n\,.$$

Step (2) Both sides are equal. Upon integrating them, one obtains

$$(\lambda_m - \lambda_n)\int_a^b \overline{u}_m u_n \rho(x)dx = p(x)W[\overline{u}_m, u_n](x)\Big|_a^b$$

where

$$W[\overline{u}_m, u_n] = \begin{vmatrix} \overline{u}_m & \overline{u}_m' \\ u_n & u_n' \end{vmatrix}\,.$$

This would be called the *Wronskian* of $\overline{u}_m$ and u_n if λ_m and λ_n were equal. The right hand side of this one-dimensional Green's identity depends only on the boundary (end) points. The idea is to point out that this right hand side vanishes for any one of the boundary conditions under consideration.

Step (3a) If one has *D-N conditions*

$$\begin{aligned} \alpha u(a) + \alpha' u'(a) &= 0 \\ \beta u(b) + \beta' u'(b) &= 0\,, \end{aligned}$$

then these D-N conditions imply

$$\begin{aligned} W(a) &= 0 \\ W(b) &= 0 \end{aligned}$$

because $1^{\text{st}} = 2^{\text{nd}}$ columns are proportional. Thus for a *regular* S-L problem

$$(\lambda_m - \lambda_n) \int_a^b \overline{u}_m(x) u_n(x) \rho(x) dx = 0\,,$$

i.e., one has *orthogonality* whenever $\lambda_m \neq \lambda_n$.
Step (3b) If one has a *periodic* S-L problem

$$\left.\begin{array}{l} p(b) = p(a) \\ u(a) = u(b) \\ u'(a) = u'(b) \end{array}\right\} \Rightarrow W(a) = W(b)\,.$$

i.e., one again has *orthogonality* whenever $\lambda_m \neq \lambda_n$.
Setp (3c) If one has a *singular* S-L problem

$$\left.\begin{array}{l} p(b) = 0 \\ W(b) = \text{finite} \end{array}\right\} \Rightarrow p(b)W(b) = 0\,.$$

Similar considerations at the other end point also yield zero. Once again one has orthogonality whenever $\lambda_m \neq \lambda_n$. To summarize, *the eigenfunctions of different eigenvalues of regular, periodic, and singular Sturm-Liouville systems are orthogonal.*

Lecture 5

Theorem 2 *(Reality of Eigenvalues) For a regular, periodic, and singular S-L system the eigenvalues are* real.
Proof: Step (1) Let u be an eigenfunction corresponding to the *complex eigenvalue* $\lambda = \mu + i\nu$. The eigenfunctions are allowed to be complex. Thus

$$\begin{array}{ccc} Lu = \lambda\rho(x)u & \text{and} & L\overline{u} = \overline{\lambda}\rho\overline{u} \\ \alpha u(a) + \alpha' u'(a) = 0 & & \alpha\overline{u}(a) + \alpha'\overline{u'}(a) = 0 \\ \beta u(a) + \beta' u'(a) = 0 & & \beta\overline{u}(a) + \beta'\overline{u'}(a) = 0 \end{array}$$

because

$$\left.\begin{array}{l} L = \overline{L},\ \rho(x) = \overline{\rho}(x) \\ \alpha, \alpha' = \overline{\alpha}, \overline{\alpha'} \\ \beta, \beta' = \overline{\beta}, \overline{\beta'} \end{array}\right\} \text{ are real}\,.$$

Step (2) We have, therefore,

$$\underbrace{\int_a^b (\overline{u}Lu - uL\overline{u})dx}_{0 = p(x)W[\overline{u}, u]\Big|_a^b} = (\lambda - \overline{\lambda}) \int_a^b \overline{u}u\rho(x)dx$$

$$0 = (\lambda - \overline{\lambda}) \int_a^b |u|^2 \rho(x)dx\,.$$

This implies that $\lambda = \overline{\lambda}$, i.e., that λ is *real*.

We now inquire as to the *number* of independent eigenfunctions corresponding to each eigenvalue. This is a question of *uniqueness*. The examples on page 15 have only *one* such eigenfunction for each eigenvalue. Consider, however, the following

Example (Periodic S-L system)

$$\begin{array}{rcll} u'' + \lambda u &=& 0 & -1 < x < 1 \\ u(-1) &=& u(1) & \\ u'(-1) &=& u'(1)\,. & \end{array}$$

We note that $p(-1) = p(1)$. Consequently, this is a periodic S-L system.

The form of the solution can be written down by inspection. Letting $\lambda = \alpha^2$, one obtains

$$u(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

without loss of generality we assume $\alpha > 0$. The two boundary conditions imply

$$2c_2 \sin \alpha = 0$$

and

$$-2\alpha c_1 \sin \alpha = 0\,.$$

Both conditions yield non-zero solutions whenever $\alpha = 0, \pi, 2\pi, \ldots$. Consequently, the eigenvalues are

$$\lambda_n = n^2\pi^2 \qquad\qquad n = 0, 1, 2, \cdots .$$

Note that for every eigenvalue (except λ_0) there are *two* eigenfunctions

$$\begin{array}{rcl} \lambda_0 & : & \dfrac{1}{2} \\ \lambda_1 & : & \cos \pi x\,, \ \ \sin \pi x \\ \vdots & & \\ \lambda_n & : & \cos n\pi x\,, \ \ \sin n\pi x\,. \end{array}$$

Such nonuniqueness is expressed by saying that each of the eigenvalues $\lambda_1, \lambda_2, \ldots$ is *degenerate*, in this example *doubly degenerate* because there are *two* independent eigenfunctions for each eigenvalue.

The next theorem states that this cannot happen for a regular S-L system. Its eigenvalues are *simple*, which is to say they are *nondegenerate*.
Note that the theorem below uses Abel's Theorem, namely Theorem 4.
Theorem 3 (Uniqueness of solutions to the regular S-L system.) *An eigenfunction of a regular Sturm-Liouville system is unique except for a constant factor, i.e., the eigenvalues of* a regular S-L problem are simple.
Proof: For the same eigenvalue λ, let u_1 and u_2 be two eigenfunctions of the regular S-L system. For a regular S-L system the b.c. are

$$\begin{array}{rcl} \alpha u_1(a) + \alpha' u_1'(a) & = & 0 \\ \alpha u_2(a) + \alpha' u_2'(a) & = & 0\,. \end{array}$$

In other words, both solutions satisfy the D-N mixed boundary conditions at the left hand endpoint. The value of the Wronskian at $x = a$ is

$$W[u_1, u_2](a) = \left| \begin{array}{cc} u_1 & u_1' \\ u_2 & u_2' \end{array} \right|_{x=a} = 0\,, \qquad\qquad \begin{pmatrix} \text{columns are} \\ \text{proportional} \end{pmatrix}.$$

Using Abel's Theorem: $p(x)W[u_1, u_2](x) = \text{constant}$, we obtain

$$\frac{u_1'(x)}{u_1(x)} - \frac{u_2'(x)}{u_2(x)} = 0 \Rightarrow u_1(x) = k u_2(x)\,.$$

This conclusion says that the solution $u_1(x)$ is unique (up to a constant multiplicative factor).

NOTE: If the endpoint condition had been the periodic boundary condition, then one cannot conclude that the eigenvalues are simple. This is because

$$\begin{array}{l} u(a) = u(b) \\ u'(a) = u'(b) \end{array} \quad \text{does } \textit{not} \text{ imply } \quad [u_1(x)u_2'(x) - u_1'(x)u_2(x)]_{x=a} = 0\,.$$

The previous uniqueness theorem used Abel's theorem, which applies to a second order linear differential equation regardless of any boundary conditions imposed on its solutions.

Theorem 4 (Abel) If u_1 and u_2 are two solutions to the *same* differential equation

$$\left[-\frac{d}{dx}p\frac{d}{dx} + q\right] u = \lambda\rho u$$

(i.e., $Lu = \lambda\rho u$), then

$$p(x)[u_1(x)u_2'(x) - u_2(x)u_1'(x)] = \text{constant}\,. \tag{1.17}$$

Remark. The expression in square brackets,

$$W = u_1u_2' - u_2u_1'$$

is called the "Wronskian" or the "Wronskian determinant".

Proof: Start with *Lagrange's identity*

$$u_2Lu_1 - u_1Lu_2 = \frac{d}{dx}p(u_1u_2' - u_2u_1') \equiv \frac{d}{dx}p(x)W[u_1, u_2]\,.$$

Use the given differential equation to conclude that the left hand side vanishes, i.e.

$$0 = \frac{d}{dx}p(x)W[u_1, u_2]\,.$$

Thus $p(x)W[u_1, u_2](x)$ is indeed a constant, independent of x.

A nice application of this theorem is that it gives us a way of obtaining a second solution to the given differential equation, if the first one is already known.

Using Abel's theorem, the Wronskian determinant can be rewritten in the form

$$u_1^2\left(\frac{u_2'}{u_1} - u_2\frac{u_1'}{u_1^2}\right) = \frac{\text{const.}}{p}$$

or

$$u_1^2 \frac{d}{dx}\left(\frac{u_2}{u_1}\right) = \frac{\text{const.}}{p}\,.$$

Integration yields the following
Corollary (Second solution)

$$u_2 = u_1(x)\int^x \frac{dx'}{p(x')u_1^2(x')} + c_1 u_1\,.$$

Thus one is always guaranteed a second solution if a first solution is known.

Exercise 1.3.1 (SCHRÖDINGER FORM: NO FIRST DERIVATIVES)

(a) SHOW that any equation of the form

$$u'' + b(x)u' + c(x)u = 0$$

can always be brought into the Schrödinger form ("no first derivatives")

$$v'' + Q(x)v = 0$$

Apply this result to obtain the Schrödinger form for

(b)
$$u'' - 2xu' + \lambda u = 0 \text{ (HERMITE EQUATION)}$$

(c)
$$x^2u'' + xu' + (x^2 - \nu^2)u = 0 \text{ (BESSEL'S EQUATION)}$$

(d)
$$xu'' + (1-x)u' + \lambda u = 0 \text{ (LAGUERRE'S EQUATION)}$$

(e)
$$(1-x^2)u'' - xu' + \alpha^2 u = 0 \text{ (TSHEBYCHEFF'S EQUATION)}$$

(f)
$$(pu')' + (q + \lambda r)u = 0 \text{ (STURM-LIOUVILLE EQUATION)}$$

(g)
$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]u = 0 \text{ (LEGENDRE EQUATION)}$$

Exercise 1.3.2 (NONDEGENERATE EIGENVALUES)
Consider the S-L eigenvalue problem

$$[Lu_n](x) \equiv \left(-\frac{d^2}{dx^2} + x^2\right) u_n(x) = \lambda_n u_n(x)\,; \quad \lim_{x\to\pm\infty} u(x) = 0\,; \tag{1.18}$$

on the infinite interval $(-\infty, \infty)$.
Show that the eigenvalues λ_n are *nondegenerate*, i.e. show that, except for a constant multiplicative factor, the corresponding eigenfunctions are unique.

Nota bene:

(i) The eigenfunctions are known as the Hermite-Gaussian polynomials. They are known to professionals in Fourier optics who work with laser beams passing through optical systems. A laser beam which is launched with cross-sectional amplitude profile $u_n(x)$ one focal length away from a lens, passes through the lens, and is then observed (on, say, a screen) one focal length after that lens, has an amplitude profile identical to the initial profile $u_n(x)$.

(ii) These eigenfunctions are also known to physicists who work with simple harmonic oscillators (e.g. vibrating molecules), in which case the eigenfunctions are the quantum states of an oscillator and the eigenvalues are its allowed energies.

Exercise 1.3.3 (EVEN AND ODD EIGENFUNCTIONS)
Consider the "parity" operator $P : L^2(-\infty, \infty) \to L^2(-\infty, \infty)$ defined by

$$P\psi(x) \equiv \psi(-x)$$

(i) For a given function $\psi(x)$, what are the eigenvalues and eigen functions of P?

(ii) Show that the eigenfunctions of the operator L defined by Eq.(1.18) are eigenfunctions of P. Do this by first computing

$$P^{-1}LP\psi(x)$$

for $\psi \in L^2(-\infty, \infty)$ and then pointing out how $P^{-1}LP$ is related to L.

Next point out how this relationship applied to an eigenfunction u_n of the previous problem leads to the result $Pu_n = \mu u_n$.

Exercise 1.3.4 (EIGENBASIS OF THE FOURIER TRANSFORM $\mathcal{F}$)
Consider the S-L eigenvalue problem

$$[Lu_n](x) \equiv \left(-\frac{d^2}{dx^2} + x^2\right) u_n(x) = \lambda_n u_n(x)\,; \quad \lim_{x\to\pm\infty} u(x) = 0\,;$$

on the infinite interval $(-\infty, \infty)$. We know that the eigenvalues are *nondegenerate* and are

$$\lambda_n = 2n + 1\,, \qquad n = 0, 1, \ldots\,.$$

Consider now the Fourier transform on $L^2(-\infty, \infty)$:

$$\mathcal{F}[u](k) \equiv \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\sqrt{2\pi}} u(x)dx\,.$$

(a) By computing

$$\mathcal{F}L\mathcal{F}^{-1}\hat{\psi}(k)$$

for arbitrary $\hat{\psi} \in L^2(-\infty, \infty)$, determine the Fourier representation

$$\mathcal{F}L\mathcal{F}^{-1} \equiv \hat{L}\ .$$

of the operator

$$L = -\frac{d^2}{dx^2} + x^2$$

(b) By viewing $\mathcal{F}$ as a map $L^2(-\infty, \infty) \to L^2(-\infty, \infty)$, compare the operators $\hat{L}$ and L.

State your result in a single English sentence and also as a mathematical equation.

(c) Use the result obtained in (b) to show that each eigenfunction u_n of the S-L operator L is also an eigenfunction of $\mathcal{F}$:

$$\mathcal{F}u_n = \mu\ u_n\,.$$

By applying the result (e) of the Fourier eigenvector Exercise on page 149 to the previous Exercise determine the only allowed values for μ. What *is* the Fourier transform of a Hermite-Gauss polynomial $u_n(x)$?

CONGRATULATIONS, you have just found an orthonormal eigenbasis of the Fourier transform operator $\mathcal{F}$ (in terms of the eigenbasis of the S-L operator L)!

Exercise 1.3.5 (HOW TO NORMALIZE AN EIGENFUNCTION) Consider the S-L system

$$\left[\frac{d}{dx}p\frac{d}{dx} - q + \lambda\rho\right] u = 0 \qquad a < x < b$$

$$\alpha u(a) + \alpha' u'(a) = 0\,; \quad \beta u(b) + \beta' u'(b) = 0\ \ .$$

Let $w(x,\lambda)$ be that unique solution to $\frac{d}{dx}p\frac{dw}{dx} + (\lambda\rho - q)w = 0$ which satisfies $\alpha w(a,\lambda) + \alpha' w'(a,\lambda) = 0$, i.e. it satisfies the left hand boundary condition. Then $w_n(x) \equiv w(x,\lambda_n)$ is an eigenfunction of the above S-L system corresponding to the eigenvalue λ_n.

Calculate the normalization integral $\int_a^b w_n^2\ \rho\ dx$ as follows:

(a) *Obtain* the preliminary formula

$$(\lambda - \lambda_n)\int_a^b w_n(x)w(x,\lambda)\rho(x)dx = p(b)W(w,w_n)|^{x=b} \quad .$$

(b) By taking the limit $\lambda \to \lambda_n$ *show* that

$$\int_a^b w_n^2\ \rho\ dx = p(b)\left[w_n'(b)\frac{dw(b,\lambda)}{d\lambda}\bigg|_{\lambda=\lambda_n} - w_n(b)\frac{d}{d\lambda}w'(b,\lambda)\bigg|_{\lambda=\lambda_n}\right] \quad ,$$

where prime denotes differentiation w.r.t. x.

Exercise 1.3.6 (ORTHONORMALIZED BESSEL FUNCTIONS)
Consider the Sturm-Liouville (S-L) problem

$$\left[-\frac{d}{dx}\ x\ \frac{d}{dx} + \frac{\nu^2}{x}\right]u = \lambda x u \quad .$$

Here u, $\dfrac{du}{dx}$ bounded as $x \to 0$, $u(1) = 0$ where ν is a real number.

(a) Using the substitution $t = \sqrt{\lambda}\,x$, *show* that the above differential equation reduces to Bessel's equation of order ν. One solution which is bounded as $t \to 0$ is $J_\nu(t)$; a second linearly independent solution, denoted by $Y_\nu(t)$, is unbounded as $t \to 0$.

(b) *Show* that the eigenvalues $\lambda_1, \lambda_2, \ldots$ of the given problem are the squares of the positive zeroes of $J_\nu(\sqrt{\lambda})$, *and* that the corresponding eigenfunctions are

$$u_n(x) = J_\nu(\sqrt{\lambda_n}\,x) \quad .$$

(c) *Show* that the eigenfunctions $u_n(x)$ satisfy the orthogonality relation

$$\int_0^1 x\ u_m(x)\ u_n(x)\ dx = 0 \quad m \neq n\,.$$

(d) For the case $\nu = 0$, apply the method of the previous problem to *exhibit* the set of orthonormalized eigenfunctions $\{u_0(x), u_1(x), u_2(x), \cdots\}$.

(e) *Determine* the coefficients in the Fourier-Bessel series expansion

$$f(x) \doteq \sum_{n=1}^{\infty} c_n \; u_n(x) \, .$$

Exercise 1.3.7 (ORTHOGONALITY OF LEGENDRE POLYNOMIALS) Consider the S-L problem

$$\left[-\frac{d}{dx}(1-x^2)\frac{d}{dx} + \frac{m^2}{1-x^2} \right] u = \lambda u$$

Here u, $\dfrac{du}{dx}$ bounded as $x \to \pm 1$. Here $m =$ integer. The solutions to this S-L problem are $u_n = P_n^m(x)$, the "associated Legendre polynomials", corresponding to $\lambda_n = n(n+1)$, $n =$ integer. *Show* that

$$\int_{-1}^{1} P_n^m(x) P_{n'}^m(x) \; dx = 0 \quad \lambda_n \neq \lambda_{n'} \, .$$

Lecture 6

Sturm's Comparison Theorem

When confronted with the regular boundary value problem

$$\left[\frac{d}{dx} p(x) \frac{d}{dx} - q(x) + \lambda \rho(x) \right] u(x) \;\; = \;\; 0 \tag{1.19}$$

$$\begin{aligned} \alpha u(a) + \alpha' u'(a) &= 0 \\ \beta u(b) + \beta' u'(b) &= 0 \end{aligned} \tag{1.20}$$

we must ask

1. How does the oscillatory nature of $u(x;\lambda)$, a solution to Eq. 1.19, depend on λ?

2. Why do the values of λ permitted by (1.19) and (1.20) form a *discrete* and *semi-infinite* sequence

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$$

with a *smallest* eigenvalue λ_0 and with $\lambda_n \to \infty$ *as* $n \to \infty$?

The "oscillatory nature" of a solution $u(x, \lambda)$ is expressed qualitatively by the location of the zeroes of its graph. One could also inquire about its behaviour between successive zeroes. However, we shall see that such an inquiry always leads to the same answer: Provided $q(x) + \lambda\rho(x)$ is positive between a pair of successive zeroes, the graph of $u(x; \lambda)$ has only a *single* maximum (or minimum). This means that $u(x, \lambda)$ can *not* oscillate between two of its successive zeroes.

Thus the most important issue is the existence and location of the zeroes, which are controlled entirely by the phase of a given solution $u(x, \lambda)$. This phase is a scalar function from which one directly constructs the solution. It is preferrable to discuss the behavior of the solution $u(x; \lambda)$ in terms of its phase because the key qualitative properties of the latter are very easy to come by. As we shall see, one only needs to solve a first order differential equation, not the second order S-L equation.

However, before establishing and solving this differential equation, let us use the second order S-L differential equation directly to determine how the zeroes of $u(x\lambda)$ are affected if the parameter λ is changed. We express this behaviour in terms of the

Sturm Comparison Theorem:
Whenever $\lambda_1 < \lambda_2$, *then between two zeroes of the nontrivial solution* $u(x, \lambda_1)$, *there lies a zero of* $u(x, \lambda_2)$.

This theorem demands that one compare the two different solutions

$$u(x, \lambda_1) \equiv u_1(x) \qquad \text{and} \qquad u(x, \lambda_2) \equiv u_2(x)$$

of Eq. 1.19 corresponding two different constants λ_1 and λ_2. The conclusion is obtained in three steps:

Step 1: Multiply these two equations respectively by u_2 and u_1, and then form their difference. The result, after cancelling out the $q(x)u_1u_2$ term, is

$$\frac{d}{dx}\left[p(x)\left(u_2\frac{du_1}{dx} - u_1\frac{du_2}{dx}\right)\right] = (\lambda_2 - \lambda_1)u_1u_2\rho(x)\,,$$

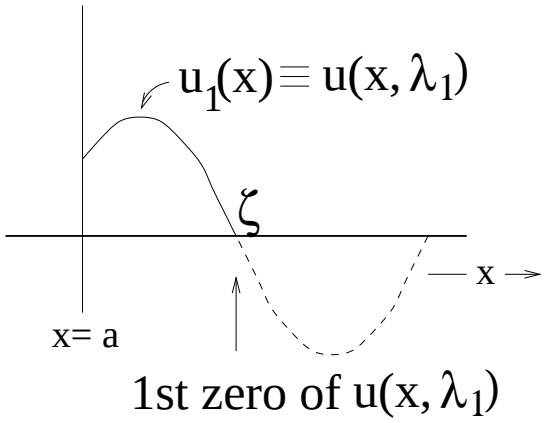


Figure 1.4: Graph of a solution $u(x, \lambda_1)$ which satisfies the mixed D-N boundary condition at $x = a$.

which is the familiar Lagrange identity, Eq.(1.15), in disguise. Upon integration one obtains

$$p(x)(u_2u_1' - u_1u_2')\Big|_a^x = (\lambda_2 - \lambda_1)\int_a^x u_1u_2\rho\,dx\,.$$

If both u_1 and u_2 satisfy the mixed Dirichlet-Neumann boundary conditions $\alpha u(a) + \alpha' u'(a) = 0$ at $x = a$, then

$$(u_2u_1' - u_1u_2')_{x=a} = -\frac{\alpha}{\alpha'}[u_2(a)u_1(a) - u_1(a)u_2(a)] = 0\,.$$

If $x = a$ is a singular point of the differential Eq.(1.19), $p(a)$ is zero. Thus, if u and u' are finite at $x = a$, then the left hand side vanishes again at the lower limit $x = a$. Thus both for a *regular* and for this *singular* Sturm-Liouville problem we have

$$p(x)\left(u_2(x)\frac{du_1(x)}{dx} - u_1(x)\frac{du_2(x)}{dx}\right) = (\lambda_2 - \lambda_1)\int_a^x u_1u_2\rho\,dx\,.$$

Step 2: Now assume that, for some range of values λ, each of the corresponding solutions $u(x, \lambda)$ satisfying the boundary condition at $x = a$, *oscillates*. In other words, as x increases beyond $x = a$, $u(x, \lambda)$ reaches a maximum, then decreases, passes through zero, reaches a minimum, increases and so on. That such a range of λ-values exists, we shall see later.

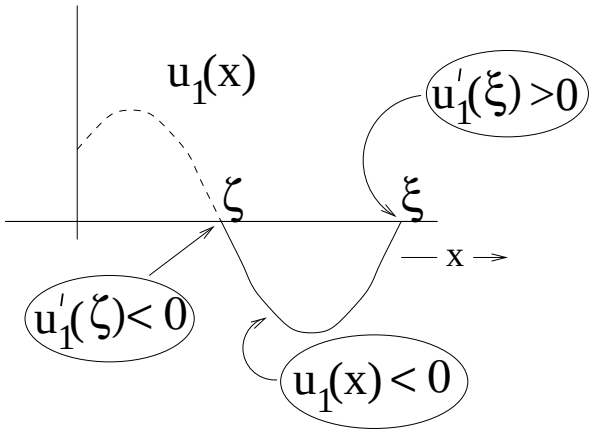


Figure 1.5: Graph of a solution $u(x, \lambda_1)$ which is zero at ζ and ξ.

Let λ_1 lie in this range, and let $x = \zeta$ be the first zero of $u_1(x)$ as in Figure 1.4. Consequently,

$$\left(p(x)u_2\frac{du_1}{dx}\right)_\zeta = (\lambda_2 - \lambda_1)\int_a^\zeta u_1 u_2 \rho\, dx\,.$$

Step 3. One must now conclude that if $\lambda_2 > \lambda_1$, then the zeroes of $u_2(x)$ are more closely spaced than those of $u_1(x)$. Why must this statement, the *Sturm Comparison Theorem*, be true?
a) Assume the contrary, i.e., assume that $u_2(x)$ has no zero in $a < x < \zeta$. See Figure 1.4. In that case $u_2(x) > 0$ for all $a < x < \zeta$. This implies

$$\left(p(x)u_2\frac{du_1}{dx}\right)_\zeta = (\lambda_2 - \lambda_1)\int_a^\zeta u_1 u_2 \rho\, dx > 0\,.$$

But $p(\zeta) > 0$, $u_2(\zeta) > 0$ and $\frac{du_1(\zeta)}{dx} < 0$, so that $\left(p(x)u_2\frac{du_1}{dx}\right)_\zeta < 0$. This is a contradiction. Hence our assumption was wrong; the function u_2 does have a zero in $a < x < \zeta$.
b) Now consider the circumstance where $u_1(x)$ has two successive zeroes at ζ and ξ: $u_1(\zeta) = u_1(\xi) = 0$. In that case one obtains

$$p(\xi)u_2(\xi)u_1'(\xi) - p(\zeta)u_2(\zeta)u_1'(\zeta) = (\lambda_2 - \lambda_1)\int_\zeta^\xi u_2 u_1 \rho\, dx\,.$$

If $u_2(x)$ does not change sign in $\zeta < x < \xi$, as in Figure 1.6, then we again have a contradiction because if $u_1(x) < 0$, then $u_1'(\zeta) < 0$, $u_1'(\xi) > 0 \Rightarrow$

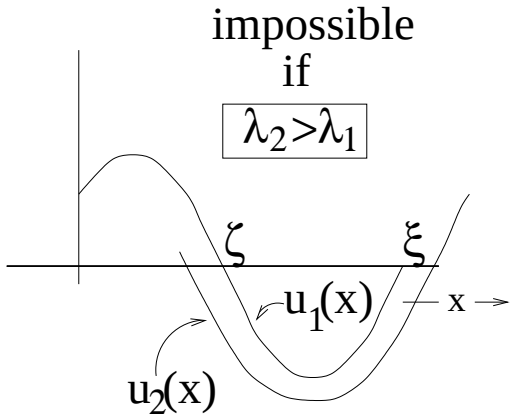


Figure 1.6: If $\lambda_2 > \lambda_1$ it is impossible that $u(x, \lambda_2)u(x, \lambda_1) > 0$ for all x in the interval $[\zeta, \xi]$.

$(r.h.s.) \times (l.h.s.) < 0$. In other words, the picture in Figure 1.6 is impossible. We conclude, therefore, that $u_2(x)$ must have a zero inside the open interval (ζ, ξ).

1.4 Phase Analysis of a Linear Second Order O.D.E.

The fundamental property of solutions to the λ-parametrized S-L equation is their oscillatory nature, i.e. the existence and the spacing of their zeros. This undulatory behavior of a solution is conceptualized by means of its *phase*, a concept formed from the observation of waves. Its mathematization in the context of a S-L d.e., more generally, a non-linear 2nd order o.d.e., is the result of a transformation to polar coordinates.

One starts with the two-dimensional domain of states, the *Poincaré phase plane*. In compliance with the given 2nd order o.d.e., one introduces rectilinear coordinates. Their values are proportional to the two state variables (u' and u) of the system governed by that 2nd order o.d. equation. As depicted in Figure 1.7, this d.e. determines the trajectory of a moving point. Its polar angle is the evolving phase of the system. It is particularly desirable to employ phase method to linear ordinary differential equations (o.d.e.'s) of second order not only for the breadth and the depth of the obtained results, but also for the ease with which these results are obtained. A major con-

tributing factor to the latter is that the method is capable of characterizing the solutions to *any* such differential equation, and it can do so *geometrically*.

1.4.1 The Prüfer System

For linear second order ordinary differential equations, the phase plane method is achieved by the so-called Prüfer substitution. It yields the phase and the amplitude of the sought after solution to the Sturm-Liouville equation.

The method to be developed applies to any differential equation having the form

$$\frac{d}{dx}\left(P(x)\frac{du}{dx}\right) + Q(x)u \;=\; 0 \qquad\qquad a < x < b \tag{1.21}$$

Here $0 < P(x)$, $P'(x)$, and $Q(x)$ are continuous.

We are interested in asking and answering the following questions:

1. How often does a solution oscillate in the interval $a < x < b$; i.e., *how many zeroes* does it have?

2. *How many maxima* and *minima* does it have between a pair of consecutive zeroes?

3. What happens to these zeroes when one changes $P(x)$ and $Q(x)$?

The questions can be answered by considering for this equation its *phase portrait* in the *Poincaré phase plane*. We do this by introducing the "phase" and the "radius" of a solution $u(x)$. This is done in three steps.

A) First apply the *Prüfer substitution*

$$\boxed{P(x)u'(x) = r(x)\cos\theta(x); \qquad u(x) = r(x)\sin\theta(x)}$$

to the quantities in Eq.(1.21). Do this by introducing the new dependent variable r and θ as defined by the formulae

$$r^2 = u^2 + P^2(u')^2\,; \qquad\qquad \theta = \arctan\frac{u}{Pu'}\,.$$

(Without loss of generality one may always assume that $u(x)$ is real. Indeed, if $u(x)$ were a complex solution, then it would differ from a real one by

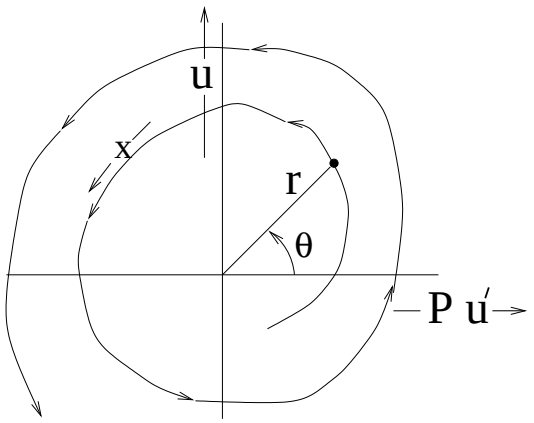


Figure 1.7: The Poincaré phase plane of the second order linear differential equation is spanned by the amplitude u and its derivative u' (multiplied by the positive coefficient P). A solution to the differential equation is represented by an x-parametrized curve. The (Prüfer) phase is the polar angle θ.

a mere complex constant.) A solution $u(x)$ can thus be pictured in this Poincaré plane as a curve parametrized by the independent variable x.

However, there is more. Two solutions to Eq.(1.21), say u_1 and u_2, give rise to two vectors,

$$\vec{R_1}(x) = (Pu_1',\, u_1) = (r_1 \cos\theta_1,\, r_1 \sin\theta_1)$$

and

$$\vec{R_2}(x) = (Pu_2',\, u_2) = (r_2 \cos\theta_2,\, r_2 \sin\theta_2)\,,$$

moving in the Poincaré plane as shown in Figure 1.8. The area of the triangle formed by these vectors is 1/2 that of the parallelogram,

$$\vec{R_2} \times \vec{R_1} = Pu_2'u_1 - Pu_1'u_2 \tag{1.22}$$
$$= r_1(x) r_2(x) \sin[\theta_1(x) \sin\theta_2(x)]. \tag{1.23}$$

This is constant because of Abel's Theorem, Eq.(1.17). This constancy as function of x is a universal property. It holds for all S-L systems, regardless of the particular boundary conditions u_1 and u_2 may be subjected to.

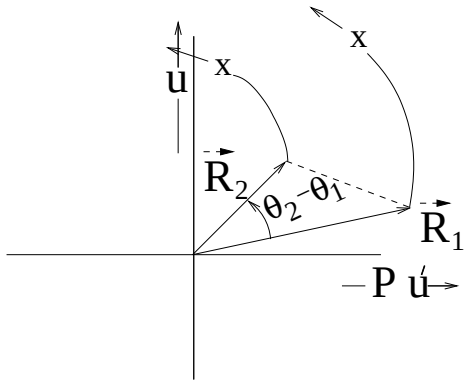


Figure 1.8: The area spanned by the vectors $\vec{R}_1$ and $\vec{R}_2$, whose position is parametrized by x, remains constant throughout its x-parametrized motion.

The transformation

$$(Pu', u) \leftrightarrow (r, \theta)$$

is a transition from rectilinear to polar coordinates in the Poincaré plane. This transformation is *non-singular* for all $r \neq 0$. Furthermore, we always have $r > 0$ for any non-trivial solutions. Why? Because if $r(x) = 0$, i.e., $u(x) = 0$ and $u'(x) = 0$ for some particular x, then by the uniqueness theorem for second order linear o.d.e. $u(x) = 0 \; \forall \, x$, i.e., we have the trivial solution.

B) Second, obtain a system of first order o.d.e. which is equivalent to the given differential Eq.(1.21).

(i) Differentiate the relation

$$\cot\theta = \frac{Pu'}{u} \quad .$$

(Side Comment: If $u = 0$, then we differentiate $\tan\theta = u/Pu'$ instead. This yields the same result.)

One obtains

$$\begin{aligned} -\csc^2\theta \frac{d\theta}{dx} &= \frac{(Pu')'}{u} - \frac{Pu'}{u^2}u' \\ &= -Q - \frac{1}{P}\frac{\cos^2\theta}{\sin^2\theta} \quad , \end{aligned}$$

or

$$\boxed{\frac{d\theta}{dx} = Q(x)\sin^2\theta + \frac{1}{P(x)}\cos^2\theta \equiv F(x,\theta)\,.} \tag{1.24}$$

This is Prüfer's differential equation for the *phase*, the *Prüfer* phase.
(ii) Differentiate the relation

$$r^2 = u^2 + (Pu')^2$$

and obtain

$$\begin{aligned} r\frac{dr}{dx} &= uu' + (Pu')(Pu')' \\ &= \frac{u}{P}Pu' - Pu'Qu \\ &= \frac{r\sin\theta}{P}r\cos\theta - r\cos\theta Qr\sin\theta \end{aligned}$$

or

$$\boxed{\frac{dr}{dx} = \frac{1}{2}\left[\frac{1}{P(x)} - Q(x)\right] r\sin 2\theta\,.} \tag{1.25}$$

This is Prüfer's differential equation for the *amplitude*.

C) Third, solve the system of Prüfer equations (1.24) and (1.25). Doing so is *equivalent* to solving the originally given equation 1.21. Any solution to the Prüfer system determines a unique solution to the equation (1.21), and conversely.

Of the two Prüfer equations, the one for the phase $\theta(x)$ is obviously much more important: it determines the qualitative, e.g. oscillatory, behavior of $u(x)$. The feature which makes the phase equation so singularly attractive is that it is a first order equation which also is independent of the amplitude $r(x)$. The amplitude $r(x)$ has no influence whatsoever on the phase function $\theta(x)$. Consequently, the phase function is governed by the simplest of all possible non-trivial differential equations: an ordinary first order equation. This simplicity implies that rather straight forward existence and uniqueness theorems can be brought to bear on this equation. They reveal the qualitative nature of $\theta(x)$ (and hence of $u(x)$) without having to exhibit detailed analytic or computer generated solutions.

(1) One such theorem says that for any initial value

$$(a, \gamma)$$

$\exists$ a unique solution $\theta(x)$ which satisfies

$$\frac{d\theta}{dx} = F(x, \theta)$$

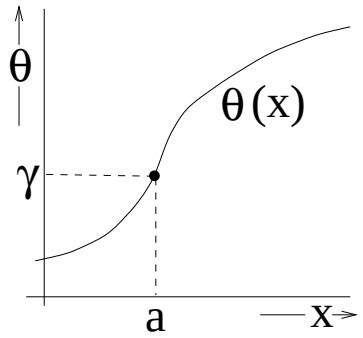


Figure 1.9: The phase function $\theta(x)$ is that unique solution to the Prüfer equation $d\theta/dx = F(\theta, x)$ whose graph passes through the given point (a, γ).

and

$$\theta(a) = \gamma \quad ,$$

provided P and Q are continuous at a. See Figure 1.9. Existence and uniqueness of $\theta(x)$ prevails even if $P(x)$ and $Q(x)$ have finite jump discontinuities at $x \neq a$.

(2) Once $\theta(x)$ is known, the *Prüfer amplitude* function $r(x)$ is determined by integrating Eq.(1.25). One obtains

$$r(x) = K \exp \int_a^x \frac{1}{2} \left[\frac{1}{P(x)} - Q(x) \right] \sin 2\theta(x)\, dx$$

where $K = r(a)$ is the initial amplitude.

(3) Each solution to the Prüfer system, Eqs.(1.24) and (1.25), depends on two constants:

$$\begin{aligned} \text{the initial amplitude} \quad K &= r(a) \\ \text{the initial phase} \quad \gamma &= \theta(a) \end{aligned}$$

Note the following important fact: Changing the constant K just multiplies the solution $u(x)$ by a constant factor. *Thus the zeroes of $u(x)$ can be located by studying only the phase d.e.*

$$\boxed{\frac{d\theta}{dx} = F(x, \theta)\,.}$$

This is a major reason why we shall now proceed to study this equation very intensively.

Vibrations, oscillations, wiggles, rotations and undulations are all characterized by a changing phase. If the independent variable is the time, then this *time, the measure of that aspect of change which permits an enumeration of states*, manifests itself physically by the advance of the phase of an oscillating system.

Lecture 7

Summary. The *phase* of a system is the most direct way of characterizing its oscillatory nature. For a linear 2^{nd} order o.d.e., this means the Prüfer phase $\theta(x)$, which obeys the first order d.e.

$$\frac{d\theta}{dx} \;=\; Q(x)\sin^2\theta + \frac{1}{P(x)}\cos^2\theta \equiv F(x,\theta) \quad . \tag{1.26}$$

It is obtained from the second order equation

$$\left[\frac{d}{dx}P(x)\frac{d}{dx} + Q(x)\right]u(x) \;=\; 0 \tag{1.27}$$

by the Prüfer substitution

$$u(x) = r(x)\sin\theta(x) \qquad Pu'(x) = r(x)\cos\theta(x)\,.$$

These equations make it clear that the zeroes and the oscillatory behavior of $u(x)$ are controlled by the phase function $\theta(x)$.

1.5 Qualitative Results

The phase is a very direct way of deducing a number of important properties of any solution to a general second order linear o.d.e. We shall do this by making a number of simple observations.

(i) The zeroes of a solution $u(x)$ to Eq. 1.19 occur where the Prüfer phase θ has the values

$$0, \pm\pi, \pm 2\pi, \ldots$$

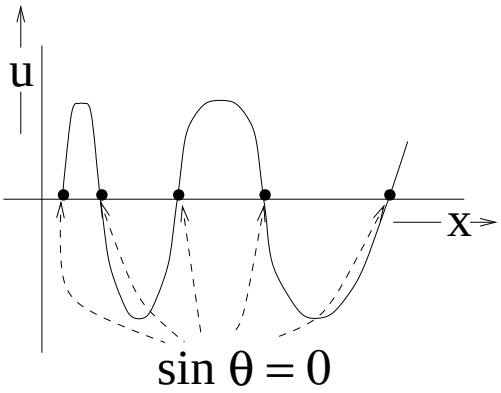


Figure 1.10: The function $u(x)$ has its zeroes whenever the phase $\theta(x)$ is an integral multiple of π.

(ii) At these points, where $\sin\theta(x) = 0$, $\theta(x)$ is an *increasing* function of x. Indeed, $\sin\theta = 0$ implies $\cos^2\theta = 1$. Consequently, the Prüfer equation yields

$$\frac{d\theta}{dx} = \frac{1}{P} > 0\,, \qquad \text{(whenever } \theta = 0, \pm\pi, \dots\text{)} \tag{1.28}$$

because $P(x) > 0$ by assumption. The positiveness of this rate of change implies that in the Poincaré phase plane, which is spanned by Pu' and u, the curve $(P(x)u'(x), u(x))$ crosses the horizontal Pu'-axis ($\theta = n\pi$) only in the *counter clockwise* sense as illustrated in Figure 1.7. In other words, the phase of the curve always goes forward, never backward (Figure 1.11) when it crosses the horizontal.

The following conclusions follow easily.

(iii) The zeroes of $u(x)$ are *isolated*, i.e. they are *separated by a finite amount* from each other. Why is this statement true? Consider two successive zeroes of $u(x)$. Call them x_n and x_{n+1}. At these points the phase has the values $\theta(x_n) = n\pi$ and $\theta(x_{n+1}) = (n+1)\pi$. The phase equation

$$\frac{d\theta}{dx} = Q(x)\sin^2\theta + \frac{1}{P(x)}\cos^2\theta$$

implies that the slopes $\left.\frac{d\theta}{dx}\right|_{x_n} = \frac{1}{P(x_n)}$ and $\left.\frac{d\theta}{dx}\right|_{x_{n+1}} = \frac{1}{P(x_{n+1})}$ both be positive. Reference to the $\theta - x$ plane, say Figure 1.12, reveals that these two inequal-

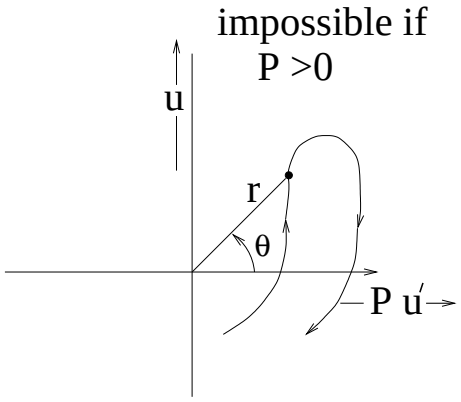


Figure 1.11: The phase can only advance, never retreat across the horizontal axis.

ities simply prevent x_n and x_{n+1} from being infinitesimally close together. The function $\theta(x)$ would become multivalued if they were.

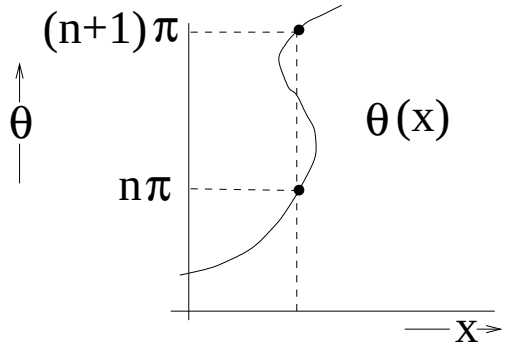


Figure 1.12: The zeroes of $u(x)$, i.e. the integral π values of θ must have a finite x-separation, otherwise $\theta(x)$ becomes multi-valued.

(iv) If $Q > 0$, then $u(x)$ has exactly one maximum (or minimum) between two successive zeroes of a given solution. Thus a sequence of maxima and minima above (or below) the x-axis is *impossible* if $Q(x) > 0$.

The reason for this impossibility is this:

1. At a maximum (or a minimum) of u one has

$$0 = Pu' = r\cos\theta \quad \Leftrightarrow \quad \cos\theta = 0\,, \quad \sin^2\theta = 1 \quad .$$

A maximum (or minimum) of u is located at $\theta = (n + \frac{1}{2})\pi$.

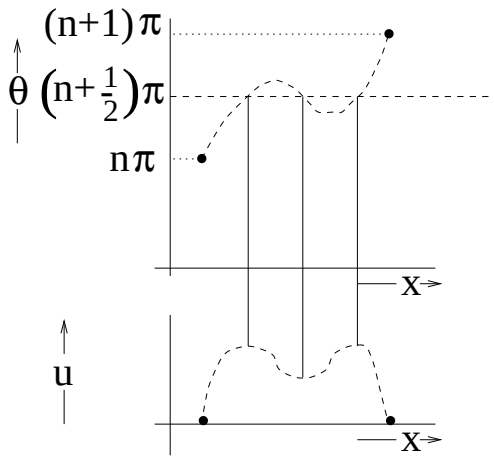


Figure 1.13: The minimum of $u(x)$ in this figure is forbidden because the corresponding slope $d\theta/dx$ at that point would have to be negative, in violation of inequality 1.29.

2. Prüfer's equation (1.24) implies, therefore,

$$\frac{d\theta}{dx} = Q\sin^2\theta + 0 = Q > 0 \quad \text{(at a MAX or a MIN)} \tag{1.29}$$

at these points. Consequently, $\theta(x)$ *can cross the line* $\theta = (n + \frac{1}{2})\pi$ *only once*, which means u has a maximum (or minimum) *only once*. If it crossed it a second time, as in Figure 1.13, the slope would have to be negative at the second crossing point, thus violating the inequality 1.29.

If $Q < 0$, then, of course, all bets are off!

1.6 Phase Analysis of a Sturm-Liouville System

Every Sturm-Liouville system has a personality, which is encoded in its *phase*.
In other words, the *phase* is the brains of the regular Sturm-Liouville system.
It is the *phase* which determines where a given solution has a maximum.
It is the *phase* which determines where a given solution is zero.
It is the *phase* which determines where a given solution oscillates.

It is the *phase* which determines how many zeroes a given solution has in its domain of definition.

Thus, when one thinks of the questions, "How do the boundary and the associated eigenvalue parameter λ control the nature of the solution to the regular Sturm-Liouville problem?" one should actually ask a more penetrating question:

"How do the boundary conditions and the associated eigenvalue parameter λ control the *phase* of the solution to the S-L problem?"

The existence of allowed (eigen)values of λ and the concomitant eigenfunction is determined entirely by the *phase*. Let us, therefore, recast the mixed D-N boundary conditions in terms of this phase.

1.6.1 The Boundary Conditions

The two D-N boundary conditions are

$$\alpha u(a) + \alpha' u'(a) = 0 \qquad \text{and} \qquad \beta u(b) + \beta' u'(b) = 0$$

at the two endpoints $x = a$ and $x = b$. We know that the phase $\theta(x, \lambda)$ satisfies the family of λ-parametrized 1^{st} order o.d.e.'s

$$\frac{d\theta}{dx} = (\lambda\rho(x) - q(x)) \sin^2\theta + \frac{1}{p(x)} \cos^2\theta \quad ,$$

where ρ, q, and p are given by the S-L equation

$$\left[\frac{d}{dx} p(x) \frac{d}{dx} + \lambda\rho(x) - q(x)\right] u = 0 \, .$$

We must now determine what conditions the two homogeneous D-N boundary conditions impose on the phase $\theta(x)$. The transformation of the D-N conditions into equivalent conditions on the phase is done with the help of the Prüfer relation

$$\tan\theta = \frac{u}{pu'} \, .$$

This determines two phase angles. At the left endpoint $x = a$, let the initial phase be $\theta(a, \lambda) = \gamma$. This phase γ is uniquely determined by the two requirements

$$\tan\gamma \equiv \frac{u(a)}{u'(a)p(a)} = -\frac{\alpha'}{\alpha p(a)} \quad \text{and} \quad \boxed{0 \le \gamma < \pi} \quad ,$$

if $\alpha \neq 0$, and by

$$\gamma = \frac{\pi}{2} \quad \text{if} \qquad \alpha = 0 \quad .$$

(It is clear that $\gamma = \frac{\pi}{2}$ expresses the case of pure Neumann condition at $x = a$.) Thus the D-N boundary condition at $x = a$ has been expressed in terms of a *single* quantity, the *initial phase*. This initial phase is required to be the same for *all* λ.

At $x = b$ we introduce the final phase angle δ. It is determined by the two requirements

$$\tan\delta = -\frac{\beta'}{\beta p(b)} \quad \text{and} \quad \boxed{0 < \delta \leq \pi}$$

if $\beta \neq 0$, and by

$$\delta = \frac{\pi}{2} \qquad \text{if} \qquad \beta = 0\,.$$

Lecture 8

Having reformulated the two D-N conditions in terms of the two angles γ and δ, we are ready to restate the S-L problem in terms of the phase function θ. This restatement is very simple.

1.6.2 The Boundary Value Problem

A solution $u(x, \lambda)$ of the S-L d.e. for $a \leq x \leq b$ will be an eigenfunction of the regular S-L boundary value problem if and only if the corresponding phase, obtained from the Prüfer d.e.

$$\frac{d\theta}{dx} = (\lambda\rho - q)\sin^2\theta + \frac{1}{p}\cos^2\theta\,,$$

satisfies the corresponding end point conditions

$$\theta(a, \lambda) = \gamma \quad \text{and} \quad \theta(b, \lambda) = \delta + n\pi \qquad n = 0, 1, \ldots$$

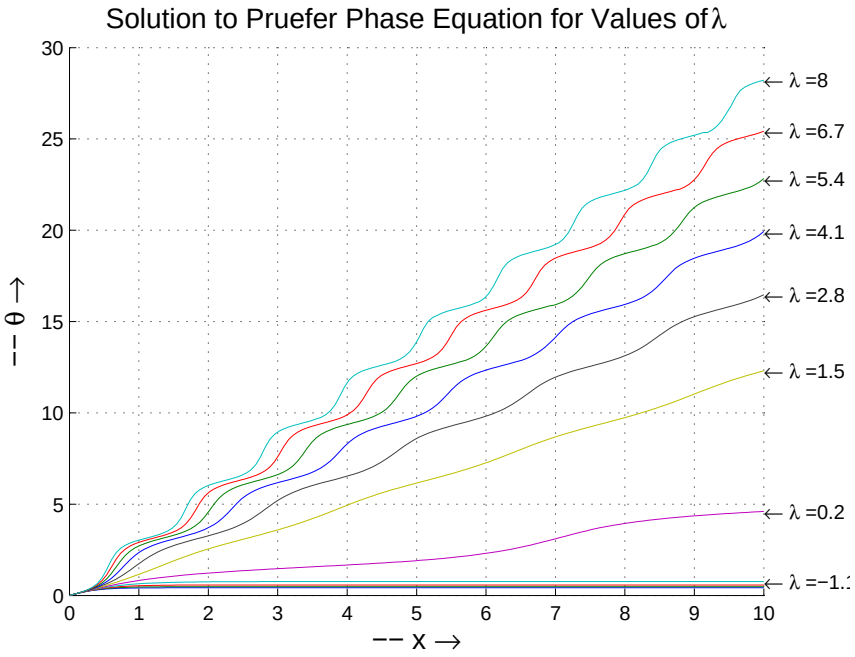
with $0 \leq \gamma < \pi$ and $0 < \delta \leq \pi$.

Note that any λ for which these endpoint conditions hold is an eigenvalue of the regular S-L problem, and conversely, that an eigenvalue of this S-L problem will yield a phase function whenever it satisfies the required end point conditions for some $n = 0, 1, 2, \ldots$.

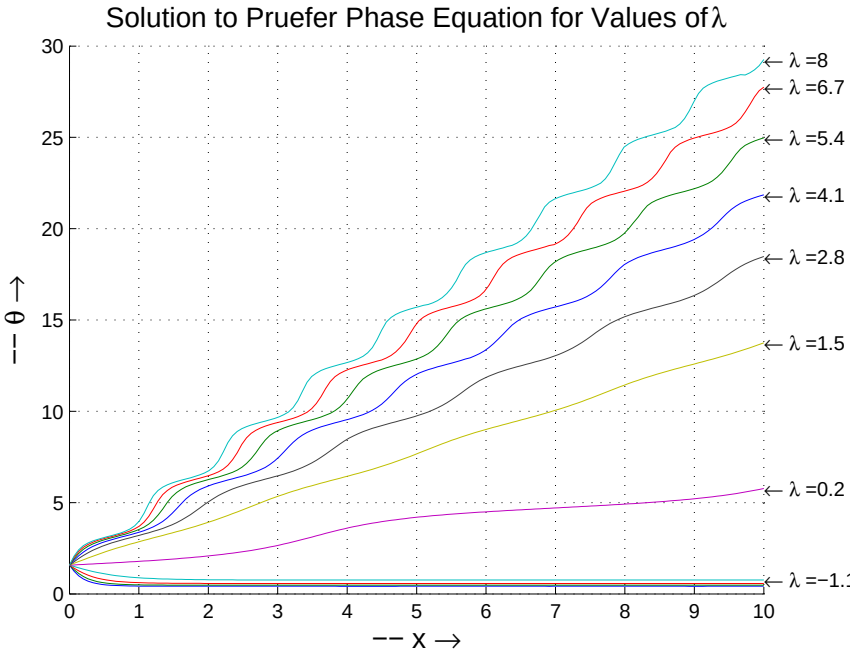
The question now is: *Does there exist a* λ which guarantees that the two end conditions are satisfied for *every* $n = 0, 1, 2, \ldots$?

1.6.3 The Behavior of the Phase: The Oscillation Theorem

The answer is yes. Indeed, let $\theta(x, \lambda)$ be that solution to the Prüfer d.e. which satisfies the initial condition $\theta(a, \lambda) = \gamma$. We have one such solution for each λ. We can draw the graphs of these solutions for various values of λ. See Figure 1.14



(a) Common initial phase $\gamma \equiv \theta(x = a, \lambda) = 0$



(b) Common initial phase $\gamma \equiv \theta(x = a, \lambda) = \pi/2$

Figure 1.14: Two families of λ-parametrized phase functions $\theta(x, \lambda)$. The starting point in each family is the same for all λ. In the first family the common starting phase is $\gamma \equiv \theta(x = a, \lambda) = 0$. In the second it is $\gamma \equiv \theta(x = a, \lambda) = \pi/2$. The S-L equations are the same, namely $u'' + \lambda u = 0$.

Note that if the function $p(x)$ of the S-L equation is the constant function, then the *slope*

$$\frac{d\theta}{dx} = \frac{1}{p} \qquad (\text{when } \theta = 0, \pi, 2\pi, \dots)$$

at every "zero" of $u(x,\lambda) = r\sin\theta(x,\lambda)$ *will be* a fixed constant, *independent of* λ. However, between a pair of successive "zeroes" the slope $\frac{d\theta}{dx}$ will be the larger, the larger λ is. Consequently, for large λ, the phase $\theta(x,\lambda)$ has an undulatory (i.e., "wavelike") behavior as $\theta(x,\lambda)$ passes through the successive zeroes of $u(x,\lambda)$.

We now ask: How does

$$\theta(x,\lambda)$$

behave as a function of λ? The answer to this question is important because it determines whether the other end point condition ($\theta(b,\lambda) = \delta + n\pi$, $n = 0, 1, \dots$) can be satisfied.

The behavior of $\theta(x,\lambda)$ as a function of λ is summarized by the following three statements, which together comprise the

Oscillation Theorem:
The solution $\theta(x,\lambda)$ of the Prüfer d.e. satisfying the initial condition

$$\theta(a,\lambda) = \gamma\,, \qquad 0 \le \gamma < \pi \qquad \forall\ \lambda$$

1. is a continuous and strictly increasing function of λ,

2. $\lim_{\lambda\to\infty} \theta(x,\lambda) = \infty$, i.e., $\theta(x,\lambda)$ is unbounded, and

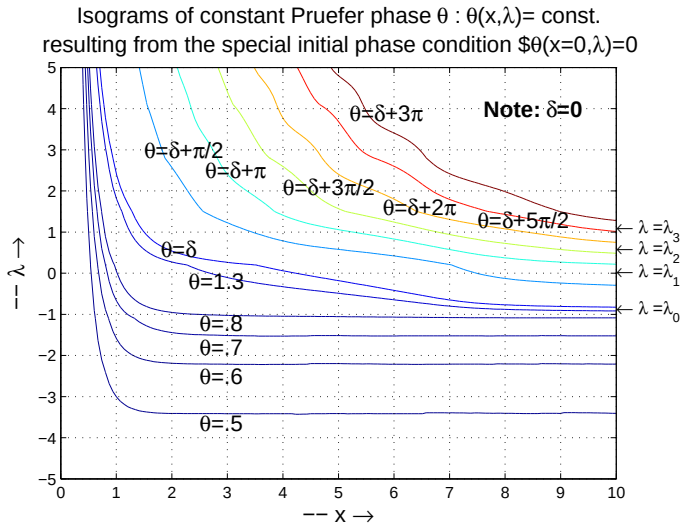
3. $\lim_{\lambda\to-\infty} \theta(x,\lambda) = 0$

for fixed x in the interval $a < x \le b$.

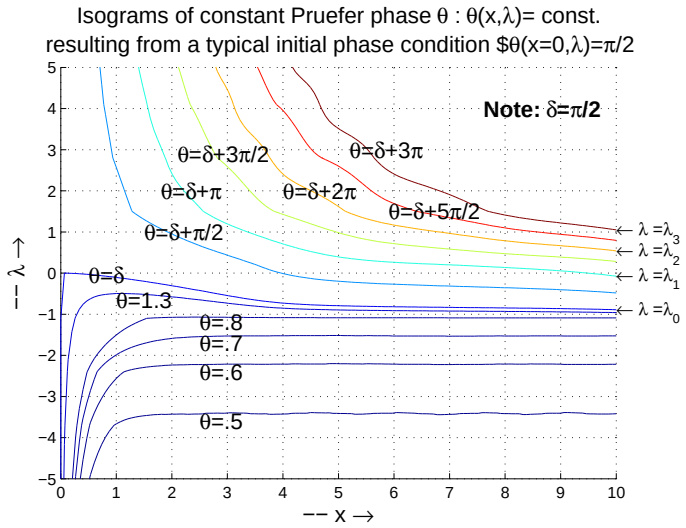
The above oscillation theorem is a statement about the global phase topograhy of the (x,λ)-domain as expressed by the scalar function $\theta(x,\lambda)$. Its domain is the vertical strip

$$\mathcal{D} = \{(x,\lambda) : a \le x \le b;\ \ -\infty < \lambda < +\infty\} \tag{1.30}$$

and its topographical features (height, gradient, critical points if any, contours, $\cdots$) are all contained in the behaviour of θ on $\mathcal{D}$.



(a) Prüfer scalar phase function contours of $\theta(x, \lambda)$ when the initial phase value (at $x = 0$) is fixed at $\theta(x = 0, \lambda) = 0$ for all λ-values



(b) Prüfer scalar phase function contours of $\theta(x, \lambda)$ when the initial phase value (at $x = 0$) is fixed at $\theta(x = 0, \lambda) = \pi/2$ for all λ-values

Figure 1.15: The (x, λ)-domain $\mathcal{D}$, Eq.(1.30), with two different contour topographies of the Prüfer phase $\theta(x, \lambda)$. The isograms in Figure (a) above result from the special initial phase value condition $\gamma \equiv \theta(x = 0, \lambda) = 0$, while those in Figure (b) result from the more typical initial phase value $\gamma \equiv \theta(x = 0, \lambda) = \pi/2$. Of all the Prüfer phase isograms, the ones having the values $\theta = \delta + n\pi, n = 0, 1, 2, \cdots$ yield the eigenvalues λ_n, which can be read off at the terminal point $x = 10$. The other isograms, $\theta = \delta + (2n+1)\frac{\pi}{2}$, and only those, yield (by inspection) the location and the number of zeros of the eigenfunction $u_n(x)$.

Instead of giving Birkhoff's and Rota's rigorous proof of this theorem, we shall increase our motivation for its validation by highlighting the geometrical meaning of the theorem's conclusions. We do this by identifying the shapes of the system of isograms of $\theta(x, \lambda)$ on $\mathcal{D}$. One thereby sees how

- they give rise to the semi-infinite eigenvalue spectrum and
- how and why they determine the location and number of zeros of the corresponding eigenfunction.

The two topographical maps (a) and (b) in Figure 1.15 tell the story. Note that the isolines of constant θ are curves $\lambda(x)$ which are homeomorphic to the horizontal lines of constant θ in Figure 1.14. There, in both (a) and (b), each line of constant θ intersects monotonically the λ-parametrized $\theta(x)$ curves.

Figures 1.15 and 1.14 contain the same information about the allowed eigenvalues λ and the number and location of zeros of the corresponding eigenfunctions. However, (a) and (b) in Figure 1.15 do so in a much more direct way.

1.6.4 Discrete Unbounded Sequence of Eigenvalues

With $x = b$ this "Oscillation theorem" tells us that $\theta(b, \lambda)$ is a function which *increases without limit* as $\lambda \to \infty$. Consequently, as λ increases from $\lambda = -\infty$, there will be a *first value*, say λ_0, for which the second boundary condition (the one at $x = b$), i.e.,

$$\theta(b, \lambda_0) = \delta$$

is satisfied. Moreover, as λ increases beyond λ_0, $\theta(b, \lambda)$ increases monotonically beyond δ until it reaches the value $\delta + \pi$. This happens at a specific value of λ, say λ_1, which is larger than λ_0,

$$\lambda_0 < \lambda_1 \ .$$

Continuing in this fashion, one finds that, regardless of how big an integer n one picks, the equation

$$\theta(b, \lambda) = \delta + n\pi$$

always has a solution for λ, which we shall call λ_n. This yields an *infinite discrete* sequence of λ's which is *monotonically increasing*

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \quad .$$

This sequence has no upper bound. Why? For any large $\Lambda > 0$ consider $\theta(b, \Lambda)$. This number lies between some pair of points, say,

$$\delta + N\pi \leq \theta(b, \Lambda) < \delta + (N+1)\pi$$

The Oscillation Theorem says that $\theta(b, \lambda)$ has the property of being a *monotonic* function of λ whose range is the *whole* positive real line. The latter property guarantees that each of the two equations,

$$\theta(b, \lambda) = \delta + N\pi$$

and

$$\theta(b, \lambda) = \delta + (N+1)\pi \quad ,$$

has a solution. The former property guarantees that each of these two solutions is unique. Call them λ_N and λ_{N+1}. The former property also guarantees that

$$\lambda_N \leq \Lambda < \lambda_{N+1} \quad .$$

Since $\Lambda > 0$ can be as large as we please, *the sequence of eigenvalues*,

$$\lambda_0 < \lambda_1 < \cdots < \lambda_N < \lambda_{N+1} < \cdots \quad ,$$

has no upper bound.

Corresponding to this sequence, there is the set of eigenfunctions

$$u_n(x) = r_n(x) \sin\theta(x; \lambda_n) \qquad n = 0, 1, 2, \ldots \, .$$

Each of these functions oscillates as a function of x. How many times does each $u_n(x)$ pass through zero in the *open* interval (a, b)? Reference to Figure 1.14 shows that $u_n(x)$ has precisely n zeroes *inside* (a, b); zeroes at the endpoints, if any, do not count. Indeed, it must have at least n zeroes because the graph of $\theta(x, \lambda)$ with λ held fixed as in Figure 1.14, must cross at least n multiples of π (dotted horizontal lines in Figure 1.14 and 1.12). On the other hand, the function $u_n(x)$ cannot have more than n zeroes because the graph of phase $\theta(x, \lambda)$ can cross each multiple of π no more than once. This fact is guaranteed by Eq.(1.28) on page 43.

To summarize, we have the following
Theorem: *Any regular S-L problem has an infinite number of solutions* $u_n(x)$ *which belong to the real eigenvalues*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \quad \text{with} \quad \lim_{n\to\infty} \lambda_n = \infty \,.$$

Furthermore, each eigenfunction $u_n(x)$

1. *has exactly* n *zeroes in the interval* $a < x < b$,
2. *is unique up to a constant multiplicative factor.*

Lecture 9

1.7 Completeness of the Set of Eigenfunctions via Rayleigh's Quotient

The fact that eigenvalues of the regular Sturm-Liouville problem form a semi-unbounded sequence, i.e., that

$$\lim_{n\to\infty} \lambda_n = \infty \,,$$

is very important. It implies that the set of eigenfunctions of the Sturm-Liouville problem

$$\begin{aligned} \mathcal{L}u &= \lambda u \\ \alpha u(a) + \alpha' u'(a) &= 0 \\ \beta u(b) + \beta' u'(b) &= 0 \quad , \end{aligned} \tag{1.31}$$

with

$$\mathcal{L} = \frac{1}{\rho(x)} \left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right] ,$$

is a *generalized Fourier basis*. In other words, they form a complete basis set for the subspace of $L^2(a,b)$ of those square-integrable functions which satisfy the given boundary conditions, Eq.(1.31). This subspace is

$$\mathcal{H} = \left\{ u : \int_a^b |u(x)|^2 \rho(x)\, dx < \infty;\ \alpha u(a) + \alpha' u'(a) = 0;\ \beta u(b) + \beta' u'(b) = 0 \right\} .$$

Recall that a set $\{u_n(x)\colon n = 0, 1, \ldots, N, \ldots\}$ is said to be *complete*, if for any vector $u \in \mathcal{H}$, the error vector

$$h_N^* = u - \sum_{n=0}^{N} c_n u_n$$

can be made to have arbitrarily small squared norm by letting $N \to \infty$, i.e.,

$$\begin{aligned} \lim_{N\to\infty} \|h_N^*\|^2 &\equiv \lim_{N\to\infty} \Big\langle u - \sum_{n=0}^{N} c_n u_n \,,\; u - \sum_{m=0}^{N} c_m u_m \Big\rangle \\ &= 0\,. \end{aligned}$$

Here

$$c_n = \langle u_n, u\rangle$$

is the nth (generalized) Fourier coefficient with the consequence that h_N^* is perpendicular to the subspace

$$W_N = \text{span}\{u_0, u_1, \ldots, u_N\}\,.$$

The subspace W_N induces $\mathcal{H}$ to be decomposed into the direct sum

$$W_N \oplus W_N^{\perp} = \mathcal{H}\,.$$

Here $W_N^{\perp}$ ("W_N perp") is the subspace of all vectors perpendicular to W_N

$$W_N^{\perp} = \{u\colon \langle u, u_n\rangle = 0 \quad n = 0, 1, \ldots, N\}\,.$$

In other words, $W_N^{\perp}$ is the space of all vectors satisfying the set of constraint conditions

$$\begin{aligned} \langle u, u_0\rangle &= 0 \\ \langle u, u_1\rangle &= 0 \\ &\vdots \\ \langle u, u_N\rangle &= 0\,. \end{aligned}$$

Our starting point for demonstrating the completeness is the Rayleigh principle. It says that the *Rayleigh quotient*

$$\frac{\langle u, \mathcal{L}u\rangle}{\langle u, u\rangle} \equiv \mathcal{R}[u]$$

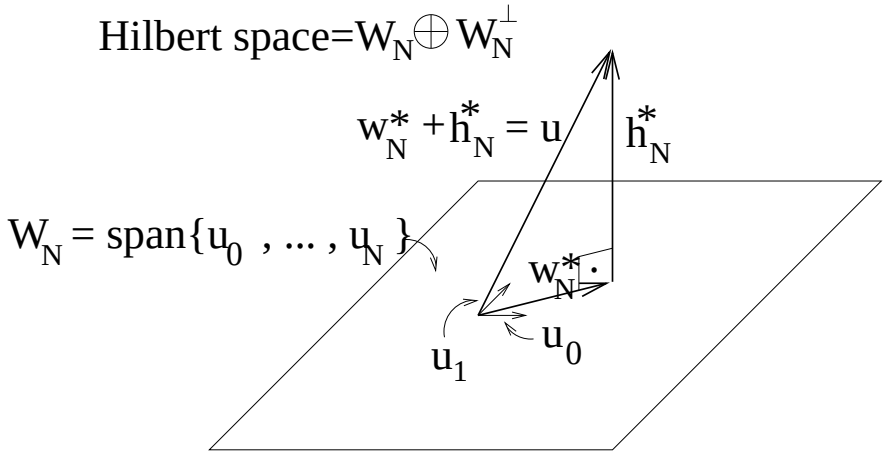


Figure 1.16: The $N+1$-dimensional subspace spanned by the eigenfunctions $u_0, u_1, \cdots, u_N$ causes the Hilbert space $L^2(a,b)$ to be decomposed into the direct sum consisting of W_N^* and the space $W_N^\perp$, which is spanned by the remaining basis vectors $u_{N+1}, u_{N+2}, \cdots$.

satisfies various minimum principles when u is restricted to lie on various subspaces $W_N^\perp$, $N = 0, 1, \ldots$. Indeed, one has

$$\begin{aligned} \lambda_0 &= \min_{u\in\mathcal{H}} \frac{\langle u, \mathcal{L}u\rangle}{\langle u, u\rangle}, \quad \text{i.e., } \frac{\langle u, \mathcal{L}u\rangle}{\langle u, u\rangle} \geq \lambda_0, \qquad \text{for all } \; u \in \mathcal{H} \\ \lambda_1 &= \min_{u\in W_0^\perp} \frac{\langle u, \mathcal{L}u\rangle}{\langle u, u\rangle}, \quad \text{i.e., } \frac{\langle u, \mathcal{L}u\rangle}{\langle u, u\rangle} \geq \lambda_1, \qquad \text{for all } \; u \in W_0^\perp \end{aligned}$$

i.e., for any $u \in \mathcal{H}$ subject to the constraint $\langle u, u_0\rangle = 0$.
More generally, the $N+1$st eigenvalue λ_{N+1} is characterized by

$$\lambda_{N+1} = \min_{u\in W_N^\perp} \frac{\langle u, \mathcal{L}u\rangle}{\langle u, u\rangle}, \quad \text{i.e., } \frac{\langle u, \mathcal{L}u\rangle}{\langle u, u\rangle} \geq \lambda_{N+1}, \quad \text{for all } \; u \in W_N^\perp$$

i.e., for any $u \in \mathcal{H}$ subject to the constraints

$$\begin{aligned} \langle u, u_0\rangle &= 0 \\ &\vdots \\ \langle u, u_N\rangle &= 0. \end{aligned}$$

The Nth error vector

$$h_N^* = u - \sum_{n=0}^{N} c_n u_n$$

satisfies the constraint conditions

$$\langle u_n, h_N^* \rangle = 0 \qquad\qquad n = 0, 1, \ldots, N\,.$$

Consequently, it satisfies the corresponding Rayleigh inequality

$$\frac{\langle h_N^*, \mathcal{L} h_N^* \rangle}{\langle h_N^*, h_N^* \rangle} \geq \lambda_{N+1}$$

or

$$\frac{\langle h_N^*, \mathcal{L} h_N^* \rangle}{\lambda_{N+1}} \geq \|h_N^*\|^2 \geq 0\,.$$

We insert the expression for h_N^* into the left hand side, and obtain

$$\begin{aligned} \ell.h.s &= \frac{1}{\lambda_{N+1}} \{ \langle u - \sum_0^N c_n u_n, \mathcal{L}u - \sum_{m=0}^N c_m \mathcal{L} u_m \rangle \} \\ &= \frac{1}{\lambda_{N+1}} \{ \langle u, \mathcal{L}u \rangle - \sum_0^N \overline{c_n} \langle u_n, \mathcal{L}u \rangle - \sum_{m=0}^N c_m \lambda_m \langle u, u_m \rangle \\ &\qquad + \sum_{n=0}^N \sum_{m=0}^N \overline{c_n} c_m \lambda_m \langle u_n, u_m \rangle\,. \end{aligned}$$

The orthonormality of eigenfunctions and the definition of the generalized Fourier coefficients guarantee that the last two sums cancel. Furthermore, by doing an integration by parts twice, and by observing that the resulting end point terms vanish because of the Dirichlet-Neumann boundary conditions, Eq. 1.31, we obtain

$$\begin{aligned} \langle u_n, \mathcal{L}u \rangle &= \int_a^b \overline{u}_n \frac{1}{\rho} \left[-\frac{d}{dx} p \frac{d}{dx} + q \right] u \rho(x) dx \\ &= -p \overline{u}_n u \,|_a^b + \int_a^b \left(p \frac{d\overline{u}_n}{dx} \frac{du}{dx} + q \overline{u}_n u \right) dx \\ &= \langle \mathcal{L} u_n, u \rangle = \lambda_n c_n\,. \end{aligned}$$

As a consequence the Rayleigh inequality becomes

$$\frac{1}{\lambda_{N+1}}\{\langle u, \mathcal{L}u\rangle - \sum_{n=0}^{N} |c_n|^2 \lambda_n\} \geq \|h_N^*\|^2 .$$

Without loss of generality one may assume that the lowest eigenvalue $\lambda_0 \geq 0$. This can always be made to come about by readjusting the λ and the function $q(x)$ in the Sturm-Liouville equation. As a result, the finite sum may be dropped without decreasing the $\ell.h.s.$ Consequently,

$$\frac{\langle u, \mathcal{L}u\rangle}{\lambda_{N+1}} \geq \|u - \sum_{n=0}^{N} c_n u_n\|^2 .$$

The numerator is independent of N. Thus

$$\lim_{N\to\infty} \|u - \sum_{n=0}^{N} c_n u_n\|^2 \leq \lim_{N\to\infty} \frac{\langle u, \mathcal{L}u\rangle}{\lambda_{N+1}} = 0$$

because $\{\lambda_N \colon N = 0, 1, \dots\}$ is an unbounded sequence. Thus we have

$$u \doteq \sum_{n=0}^{\infty} c_n u_n ,$$

The function u is an arbitrary square integrable function satisfying the the given mixed Dirichlet-Neuman end point conditions. Consequently, the Sturm-Liouville eigenfunctions form a (complete) generalized Fourier basis indeed.

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Chapter 2

Infinite Dimensional Vector Spaces

Lecture 10

The process of extending the algebraic and geometrical methods of linear algebra from matrices to differential or integral operators consists of going from a finite dimensional vector space, typically R^n, to an infinite dimensional vector space, typically a function space.

However, a vector space of functions has certain idiosyncrasies precisely because its dimension is infinite. These peculiarities are so important that we must develop the framework in which they arise.

One of the most useful, if not the most useful, framework is the theory of *Hilbert spaces*, the closest thing to the familiar finite-dimensional Euclidean spaces. In passing we shall also mention *metric spaces* and *Banach spaces*.

We shall see that infinite dimensional vector spaces are a powerful way of organizing the statement and solution of boundary value problems. In fact, these spaces are the tool of choice whenever the linear superposition principle is in control. This happens in signal processing, in quantum mechanics, electromagnetic wave theory, and elsewhere.

The most notable peculiarity associated with infinite dimensional vector spaces is the issue of *completeness*.

From the viewpoint of physics and engineering, completeness is an issue of precision in measurements. We would like to have at our disposal mathematical concepts which are such that they are capable of mathematizing

natural phenomenona no matter how advanced the technolgy of measuring their properties.

One of the most useful infinite dimensional vector spaces is Hilbert space. To define it, we must have at our disposal the constellation of concepts on which it is based. Let us identify the components of the constellation.

2.1 Inner Product Spaces

An *inner product space* is a vector space, say $\mathcal{H}$, together with a complex bilinear function $\langle\ ,\ \rangle$ having the following properties:

(i) $\langle f, g\rangle = \overline{\langle g, f\rangle}$ where $f, g \in \mathcal{H}$
(ii) $\langle f, \alpha_1 g_1 + \alpha_2 g_2\rangle = \alpha_1\langle f, g_1\rangle + \alpha_2\langle f, g_2\rangle$
where α_1 and α_2 are complex numbers
(iii) $\langle f, f\rangle > 0$ if $f \neq \vec{0}$
and $\langle f, f\rangle = 0 \Leftrightarrow f = \vec{0}$.

Comments:

(a) The condition $\langle f, g\rangle = \overline{\langle g, f\rangle}$ is quite necessary, otherwise there would be conflict with (iii). Indeed, if $i = \sqrt{-1}$, then

$$\begin{aligned}\langle if, if\rangle &= i\langle if, f\rangle = i\overline{\langle f, if\rangle} = i(-i)\overline{\langle f, f\rangle} \\ &= \langle f, f\rangle > 0 \quad .\end{aligned}$$

In other words, condition (i) guarantees that the *positive definiteness* condition (iii) is preserved.

(b) With the help of (i), condition (ii) is equivalent to

$$\langle \alpha_1 f_1 + \alpha_2 f_2, g\rangle = \overline{\alpha}_1\langle f_1, g\rangle + \overline{\alpha}_2\langle f_2, g\rangle\,. \tag{2.1}$$

Thus we see that a complex scalar (say, α_1 or α_2) in the first factor of the inner product gets complex conjugated when it gets separated from the inner product as a multiplicative factor. One says that $\langle\ ,\ \rangle$ is *linear* in the second argument and *antilinear* in the first argument.

(c) The square root of $\langle f, f\rangle$, $\sqrt{\langle f, f\rangle} \equiv \|f\|$, is called the *norm* of the vector f. It is always understood that the norm is finite. In particular $\langle f, f\rangle < \infty$

(d) The inner product satisfies the *Cauchy-Schwarz inequality*

$$|\langle f, g\rangle| \leq \|f\| \, \|g\| \ .$$

This inequality has a nice geometrical interpretation for real inner product spaces. In that case $\langle f, g\rangle = \langle g, f\rangle$ is the familiar inner product and

$$-1 \leq \frac{\langle f, g\rangle}{\|f\|\|g\|} \equiv \cos(\text{angle between } f \text{ and } g) \leq 1 \quad .$$

The Cauchy-Schwarz inequality follows from the fact that for any complex λ

$$\begin{aligned} 0 \leq \langle \lambda f + g, \lambda f + g\rangle &= |\lambda|^2 \|f\|^2 + \|g\|^2 \\ &+ \overline{\lambda}\langle f, g\rangle + \lambda\langle g, f\rangle \ . \end{aligned}$$

Letting $\lambda = x\frac{\langle f,g\rangle}{|\langle f,g\rangle|}$ we obtain for all real x

$$0 \leq x^2\|f\|^2 + 2x|\langle f, g\rangle| + \|g\|^2 \quad .$$

Consequently, the discriminant,

$$|\langle f, g\rangle|^2 - \|f\|^2\|g\|^2 \, ,$$

of this quadratic expression must be negative or zero, otherwise this expression would be negative for some values of x. It follows that

$$|\langle f, g\rangle| \leq \|f\| \, \|g\| \, .$$

(e) The inner product implies the *triangle inequality*

$$\|f \pm g\| \ \leq \ \|f\| + \|g\| \quad . \tag{2.2}$$

This inequality readily follows from the properties of the inner product (Why?)

2.2 Normed Linear Spaces

There exist other structures which a vector space may have. A *norm* on the vector space $\mathcal{V}$ is a linear functional, say $p(f)$, with the following three properties:

1. Positive definiteness: $p(f) > 0$ for all nonzero vectors f in $\mathcal{V}$, and $p(f) = 0 \Leftrightarrow f = \vec{0}$.

2. Linearity: $p(\alpha f) = |\alpha| p(f)$ for all vectors f and for all complex numbers α.

3. Triangle inequality: $p(f + g) \leq p(f) + p(g)$ for all vectors f and g in $\mathcal{V}$.

Such a function is usually designated by $p(f) = \|f\|$, a *norm* of the vector f. The existence of such a norm gives rise to the following definition:

A linear space $\mathcal{V}$ equipped with a norm $p(f) = \|f\|$ is called a *normed linear space*.

Example 1: Every inner product of an inner product space determines the norm given by

$$\|f\| = (\langle f, f \rangle)^{\frac{1}{2}} \quad ,$$

which, as we have seen, satisfies the triangle inequality,

$$\|f + g\| \leq \|f\| + \|g\| \, .$$

Thus *an inner product space is always a normed linear space with the inner product norm.* However, a normed linear space is not necessarily an inner product space.

Lecture 11

Example 2: Consider the vector space of $n \times n$ matrices $A = [a_{ij}]$. Then

$$\|A\| = \max_{i,j} |a_{ij}|$$

is a norm on this vector space.

Example 3: Consider the vector space of all infinite sequences

$$x = (x_1, x_2, \ldots, x_k, \ldots)$$

of real numbers satisfying the convergence condition

$$\sum_{k=1}^{\infty} |x_k|^p < \infty$$

where $p \geq 1$ is a real number. Let the norm be defined by

$$\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} .$$

One can show that (*Minkowski's inequality*)

$$\left(\sum_{k=1}^{\infty} |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p \right)^{\frac{1}{p}} ,$$

i.e., that the triangle inequality,

$$\|x + y\| \leq \|x\| + \|y\| ,$$

holds. Hence $\| \cdot \|$ is a norm for this vector space. The space of p-summable $\left(\sum_{1}^{\infty} |x_k|^p < \infty \right)$ real sequences equipped with the above norm is called ℓ^p and the norm is called the ℓ^p-norm.

This ℓ^p-norm gives rise to geometrical objects with unusual properties. consider the following

Example 4: The surface of a unit sphere centered around the origin of a linear space with the ℓ^p-norm is the locus of points $\{(x_1, x_2, \cdots\}$ for which

$$\left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} = 1 .$$

Consider the intersection of this sphere with the finite dimensional subspace R^n, which is spanned by $\{(x_1, x_2, \cdots, x_n\}$.

a) When $p = 2$, this intersection is the locus of points for which

$$|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = 1 \quad \text{(unit sphere in } R^n \text{ with } \ell^2\text{-norm)}$$

This is the familiar $(n-1)$-dimensional unit sphere in n-dimensional Euclidean space whose distance function is the *Pythagorean* distance

$$d(x, y) = \left\{ |x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2 \right\}^{\frac{1}{2}} .$$

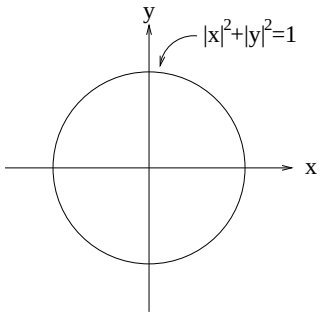

Figure 2.1: Circle in R^2 endowed with the Pythagorean distance function of ℓ^2.

b) When $p = 1$, this intersection is the locus of points for which

$$|x_1| + |x_2| + \cdots + |x_n| = 1 \quad \text{(unit sphere in } R^n \text{ with } \ell^1\text{-norm)}$$

This is the $(n-1)$-dimensional unit sphere in n-dimensional vector space endowed with a different distance function, namely one which is the sum of the differences

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| \ ,$$

between the two locations in R^n, instead of the sum of squares. This distance function is called the *Hamming* distance, and one must use it, for example, when travelling in a city with a rectangular grid of streets. With such a distance function a circle in R^2 is a square standing on one of its vertices. See Figure 2.2. A 2-sphere in R^3 is a cube standing on one of its vertices, etc.

c) When $p \to \infty$, this intersection is the locus of points for which

$$\begin{aligned} \lim_{p\to\infty} \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} &= 1 \quad \Longrightarrow \\ \text{Max}\{|x_1|, |x_2|, \cdots, |x_n|\} &= 1 \quad \text{(unit sphere in } R^n \text{ with } \ell^\infty\text{-norm)} \ . \end{aligned}$$

Such a unit sphere R^n is based on the distance function

$$\begin{aligned} d(x, y) &= \lim_{p\to\infty} \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{\frac{1}{p}} \\ &= \text{Max}\{|x_1 - y_1|, |x_2 - y_2|, \cdots, |x_n - y_n|\} \ . \end{aligned}$$

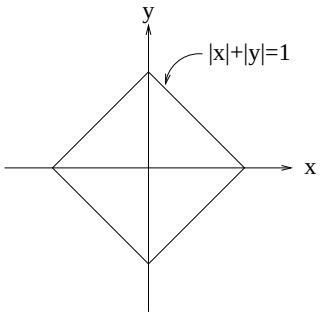

Figure 2.2: Circle in R^2 endowed with the Hamming distance function of ℓ^1.

This is called the *Chebyshev* distance. It is simply the maximum coordinate difference regardless of any other differences. With such a distance function a circle in R^2 is a square. See Figure 2.3. A 2-sphere in R^3 is a cube, etc.

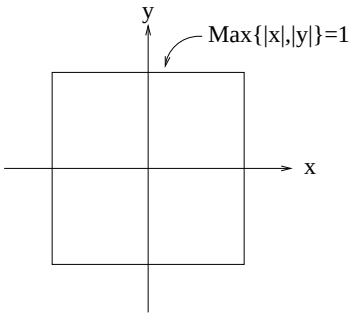

Figure 2.3: Circle in R^2 endowed with the Chebyshev distance function of ℓ^∞.

2.3 Metric Spaces

Inner product spaces as well as normed spaces have a distance function, namely the norm of the difference,

$$\|f - g\| \equiv d(f, g) \quad ,$$

between two vectors f and g. This norm of the difference is called the *distance* between f and g. Applying this formula to the three pairs of points

of a generic triangle, one obtains the *triangle inequality*

$$\|f-h\| \leq \|f-g\| + \|g-h\|$$

i.e., $d(f,h) \leq d(f,g) + d(g,h)$.

The importance of a distance function and the triangle inequality is that it can also be applied to certain *nonlinear* spaces, which have no zero element ("origin"). Such spaces are called *metric spaces*. More precisely we have the following definition.

By a *metric space* is meant a pair (X,d) consisting of set X and a *distance function* d, i.e., a single-valued, nonnegative, real function $d(f,g)$ defined for $f,g \in X$ which has the following three properties.

1. Positive definiteness: $d(f,g) \geq 0$ for all f and g in X, and $d(f,g) = 0$ if and only if $f = g$.
2. Symmetry: $d(f,g) = d(g,f)$.
3. Triangle inequality: $d(f,h) \leq d(f,g) + d(g,h)$.

The distance function $d(\ ,\ )$ is called the *metric* of the *metric space*. *All* inner product spaces are metric spaces with

$$d(f,g) = \|f-g\| \quad .$$

All normed linear spaces are metric spaces with

$$d(f,g) = \|f-g\| \, .$$

However not all metric spaces are normed linear spaces.

Example 1: The two dimensional surface of a sphere

$$X = \{(x,y,z)\colon x^2+y^2+z^2-1\} \quad (\equiv S^2)$$

is not a vector space. It is, however, a metric space whose distance function is the (shortest) length of the great circle passing between a pair of points.

Exercise 2.3.1 (DISTANCE FUNCTIONS AS METRICS)
Show that (a) the Hamming distance, (b) the Pythagorean distance, and (c) the Chebyshev distance each satisfy the triangle inequality.

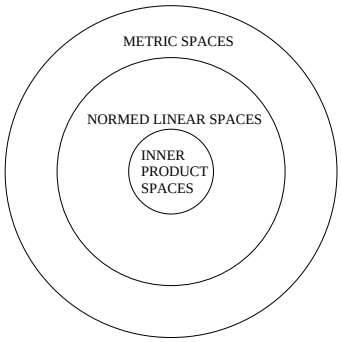

Figure 2.4: Hierarchy of linear and nonlinear spaces

2.4 Complete Metric Spaces

Lecture 12

2.4.1 Limit of a Sequence

A sequence of elements $f_1, f_2, \ldots$ in a vector space, or more generally in a metric space, is said to *converge* to the element f if

$$\lim_{n\to\infty} f_n = f.$$

The element f is called the *limit* of the sequence $\{f_n\}$. The meaning of this is that the distance

$$d(f_n, f) = \|f_n - f\|$$

between f and f_n can be made arbitrarily small by making n sufficiently large. To summarize, a *convergent* sequence is one which *converges to a limit*.

Side Comment: It is easy to show that this limit is unique, a property which applies to all metric spaces.

2.4.2 Cauchy Sequence

It is clear that the elements $f_1, f_2, \ldots, f_m, \ldots, f_n, \ldots$ become closer and closer in some sense. In fact, from the triangle inequality for a vector space

$$\|f + g\| \leq \|f\| + \|g\|$$

one finds that

$$\begin{aligned} \|f_n - f_m\| &= \|(f_n - f) + (f - f_m)\| && (2.3) \\ &\leq \|f_n - f\| + \|f - f_m\| \to 0 \ \text{ as } m, n \to \infty && (2.4) \end{aligned}$$

or more generally

$$d(f_n, f_m) \leq d(f_n, f) + d(f, f_m) \to 0 \ \text{ as } m, n \to \infty$$

in a metric space.

Consequently

$$\lim_{n,m\to\infty} \|f_n - f_m\| = 0 \ \text{ in a vector space}$$

or

$$\lim_{n,m\to\infty} d(f_n, f_m) = 0 \ \text{ in a metric space}.$$

A sequence $\{f_n\}$ whose elements satisfy this limit condition, i.e., whose elements get arbitrarily close together for sufficiently large n and m, is a *Cauchy sequence*. Thus

$$\{f_n\} \ \text{ has a limit } \ \Rightarrow \{f_n\} \ \text{ is a Cauchy sequence}.$$

Thus every convergent sequence is a Cauchy sequence, i.e., "every convergent sequence also converges in the Cauchy sense".

2.4.3 Cauchy Completeness: Complete Metric Space, Banach Space, and Hilbert Space

Can one turn the last sentence around? In other words, is every Cauchy sequence a convergent sequence? Put differently, if $\{f_n\}$ is a Cauchy sequence, is it true that $\{f_n\}$ has a limit? What is, in fact, meant by this question is whether $\{f_n\}$ has a limit in the same space to which the elements f_n belong. The answer is this: in a finite dimensional complex vector space, a Cauchy sequence always has a limit in that vector space; in other words, a *finite dimensional vector space is complete*. However, such a conclusion is no longer true for many familiar infinite dimensional vector spaces.

Example: Consider the inner product space

$$C[a, b] = \{f(x) \colon f \text{ is continuous on } a \leq x \leq b\}$$

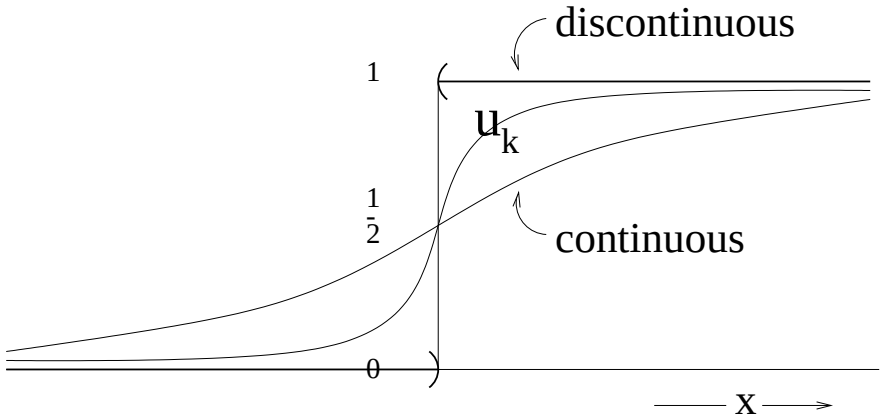

Figure 2.5: Discontinuous function as a limit of continuous functions.

with inner product $\langle f, f\rangle = \int_a^b \overline{f} f \; dx \; = \|f\|^2$.
Claim: $C[a, b]$ is "incomplete".
Discussion: Consider the sequence of continuous functions

$$u_k(x) = \frac{1}{2} + \frac{1}{\pi}\arctan kx \qquad -1 \le x \le 1.$$

From Fig. 2.5 we see that:

1. $\lim\limits_{k,p\to\infty} \|u_k - u_p\|^2 = \lim\limits_{k,p\to\infty} \int_{-1}^{1} (u_p - u_k)^2 \, dx = 0$; in other words, the sequence $\{u_k\}$ is a *Cauchy sequence*.

2. For fixed x
$$\lim_{k\to\infty} u_k(x) = v(x) = \begin{cases} 1 & 0 < x \le 1 \\ \frac{1}{2} & x = 0 \\ 0 & -1 \le x < 0 \end{cases}$$
which is a *discontinuous* function, i.e., $v \notin C[-1, 1]$. Furthermore, we say that the sequence of functions $\{u_k \; : k = 1, 2, \cdots\}$ *converges pointwise* to the function v.

3. $\lim\limits_{k\to\infty} \|v - u_k\|^2 = \lim\limits_{k\to\infty} \int_{-1}^{1} (v(x) - u_k)^2 \, dx = 0$.

4. One can show that $\nexists$ *any* continuous function w such that
$$\|w - u_k\| \to 0 \;\; \text{as} \;\; k \to \infty \, .$$

We say that $C[a,b]$, the space of *continuous* square integrable functions ($\int_a^b |f|^2\,dx < \infty$), is *Cauchy incomplete*, or $C[a,b]$ is Cauchy *incomplete relative* to the given norm $\|f\| = \sqrt{\langle f, f\rangle}$. This is so because we have found a Cauchy sequence of functions $\{u_n\}$ in the inner product space $C[a,b]$ with the property that

$$\lim_{n\to\infty} u_n = v \notin C[a,b]\,.$$

In other words, the limit of the Cauchy sequence does not lie in the inner product space. This is just like the set of rationals which is not extensive enough to accomodate the norm $|x-y|$: there are holes (but no gaps) in the space. These holes are the irrational numbers, which are not detected by $|\cdots|$ when it is used to determine whether or not an infinite sequence is convergent.

Example: The 1-dimensional vector space of rationals over the field of rationals is Cauchy incomplete.

Problem: Why is it that the real line equipped with the distance

$$d(x,y) = |\arctan x - \arctan y|$$

is an incomplete metric space?

In order to remove this incompleteness deficiency, one enlarges the space so that it includes the limit of any of its Cauchy sequences. A space which has been enlarged in this sense is said to be *Cauchy complete*. This enlarged space is called a *complete metric space*.

The Cauchy completion of the rationals are the *reals*. The Cauchy completion of an inner product space is a *Hilbert space* The Cauchy completion of a normed linear space is a *Banach space*.

2.5 Hilbert Spaces

Lecture 13

An infinite dimensional inner product space which is Cauchy-complete is called a Hilbert space.

In the realm of infinite dimensional vector spaces, a Hilbert space is the next best thing to an Euclidean space, i.e., a finite dimensional inner product space. The single most useful property of these spaces is that they permit the introduction of an orthonormal basis.

The first and most important way of specifying such a basis is to introduce a Hermitian matrix or operator. Its eigenvectors form an orthonormal basis. In fact, this is why a Hilbert space was invented in the first place: to accomodate the eigenvalue problem of a Hermitian operator,

$$Au = \lambda u$$

The Sturm-Liouville eigenvalue problem, Section 1.3.3, page 19, is a premiere example of this. It arises in the mathematization of boundary value problems, for example.

The second way of specifying such a basis is by means of the *Gram-Schmidt* orthogonalization process. From a given linearly independent set of vectors, one constructs by an iterative process a corresponding set of orthonormal vectors.

Let us, therefore, assume that we have acquired by one of these, or by some other method, the system of orthonormal elements

$$\{u_1, u_2, \ldots, u_n, \ldots : \langle u_i, u_j \rangle = \delta_{ij}\}$$

of the Hilbert space $\mathcal{H}$.

If such a system of o.n. basis vectors has been obtained, it is natural to ask: Does the set of such orthonormal elements constitute a *basis* for the whole space $\mathcal{H}$? In other words, for *any* $f \in \mathcal{H}$, can one, in some sense, claim that

$$f = c_1 u_1 + c_2 u_2 + \cdots \qquad \text{(Generalized Fourier series)}$$

for an appropriate choice of c_i? This is an existence question. Furthermore, given that such constants have been constructed, are they unique?

For a finite dimensional vector space these questions have an implicit affirmative answer because of the finiteness of the dimension. However, for an infinite dimensional vector space there is cause to worry about existence. The vector f might point into a direction which is so peculiar that not even the given infinite number of basis vectors is sufficient to give a 100% accurate representation of f in terms of these vectors. There is a sense in which this worry is justified if the vector is a discontinuous function. This fact is highlighted by the Fourier Theorem in chapter 3 on page 124. However, there is another sense in which the representation is always 100% accurate, with the result that the answer to the above questions is in the affirmative.

2.5.1 Two Prototypical Examples

The two most important Hilbert spaces in physics (e.g quantum mechanics), engineering (e.g. signal processing, vibrations), and mathematics (e.g. solutions to differential equations) are:

1. The vector space of *square summable sequences*

$$\ell^2 = \{U = (c_1, c_2, \dots) \colon c_i \in \{\text{ complex numbers }\},\ i = 1, 2, \dots\}$$

with squared norm given by the inner product

$$\|U\|^2 = \langle U, U\rangle = \sum_{i=1}^{\infty} |c_i|^2 .$$

2. The vector space of *square integrable functions*

$$L^2(a,b) = \{f \colon \int_a^b \overline{f} f \rho(x) dx < \infty\,;\ \rho(x) > 0\}.$$

The positive function $\rho(x)$ is given. It is called a *weight function*.

This vector space has the following three properties.

(a) $L^2(a,b)$ is an *inner product* space

$$\begin{array}{rcll} \langle f, g\rangle & \equiv & \int_a^b \overline{f} g \rho(x)\, dx & \text{Physics convention} \\ & \equiv & \int_a^b f \overline{g} \rho(x)\, dx & \text{Mathematics convention} \\ & \Updownarrow & & \\ \langle f, f\rangle & \equiv & \int_a^b |f|^2 \rho(x)\, dx & \text{which is the squared norm of } f \end{array}$$

Comment: $\rho(x) > 0 \Rightarrow \sqrt{\rho}$ can be absorbed into the functions, so that instead of $\{f(x)\}$ one has $\{h(x)\} = \{f(x)\sqrt{\rho(x)}\}$ with the squared norm

$$\langle h, h\rangle = \int_a^b |h|^2\, dx.$$

Conclusion: We still have the *same* inner product space.

(b) $L^2(a,b)$ is *closed* under addition:

i. Expanding the inner product of a sum with itself, we have

$$\begin{aligned}\|f+g\|^2 = \langle f+g, f+g\rangle &= \|f\|^2 + \|g\|^2 + \langle f,g\rangle + \langle g,f\rangle \\ &= \|f\|^2 + \|g\|^2 + 2\mathrm{Re}\,\langle f,g\rangle \\ &\leq \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\|,\end{aligned} \tag{2.5}$$

where we used the Cauchy-Schwarz inequality.

ii. Recall that

$$0 \;\leq\; (\|f\| - \|g\|)^2 = \|f\|^2 + \|g\|^2 - 2\|f\|\,\|g\|. \tag{2.6}$$

iii. Adding this non-negative quantity to the r.h.s. of (2.5) increases it. Consequently, Eqs.(2.5) and (2.6) $\Rightarrow \|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$.
Thus we have

$$f, g \;\text{ square integrable} \Rightarrow f+g \;\text{ is square integrable}\,,$$

i.e. L^2 is indeed closed under addition.

(c) $L^2(a,b)$ is Cauchy complete.

2.5.2 Hilbert Spaces: Their Coordinatizations

Lecture 14

The importance of L^2 derives from the fact that its elements refer, among others, to the finite energy states of an archetypical system, a vibrating system or the finite energy signals of an ensemble of messages. These states/signals are mathematized in terms of functions. On the other hand, for the purpose of measurement they need to be, or have been, represented by sequences of numbers, i.e. by elements of ℓ^2. Without measurement data expressed in terms of these elements, these states would be mere floating abstraction disconnected from the physical world. It is the elements of ℓ^2 which ground such finite energy states in observations and measurements. Granted their epistemic foundation in the physical world, what is the role of L^2 and ℓ^2 in the structure of linear mathematics?

To this end recall that, given an n-dimensional vector space V, then a choice of basis determines an isomorphism F which relates V to R^n, the space of n-tuples, a coordinate realization of V:

$$\begin{array}{ccc} V & \xrightarrow{\text{F}} & R^n \\ u & \rightsquigarrow & F(u) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \end{array}$$

Here

(i) F is induced by a given system of orthonormal spanning vectors u_k. This means that once $\{u_k\}$ is given, F is determined: for any u in V F yields a unique $F(u)$, the array expansion coefficients c_k.

(ii) F is linear. This means that F is mathematized by means of a matrix.

(iii) F is one-to-one.

(iv) F is onto,

where "onto" means that, given any $v \in R^n$, one can solve

$$F(u) = v \tag{2.7}$$

for $u \in V$, while "one-to-one" means that such a solution is unique. In brief, F is an isomorphic relation between V and R^n, and it is induced by $\{u_k\}_{k=1}^n$.

The extension of the idea of such an isomorphism to infinite-dimensional Hilbert spaces results in the claim that a Hilbert space $\mathcal{H}$ has ℓ^2 as *its* coordinate realization. Relating these two spaces is the isomorphism $\mathcal{F}$,

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mathcal{F}} & \ell^2 \\ f & \rightsquigarrow & \mathcal{F}[f] = \{c_1, c_2, \cdots\} \end{array}$$

Here, as in the finite dimensional case,

(i) $\mathcal{F}$ is induced by a given system of orthonormal vectors u_k.

(ii) $\mathcal{F}$ is linear.

(iii) $\mathcal{F}$ is one-to-one.

(iv) $\mathcal{F}$ is onto.

An isomorphism is a two-way map $\mathcal{H} \longrightarrow \ell^2$ and $\mathcal{H} \longleftarrow \ell^2$. In finite dimensions its validation is achieved by algebraic manipulations on a basis-induced system of equations.

In infinite dimensions, however, the method for validation is necessarily different. It is a three-step process:

I. Identify an infinite system of o.n. vectors. This sytem induces a linear map $\mathcal{F}$ from $\mathcal{H}$ to ℓ^2.

II. Apply Bessel's least squares theorem to the system of o.n. vectors. This application starts by constructing the linear map,

$$\mathcal{F}: \ \mathcal{H} \longrightarrow \ell^2 \ . \tag{2.8}$$

The map $\mathcal{F}$ is the unifying kingpin in the whole subsequent development, as summarized by the three theorems 1.5.1, 1.5.2, and 1.5.3 below.

According to Bessel's least squares theorem (1.5.1 below), even if more than one $f \in \mathcal{H}$ yields the same $\{c_k\} \in \ell^2$, each one of them has the same optimal (least squared error) representative in the subspace spanned by those o.n. vectors. This representative is the result of Bessel's least squares error analysis. If the least squared error is zero, i.e. Parseval' identity is satisfied, Bessel's theorem guarantees that there is only a single function f having the particular least squares representation induced by the system of o.n. vectors. In other words, the map $\mathcal{F}$ is one-to-one (Theorem 1.5.2).

III. Use the Riesz-Fischer Theorem (1.5.3). Its gist is the fundamental feature that, for a system with an infinite number of o.n. vectors, this map is always onto. In other words, for every element $\{c_k\} \in \ell^2$ there is at least one element $f \in \mathcal{H}$.

In summary, Fischer and Riesz guarantee that $\mathcal{F}$ is *onto*, while Bessel and Parseval guarantee that it is *one-to-one*. In brief, $\mathcal{F}$ is an isomorphism. Put differently, as exemplified by the contributions of these four workers, human knowledge of mathematical methods is not a mere collection, but a

structure. Their works, even though developed separately, does not amount to a juxtaposition. By applying the concept of an isomorphism to their works one forms a unified structure, the *coordinatization of a Hilbert space*.

Bessel and Parceval

Bessel's fundamental optimization process, in particular the construction of of the linear map Eq.(2.8), is summarized by the following

Theorem 2.5.1 (Bessel)
(Least squares approximation via subspaces) Given an o.n. system

$$u_1, u_2, \ldots, u_k, \ldots$$

in the Hilbert space $\mathcal{H} \subseteq L^2$, let f be an arbitrary element of $\mathcal{H}$. Then
(i) the expression

$$\left\| f - \sum_{k=1}^{N} a_k u_k \right\|^2 \equiv E_N^2(a_1, \ldots, a_N)$$

has a minimum, for

$$a_k = \langle u_k, f \rangle \equiv c_k \quad k = 1, \cdots, N \ .$$

(ii) This minimum equals

$$\|f\|^2 - \sum_{k=1}^{N} |c_k|^2 = E_N^2(c_1, \ldots, c_N) \quad N = 1, 2, \cdots \ .$$

with its associated hierarchy

$$\|f\|^2 \geq E_1^2(c_1) \geq E_2^2(c_1, c_2) \geq \cdots \geq E_N^2(c_1, \cdots, c_N) \geq \cdots \geq 0 \ .$$

(iii) Moreover,

$$\sum_{k=1}^{\infty} |c_k|^2 \leq \|f\|^2 \, ,$$

a result known as Bessel's inequality.

Note: This theorem introduces *en passant* two new concepts which are key to the subsequent development:

- The coefficients
$$c_k = \langle u_k, f \rangle$$
are called the (generalized) *Fourier coefficients*. They form the image of the function f under the *linear* transformation
$$\mathcal{F}: \ \mathcal{H} \to \ell^2 \tag{2.9}$$
$$f \rightsquigarrow \mathcal{F}[f] = \{c_k\}_{k=1}^{\infty} \tag{2.9'}$$

- The sequence of sums
$$\sum_{k=1}^{N} c_k u_k \equiv S_N \quad N = 1, 2, \cdots ,$$
is called the sequence of *partial Fourier series of* f, with S_N being the N^{th} *partial Fourier series*.

Nota bene: The function $E_N^2(a_1, \ldots, a_N)$ is called *Gauss's mean squared error*. Minimizing it by setting
$$\frac{\partial E_N^2}{\partial \overline{a}_k} = 0 \qquad\qquad k = 1, \ldots, N$$
yields the N Fourier coefficients
$$a_k = \langle u_k, f \rangle \equiv c_k \qquad\qquad k = 1, \ldots, N$$
as the *solution* to this equation (try it!). The word "mean" in *Gauss's mean squared error* arises from its defining property,
$$E_N^2 = \int_a^b |f(x) - \sum_{k=1}^{N} a_k u_k(x)|^2 \rho(x)\, dx \ .$$
The integrand $|f(x) - \sum_{k=1}^{N} a_k u_k(x)|^2$ is the error at x, while the integral is $(b-a)$ times the (weighted) "mean" of this quantity, in compliance with the mean value theorem of integral calculus.
Proof: The Gaussian mean squared error function
$$\langle f - \sum_{k=1}^{N} a_k u_k \,,\ f - \sum_{\ell=1}^{N} a_\ell u_\ell \rangle \equiv E_N^2$$

is a quadratic expression in the complex unknowns a_k. As usual, in such expressions *completing the square* will yield the minimum value at a glance. Multiplying out the inner product yields

$$E_n^2 = \|f\|^2 - \sum_{k=1}^{N} \overline{a}_k \langle u_k, f\rangle - \sum_{\ell=1}^{N} a_\ell \langle f, u_\ell\rangle + \sum_{k=1}^{N}\sum_{\ell=1}^{N} \overline{a}_k a_\ell \langle u_k, u_\ell\rangle \,.$$

By (i) introducing the Fourier coefficients

$$c_k = \langle u_k, f\rangle$$

of f *relative to* the system $\{u_k\}$, (ii) using the orthonormality of the u_k's yields, and (iii) adding and subtracting $\sum\limits_{k}^{N} |c_k|^2$, one obtains

$$\begin{array}{rcccccccc} E_N^2 & = & \|f\|^2 & - & \sum_{k=1}^N \overline{a}_k c_k & - & \sum_{\ell=1}^N a_\ell \overline{c}_\ell & + & \sum_{k=1}^N \overline{a}_k a_k \\ & & & - & \sum_{k=1}^N |c_k|^2 & & & + & \sum_{k=1}^N \overline{c}_k c_k \\ & = & \|f\|^2 & - & \sum_{k=1}^N |c_k|^2 & + & \sum_{k=1}^N |a_k - c_k|^2 & & \end{array}$$

This expression is the key to validating the three conclusions of the theorem.

(i) E_N^2 achieves its minimum when

$$a_k = c_k \ .$$

Thus $\mathcal{F}[f] = \{c_k\}_{k=1}^{\infty}$ is *linear*.

(ii) The minimum value of E_N^2 is

$$\begin{aligned} E_N^2(c_1, \cdots, c_N) &= \|f - S_N\|^2 \\ &= \|f\|^2 - \sum_{k=1}^{N} |c_k|^2 \ . \end{aligned} \tag{2.10}$$

(iii) The fact that this holds for all integers N implies

$$\sum_{k=1}^{\infty} |c_k|^2 \le \|f\|^2 \ ,$$

which is also called *Bessel's inequality*.

Bessel's Inequality: Its Geometrical Meaning

This theorem can also be summarized geometrically as follows:

1. The set of linear combinations

$$\text{span}\,\{u_1, \ldots, u_N\} \equiv W_N \subset \mathcal{H} \subseteq L^2$$

is a *subspace* of L^2, and the N^{th} *partial Fourier sum*

$$\sum_{k=1}^{N} c_k u_k \equiv w_N^* \tag{2.11}$$

is the *orthogonal projection* of f onto W_N. The squared length of w_N^* is

$$\begin{aligned} \|w_N^*\|^2 &= \langle \sum_{k=1}^{N} c_k u_k, \sum_{k=1}^{N} c_k u_k \rangle \\ &= \sum_{k=1}^{N} |c_k|^2 \ , \end{aligned}$$

which is the Pythagorean theorem in W_N.

2. This projection of f onto W_N is linear. It is given by

$$w_N^* = \sum_{k=1}^{N} u_k \langle u_k, f \rangle \equiv P_{W_N} f \quad (\in W_N) \ . \tag{2.12}$$

It is depicted in Figure 2.6, and it has the property that

$$P_{W_N} P_{W_N} f = P_{W_N} f \quad \text{for all } f \in \mathcal{H} \ .$$

This expresses the fact that P_{W_N} is the identity operator on W_N. On the other hand, in light of Bessel's inequality, P_{W_N} *shortens* f if $f \notin W_N$.

3. The triangle formed by f, $w_N^* \in W_N$, and the error vector

$$h_N^* = f - w_N^*; \qquad w_N^* = \sum_{k=1}^{N} c_k u_k$$

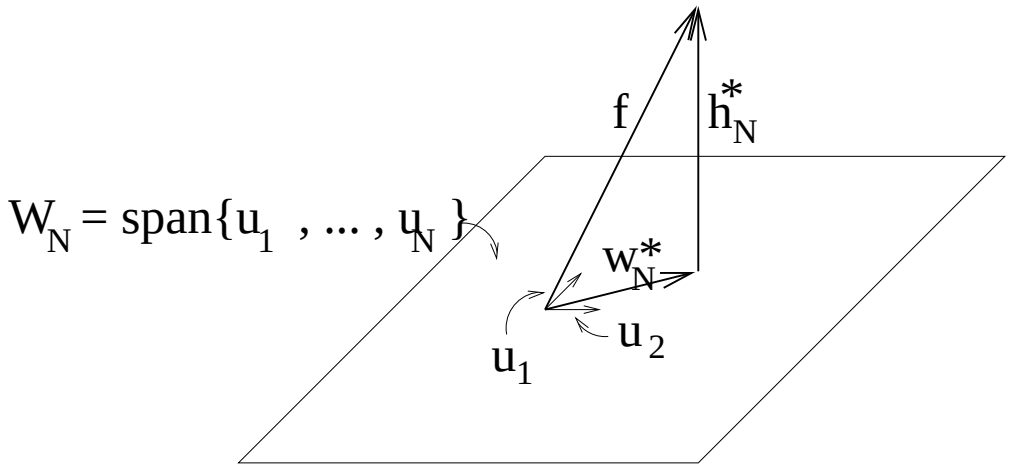

Figure 2.6: The N-dimensional subspace W_N of the ambient Hilbert space $\mathcal{H} = L^2$. The least squares approximation w_N^* is the orthogonal projection of the vector f onto W_N. The difference between the given vector f and its projection onto the subspace is the error vector h_N^*.

is a right triangle: the sides of the triangle obey the Pythagorean theorem in the ambient Hilbert space:

$$\|h_N^*\|^2 = \|f - \sum_{k=1}^{N} c_k u_k\|^2 \tag{2.13}$$

$$= \langle f - \sum_{k=1}^{N} c_k u_k, f - \sum_{k=1}^{N} c_k u_k \rangle$$

$$= \|f\|^2 - \sum_{k=1}^{N} |c_k|^2 \tag{2.14}$$

$$= \|f\|^2 - \|w_N^*\|^2$$

This evidently yields

$$\|f\|^2 \geq \|w_N^*\|^2$$

which is the finite-dimensional version of *Bessel's inequality*.

Using the optimal (= "least square") approximation

$$w_N^* = \sum_{i=1}^{N} c_k u_k \,, \qquad\qquad c_k = \langle u_k, f \rangle$$

one obtains

$$\|f\|^2 \geq \sum_{i=1}^{N} \overline{c_k} c_k \; ; \qquad c_k \text{ determined by the least squares approximation,}$$

which for our square integrable functions is

$$\int_a^b |f(x)|^2 \rho(x) dx \geq \sum_{i=1}^{N} |c_k|^2 \, .$$

Geometrically this inequality says

$$(\text{length of vector})^2 \geq \left(\begin{array}{c} \text{length of its projection} \\ \text{onto the subspace } W_N \end{array} \right)^2 .$$

Consider now a sequence of subspaces

$$W_1 \subset W_2 \subset \cdots \subset W_N \subset W_{N+1} \subseteq \cdots \, ,$$

the respective optimal approximations to the given function f, and the corresponding sequence of least square errors

$$\|h_N^*\|^2 = \|f - \sum_{i=1}^{N} c_k u_k\|^2 \, , \qquad N = 1, 2, \ldots \quad .$$

This sequence not only reveals the quality of each partial sum approximation. If $\|h_N^*\|^2$ approaches zero as N tends to infinity, then this very fact also reveals something about $\{u_k\}$. Indeed, whenever $\|h_N^*\|^2 \to 0$, the o.n. system is one which constitutes a basis for $\mathcal{H}$, meaning that Eq.(2.15) or (2.17) is satisfied.

4. With $\|h_N^*\|^2$ as the shortest squared distance between f and W_N, the error vector h_N^* is perpendicular to W_N:

$$\langle u_k, h_N^* \rangle = 0, \quad k = 1, 2, \cdots, N.$$

The Orthonormal System

Highlighting the fact that any o.n. system induces a linear transformation into ℓ^2 is only the first step in using a Hilbert space to conceptualize the measured properties of things in science. Is the chosen o.n. system appropriate for this task? There are many kinds of o.n. systems and their concomitant linear transformations. The necessity of grounding a Hilbert space in the measurements and observations of the physical world requires answers to two key questions about any o.n. system and its linear transformations:

1. Is it onto ("surjective")?
2. Is it one-to-one ("injective")?

Their answers require the following
Definition.
A system of o.n. vectors $\{u_k : k = 1, 2, \cdots\}$ *is said to be* **closed** *whenever*

$$\sum_{k=1}^{\infty} |c_k|^2 = \|f\|^2 \tag{2.15}$$

for every f *in* $\mathcal{H}$.
Under such a circumstance one refers to this relation as **Parseval's identity.**[1]
By taking the limit as $N \to \infty$ of the right hand sides of Eqs.(2.13) and (2.14), one obtains the result that

$$\boxed{\sum_{k=1}^{\infty} |c_k|^2 = \|f\|^2 \Longleftrightarrow \lim_{N\to\infty} \|f - \sum_{k=1}^{N} c_k u_k\|^2 = 0} \tag{2.16}$$

whenever $c_k = \langle u_k, f\rangle$. This equivalence is the mathematization of the portal between any Hilbert space and its coordinate realization by an "appropriate" system of o.n. vectors. That the two equalities imply each other is the result of a mere algebraic evaluation of the involved inner products.

Exercise 2.5.1 (PARSEVAL'S IDENTITY AND FOURIER SERIES)
Let

$$c_k \equiv \langle u_k, f\rangle \quad k = 1, 2, \cdots$$

[1] It is called an *identity*, and not an equation, because it holds for all f's.

be the Fourier coefficients of $f \in L^2$ relative to the orthonormal system $\{u_k\}_{k=1}^{\infty} \subset L^2$.

Prove:

$$\sum_{k=1}^{\infty} |c_k|^2 = \|f\|^2 \Longleftrightarrow \|f - \sum_{k=1}^{\infty} c_k u_k\|^2 = 0 \ .$$

The concept "appropriate" is too broad and not particular enough to identify the kind of system worthy of thorough study. This deficiency is remedied by the concept having the following

Definition

A system of o.n. vectors $\{u_k\}_{k=1}^{\infty}$ is said to be **complete** whenever

$$\lim_{N\to\infty} \|f - \sum_{k=1}^{N} c_k u_k\|^2 = 0 \tag{2.17}$$

i.e.

$$f \doteq \sum_{k=1}^{\infty} c_k u_k$$

Comment

This is to be compared with pointwise equality, which is expressed by the statement that

$$f(x) = \sum_{k=1}^{\infty} c_k u_k(x) \quad . \tag{2.18}$$

The difference between "$\doteq$" and "$=$" manifests itself only when the square-summable sequence $\{c_k\}$ yields a function which has one or more discontinuities, then one does not have pointwise equality, Eq.(2.18). Instead, one has the weaker condition, Eq.(2.17). This condition does not specify the value of f at the point(s) of discontinuity. Instead, it specifies an *equivalence class* of functions, all having the same graph everywhere except at the point(s) of discontinuity.

Completeness Relation For $\mathcal{H} \subseteq L^2(a,b)$

An "appropriate" system is one for which either equation in Eq.(2.16), i.e.

$$\sum_{k=1}^{\infty} \langle f, u_k\rangle\langle u_k, f\rangle = \langle f, f\rangle \Longleftrightarrow f \doteq \sum_{k=1}^{\infty} u_k \langle u_k, f\rangle \tag{2.19}$$

holds for all $f \in \mathcal{H}$. Moreover, the statement

$$\langle f, f\rangle = \sum_{k=1}^{\infty} \langle f, u_k\rangle\langle u_k, f\rangle \quad \forall f \in \mathcal{H}$$

implies and is implied by

$$\langle f, g\rangle = \sum_{k=1}^{\infty} \langle f, u_k\rangle\langle u_k, g\rangle \quad \forall f \text{ and } g \in \mathcal{H}.$$

Explicitly, one has

$$\int_a^b \overline{f}(x)g(x)\rho(x)dx = \sum_{k=1}^{\infty} \int_a^b \overline{f}(x)u_k(x)\rho(x)dx \int_a^b \overline{u}_k(x')g(x')\rho(x')dx' .$$

This can be rewritten in terms of the Dirac delta function (which is developed in Section 3.2 starting on page 135) as

$$\int_a^b \int_a^b \overline{f}(x)\delta(x-x')g(x')\rho(x)dxdx' = \int_a^b \int_a^b \overline{f}(x) \sum_{k=1}^{\infty} u_k(x)\overline{u}_k(x')\rho(x)\rho(x')g(x')dxdx' .$$

This holds for all $f, g \in \mathcal{H} = L^2(a, b)$. Consequently, we have the following alternate form for the completeness of the set of orthonormal functions

$$\frac{\delta(x-x')}{\rho(x')} = \sum_{k=1}^{\infty} u_k(x)\overline{u}_k(x')$$

or

$$\frac{\delta(x-x')}{\rho(x')} = \sum_{k=1}^{\infty} |u_k(x)\rangle\langle u_k(x')|$$

in quantum mechanical notation.

Usually the orthonormal functions u_k are the eigenfunctions of some operator (for example, the Sturm-Liouville operator + boundary conditions, which we have met in chapter 1 on page 20). The Dirac delta function

$$\frac{\delta(x-x')}{\rho(x')} = \frac{\delta(x-x')}{\rho(x)}$$

is the identity operator on the Hilbert space $\mathcal{H}$. Consequently, the alternate form of the completeness relation

$$\frac{\delta(x-x')}{\rho(x)} = \sum_{k=1}^{\infty} u_k(x)\overline{u}_k(x') \tag{2.20}$$

can be viewed as a *spectral representation* of the *identity operator* in $\mathcal{H}$.

Thus the *completeness* of a set $\{u_k\}_{k=1}^{\infty}$ refers to the fact that it contains sufficiently many u_k's of the right kind so that the identity transformation (2.20) can be represented in terms of them. Equivalently, the u_k's form a spanning set for $\mathcal{H} \subseteq L^2$.

On the other hand, in the quest for alternative mathematical precision, some mathematicians asked and answered the following important question: Is an o.n. system $\{u_k\}_{k=1}^{\infty}$ unique whenever it gives rise to Parseval's identity? In other words, can such a $\{u_k\}$ be a proper subset of any other orthonormal set in $\mathcal{H}$? The reason that the answer is "no" is that they call $\{u_k\}$ a **maximal** o.n. sequence.

Band-Limited L^2 Message Spaces

Example (*Complete vs. incomplete system of o.n. band-limited L^2 signal functions.*)
Consider the following three Hilbert spaces:

$$\mathcal{H}_{[0,\varepsilon]} = \left\{ f \in L^2(-\infty,\infty) : \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{\sqrt{2\pi}} f(x)dx = 0;\ \omega \notin [0,\varepsilon] \right\}, \tag{2.21}$$

$$\mathcal{H}_{[\varepsilon,2\varepsilon]} = \left\{ g \in L^2(-\infty,\infty) : \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{\sqrt{2\pi}} g(x)dx = 0;\ \omega \notin [\varepsilon,2\varepsilon] \right\}, \tag{2.22}$$

$$\mathcal{H}_{[0,2\varepsilon]} = \left\{ h \in L^2(-\infty,\infty) : \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{\sqrt{2\pi}} h(x)dx = 0;\ \omega \notin [0,2\varepsilon] \right\} \tag{2.23}$$

In mathematical engineering each one refers to a set of band-limited signals: The Fourier amplitudes of the f's, g's, and h's are non-zero only in the frequency windows $[0,\varepsilon]$, $[\varepsilon,2\varepsilon]$, and $[0,2\varepsilon]$ respectively.

Each of these spaces is a type of function space. Indeed, each of the f's, g's, and h's is a particular signal function. The difference between the three

is the difference in how a particular signal function is put into mathematical form ("mathematized"). Although there are many ways of doing this, here we shall do it by means of Fourier series: Consider the following three systems of o.n. vectors:

$$\{u_k(x)\} = \{P^{\varepsilon}_{0k}(x) : \ k = 0, \pm 1, \cdots\} \tag{2.24}$$

$$\{u_{k'}(x)\} = \{P^{\varepsilon}_{1k'}(x) : \ k' = 0, \pm 1, \cdots\} \tag{2.25}$$

$$\{u_{k''}(x)\} = \{P^{2\varepsilon}_{0k''}(x) : \ k'' = 0, \pm 1, \cdots\}. \tag{2.26}$$

Borrowing from Section 3.4.2 the o.n. wave packets

$$P^{\varepsilon}_{j\ell}(x) = \int_{j\varepsilon}^{(j+1)\varepsilon} \frac{e^{-2\pi i \ell \omega/\varepsilon}}{\sqrt{\varepsilon}} \frac{e^{i\omega x}}{\sqrt{2\pi}} \, d\omega \ , \tag{2.27}$$

and affixing superscripts ε and 2ε to them to indicate their banwidth, we note that each of the $P^{2\varepsilon}_{0k''}$'s has twice the frequency bandwidth of the P^{ε}_{0k}'s and of the $P^{\varepsilon}_{0k'}$'s.

Messages are represented by Fourier series in their respective message spaces:

$$f(x) = \sum_k c_k u_k(x) \in \mathcal{H}_{[0,\varepsilon]}$$

$$g(x) = \sum_{k'} c'_{k'} u'_{k'}(x) \in \mathcal{H}_{[\varepsilon,2\varepsilon]}$$

$$h(x) = \sum_{k''} c'_{k''} u''_{k''}(x) \in \mathcal{H}_{[0,2\varepsilon]}$$

Thus the three systems induce the following three respective Fourier maps

$$\mathcal{F}_{[0,\varepsilon]} : \begin{cases} L^2 & \to & \ell^2 \\ \mathcal{H}_{[0,\varepsilon]} & \leftrightarrow & \ell^2 \\ f & \rightsquigarrow & \mathcal{F}_{[0,\varepsilon]}[f] = \{\langle u_k, f\rangle \equiv c_k\} \end{cases}$$

$$\mathcal{F}_{[\varepsilon,2\varepsilon]} : \begin{cases} L^2 & \to & \ell^2 \\ \mathcal{H}_{[\varepsilon,2\varepsilon]} & \leftrightarrow & \ell^2 \\ g & \rightsquigarrow & \mathcal{F}_{[\varepsilon,2\varepsilon]}[f] = \{\langle u'_{k'}, g\rangle \equiv c'_{k'}\} \end{cases}$$

$$\mathcal{F}_{[0,2\varepsilon]} : \begin{cases} L^2 & \to & \ell^2 \\ \mathcal{H}_{[0,2\varepsilon]} & \leftrightarrow & \ell^2 \\ h & \rightsquigarrow & \mathcal{F}_{[0,2\varepsilon]}[f] = \{\langle u''_{k''}, h\rangle \equiv c''_{k''}\} \end{cases}$$

Each of the three systems, Eqs.(2.24)-(2.26) is complete in their respective subspaces $\mathcal{H}_{[0,\varepsilon]}$, $\mathcal{H}_{[\varepsilon,2\varepsilon]}$, and $\mathcal{H}_{[0,2\varepsilon]}$. This is because the respective Parseval identities:

$$\begin{aligned}\|f\|^2 &= \sum_k |c_k|^2 \\ \|g\|^2 &= \sum_{k'} |c'_{k'}|^2 \\ \|h\|^2 &= \sum_{k''} |c''_{k''}|^2\end{aligned}$$

are satisfied.
However, the system (2.24) is *incomplete* in $\mathcal{H}_{[\varepsilon,2\varepsilon]}$. This is because the Fourier coefficients of any $g \in \mathcal{H}_{[\varepsilon,2\varepsilon]}$ w.r.t. (2.24) vanish identically,

$$\begin{aligned}\langle u_k, g\rangle &= \langle u_k, \sum_{k'} c'_{k'} u'_{k'}\rangle \\ &= \sum_{k'} c'_{k'} \langle P^{\varepsilon}_{0k}, P^{\varepsilon}_{1k'}\rangle \\ &= \sum_{k'} c'_{k'} \qquad \underbrace{\langle \hat{P}^{\varepsilon}_{0k}, \hat{P}^{\varepsilon}_{1k'}\rangle}_{=\text{zero; see Eq.(3.83).}} \\ &= 0\ .\end{aligned}$$

As a consequence, Parseval's identity is strongly violated:

$$\|g\|^2 > \sum_k |\langle u_k, g\rangle|^2\ (=0).$$

This violation is due to the following: even though g is a non-trivial message (i.e. a Fourier series) in $\mathcal{H}_{[\varepsilon,2\varepsilon]}$, an attempt to represent g as a Fourier series in $\mathcal{H}_{[o,\varepsilon]}$ results in failure: relative to $\mathcal{H}_{[0,\varepsilon]}$ the system of o.n. vectors, g gets represented by a Fourier series with coefficients all zero. In other words, $\mathcal{F}_{[o,\varepsilon]}$ maps g and the zero vector into the origin of ℓ^2. Thus $\mathcal{F}_{[o,\varepsilon]}$ is not one-to-one if its domain includes $\mathcal{H}_{[\varepsilon,2\varepsilon]}$. But this can happen only if Parseval's identity is violated.

The relation between the completeness of a system of o.n. vectors and its concomitant Fourier map $\mathcal{F}$ can therefore be summerized by the following

Theorem 2.5.2 ($\mathcal{F}$ Is One-To-One)
Given:
Let $\{u_k\}$ be an orthonormal system on a Hilbert space $\mathcal{H}$. Let $\{c_k = \langle u_k, f\rangle\}$ be the set of corresponding Fourier coefficients of some $f \in \mathcal{H}$.
Conclusion:
The Fourier representation map

$$\begin{aligned} \mathcal{F}:\ \mathcal{H} &\to \ell^2 \\ f &\rightsquigarrow \mathcal{F}[f] = \{c_k\} \end{aligned}$$

is one-to-one if and only if $\{u_k\}$ is complete in $\mathcal{H}$.

Proof: (*in three steps*)

1: Let f and g have the same image in ℓ^2:

$$\mathcal{F}[f] = \mathcal{F}[g].$$

2: Let $h = f - g$. Then

$$\mathcal{F}[h] \equiv \{\langle u_k, h\rangle\} = \{0, 0, \cdots, 0, \cdots\}$$

3: $\{u_k\}$ is complete if and only if

$$\sum_k |\langle u_k, h\rangle|^2 = \|h - \sum_k u_k \langle u_k, h\rangle\|^2 = \|h\|^2 = \|f - g\|^2$$

Thus

$$f \doteq g$$

i.e. $\mathcal{F}$ is one-to-one indeed.

Whichever way one singles out such a system of o.n. vectors, in linear mathematics one refers to such a system as being a **spanning set** for the vector space $\mathcal{H}$. Such a set is an $\mathcal{H}$ **basis** because the orthogonality of its elements makes it also a linearly independent set.

In summary, when Bessel's inequality becomes an equality, in which case one has Parseval's identity on L^2, then Eq.(2.16) tells us that

$$f \doteq \sum_{k=1}^{\infty} c_k u_k \ .$$

Here

$$c_k = \langle u_k, f \rangle$$

are simply the generalized Fourier coefficients.
Thus Bessel-based Parseval's identity, which is a statement about these coefficients, achieves two goals at once:

- It yields the *linear* map $\mathcal{F}$ from L^2 into ℓ^2: for each f there is a unique $\{c_k\} \in \ell^2$, and
- it yields an o.n. system $\{u_k\}_{k=1}^{\infty}$ which is *complete* in $\mathcal{H}$.

This achievement completes the first two steps in coordinatizing a Hilbert space.

Lecture 15

Riesz and Fischer

If the elements of ℓ^2 comprise measured laboratory data, then the introduction of an o.n. system is the means of relating this data to a theory as expressed by the elements of a particular Hilbert space. If this relation is expressed by the linear map $\mathcal{F}$, and if the physical context demands that this $\mathcal{F}$ accomodate a range of measurements, then it is necessary to ascertain whether the range space of the chosen $\mathcal{F}$ is sufficiently "large" (high dimensional) to cover the whole space of measurements, ℓ^2. In other words, is $\mathcal{F}$ an *onto* map? Riesz and Fischer took the decisive step in answering this question in the affirmative. Their line of reasoning is summarized by means of Riesz-Fischer Theorem 1.5.3 on page 92

That third step was a non-trivial task, and they achieved it in two entirely different ways. But interestingly enough, they published it in the same year (1907) in the same issue of the same mathematics journal.

Ernst Fischer's approach, which we also shall adopt in the R-F Theorem, consisted of showing that the inner product space L^2 is Cauchy complete. By contrast Frigyes Riesz's approach was to solve a system of integral equations. He showed that one can solve

$$\mathcal{F}[f] = \{c_k\}_{k=1}^{\infty}$$

for the function f. That is to say, the system of integral equations

$$\begin{aligned}
\int \overline{u_1}\, f\, \rho\, dx &= c_1 \\
\int \overline{u_2}\, f\, \rho\, dx &= c_2 \\
&\vdots \\
\int \overline{u_k}\, f\, \rho\, dx &= c_k \\
&\vdots
\end{aligned}$$

can always be solved for an $f \in L^2$ such that

$$\|f_N - f\|^2 \to 0 \quad \text{as } N \to \infty \ .$$

Thus both Riesz and Fischer show that to every element $\{c_1, c_2, \cdots, c_k, \cdots\} \in \ell^2$ there corresponds an element $f \in L^2(a,b)$ with the numbers $c_1, c_2, \cdots, c_k, \cdots$ as its generalized Fourier coefficients. In brief, they show that $\mathcal{F}$, Eq.(2.29), is an onto map.

With this historical background, these achievements of Riesz and Fischer's, restated in terms of the modern geometrical inner product framework, are summarized by what is nowadays known as the

Theorem 2.5.3 (Riesz-Fischer: $\mathcal{F}$ is Onto)

Given: (i) An orthonormal system $\{u_k\}$ in the (Cauchy complete) L^2

(ii) A sequence of numbers $c_1, c_2, \ldots, c_k, \cdots$ with the property that

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty \ .$$

Conclusion: (a) There exists an element $f \in L^2(a,b)$ with $c_1, c_2, \ldots, c_k, \cdots$ as its Fourier coefficients, i.e. such that

(b) $$\|f\|^2 = \sum_{k=1}^{\infty} |c_k|^2$$

(c) with

$$c_k = \langle u_k, f \rangle, \quad k = 1, 2, \cdots \tag{2.28}$$

Comment 1: Items (a) and (c) imply that

$$\begin{array}{ccc} & \mathcal{F} & \\ L^2(a,b) & \longrightarrow & \ell^2 \\ f & \rightsquigarrow & \mathcal{F}[f] = \{c_k\}_{k=1}^{\infty} \end{array} \tag{2.29}$$

with the property that it is *onto*[2] and *linear*.

Comment 2: Item (b) implies that this $\mathcal{F}$ is an *isometry*. This fact is defined explicitly on page 100

Comment 3: In light of the fact that (b) implies[3] that

$$\lim_{N\to\infty} \|f - \sum_{k=1}^{N} c_k u_k\|^2 = 0$$

one concludes that system $\{u_k\}$ of orthonormal elements is a *complete*, i.e. a *spanning* set for L^2:

$$f \doteq \sum_{k=1}^{\infty} c_k u_k$$

Proof: Starting with $\{c_k\}$ we must first show that there exists a function $f \in L^2$ having the two properties (a) and (c). Then we must show that this function satifies (b).

The existence follows from the following three step construction:

---

[2]This claim is also stated by saying that the *preimage* $\mathcal{F}^{-1}[\{c_k\}]$ of $\{c_k\}$, namely, the set of elements

$$\mathcal{F}^{-1}[\{c_k\}] = \{f : \mathcal{F}[f] = \{c_k\}\} \ ,$$

is non-empty.

[3]In fact, one has

$$\sum_{k=1}^{\infty} |c_k|^2 = \|f\|^2 \quad \Leftrightarrow \lim_{N\to\infty} \|f - \sum_{k=1}^{N} c_k u_k\|^2 = 0 \ .$$

For the validity of this equivalence see Exercise 1.5.1 on page 85

1. Concider the following sequence of sums in L^2:

$$f_N = \sum_{k=1}^{N} c_k u_k, \qquad N = 1, 2, \cdots \tag{2.30}$$

This sequence is a Cauchy sequence, i.e. it satisfies the Cauchy criterion. Indeed,

$$\begin{aligned} \|f_{N+P} - f_N\|^2 &= \|c_{N+1}u_{N+1} + \cdots c_{N+P}u_{N+P}\|^2 \\ &= \sum_{k=N+1}^{N+P} |c_k|^2 \ \to 0 \text{ as } N \to \infty \end{aligned}$$

because

$$\sum_{k=1}^{\infty} |c_k|^2$$

converges[4]. So $\{f_N\}_1^\infty$ is a Cauchy sequence indeed.

2. The fact that L^2 is Cauchy complete w.r.t. $\|\cdots\|^2$ implies that there exists a function $f \in L^2$ such that

$$\lim_{N\to\infty} \|f - f_N\|^2 = 0 \ . \tag{2.31}$$

This validates part (a) of the theorem.

3. Is there a relation between the given c_k's and the inner products of the u_k's and that function f? Consider any one of the inner products, say,

$$\langle u_\ell, f\rangle = \langle u_\ell, f_N\rangle + \langle u_\ell, f - f_N\rangle$$

[4]Let us say it converges to S. In that case one has

$$\begin{aligned} \sum_{k=N+1}^{N+P} |c_k|^2 &= \sum_{k=1}^{N+P} |c_k|^2 - S + S - \sum_{k=1}^{N} |c_k|^2 \\ &\le \left| \sum_{k=1}^{N+P} |c_k|^2 - S \right| + \left| S - \sum_{k=1}^{N} |c_k|^2 \right| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for sufficiently large } N \text{ and } N+P. \end{aligned}$$

For the first term one has

$$\langle u_\ell, f_N \rangle = c_\ell$$

whever $N > \ell$. The second term goes to zero as $N \to \infty$. Indeed, applying the Cauchy-Schwarz inequality, one has

$$|\langle u_\ell, f - f_N \rangle|^2 \leq \|u_\ell\|^2 \cdot \|f - f_N\|^2 \ .$$

In compliance with the Cauchy completeness, Eq.(2.31), the r.h.s. of this inequality goes to zero in the limit as $N \to \infty$. One therefore has

$$\begin{aligned} \langle u_\ell, f \rangle &= \lim_{N\to\infty} \{\langle u_\ell, f_N \rangle + \langle u_\ell, f - f_N \rangle\} \\ &= c_k \ . \end{aligned}$$

In other words, c_ℓ is the ℓ^{th} Fourier coefficient of f relative to the o.n. system $\{u_k\}$. This validates part (c) of the theorem.

Comment: This line of reasoning, which starts with $\langle u_\ell, f \rangle$ and ends with $\langle u_\ell, f \rangle = c_\ell$, applies to every component of the given $\{c_k\}_{k=1}^{\infty} = \{c_1, c_2, \cdots, c_k, \cdots\}$, a vector in ℓ^2. A linear algebraist would say that Riesz and Fischer, by following definite (mental) procedures applied to established knowledge (including taking note of the fact that the limit f is unique [5]), have formed a new concept: the linear mapping

$$\ell^2 \to L^2 \tag{2.32}$$

$$\{c_k\} \rightsquigarrow f = \lim_{N\to\infty} \sum_{k=1}^{N} c_k u_k \ . \tag{2.32'}$$

This mapping is the inverse of the mapping (2.29) and is symbolized by $\mathcal{F}^{-1}$.

4. The limiting behaviour of f_N, $\|f - f_N\| \to 0$ as $N \to \infty$ implies

$$\lim_{N\to\infty} \left\langle f - \sum_{k=1}^{N} c_k u_k, f - \sum_{\ell=1}^{N} c_\ell u_\ell \right\rangle = 0 \ , \tag{2.33}$$

[5] See the "Side Comment" on page 69.

which in turn is equivalent to

$$\|f\|^2 = \sum_{k=1}^{\infty} |c_k|^2 \ . \tag{2.34}$$

This is because (i) the c_k's are the Fourier coefficients of f relative to the $\{u_k\}$, and (ii) the u_k's are mutually orthonormal. This validates part (b) of the theorem.

Corollary 1
The map $\mathcal{F}^{-1} : \ell^2 \to L^2$ is one-to-one.
It is a consequence of Parseval's identity, Eq.(2.34), that (2.32) is a one-to-one map. This is because $f{=}0$ yields $\{c_k = 0\}_{k=1}^{\infty}$ as the only preimage under (2.32).
Corollary 2 *The map $\mathcal{F} : L^2 \to \ell^2$ is an isomorphism.*
Theorems 1.5.3 combined with 1.5.2, i.e. the map $\mathcal{F}$ induced by a complete infinite system of o.n. vectors is onto *and* one-to-one, leads to the definition that $\mathcal{F}$ is an isomorphism.
Corollary 3 *The map $\mathcal{F} : L^2 \to \ell^2$ is an isometry.*
Equation (2.34) holds for all $\{c_k\}$'s and f's. Consequently,

$$\sum_{k=1}^{\infty} \overline{c_k}\, d_k = \langle f, g \rangle \ .$$

This validates that $\mathcal{F}$ is an *isometry*, i.e. an *isometric* isomorphism, between ℓ^2 and L^2.
Proposition (Coordinatization of Hilbert Space $\mathcal{H}$)
Let $\mathcal{H}$ be an "infinite" dimensional Cauchy-complete inner product space, i.e. given any positive integer N, $\mathcal{H}$ contains N o.n. vectors. Then $\mathcal{H}$ has l^2 as its coordinate realization. In brief, $\mathcal{H}$ is coordinatized by the isometric mapping, Eq.(2.29), which is induced by this system of o.n. vectors.

2.5.3 Isomorphic Hilbert Spaces

Lecture 16

Parseval's identity, Eq.(2.15), *is a remarkable for its diverse and non-trivial implications!* One of its consequences is the generalized Fourier expansion

$$f(x) \doteq \sum_{k=1}^{\infty} c_k u_k$$

with $c_k = \langle u_k, f\rangle$. Indeed, starting with Parseval's mathematically simple identity,

$$0 = \langle f, f\rangle - \lim_{N\to\infty} \sum_{k=1}^{N} \overline{c}_k c_k \,,$$

subtract and add the limit of the sum

$$\sum_{\ell=1}^{N} c_\ell \langle f, u_\ell\rangle = \sum_{\ell=1}^{N}\sum_{k=1}^{N} c_\ell \overline{c}_k \langle u_k, u_\ell\rangle.$$

One obtains

$$\begin{aligned}
0 &= \langle f, f\rangle - \lim_{N\to\infty} \Big\{ \sum_{k=1}^{N} \overline{c}_k \langle u_k, f\rangle - \sum_{\ell=1}^{N} c_\ell \langle f, u_\ell\rangle \\
&\qquad + \sum_{\ell=1}^{N}\sum_{k=1}^{N} c_\ell \overline{c}_k \langle u_k, u_\ell\rangle \Big\} \\
0 &= \langle f - \lim_{N\to\infty} \sum_{k=1}^{N} c_k u_k, f - \lim_{N\to\infty} \sum_{\ell=1}^{N} c_\ell u_\ell\rangle \\
0 &= \lim_{N\to\infty} \|f - \sum_{k=1}^{N} c_k u_k\|^2 \,,
\end{aligned}$$

which is what is meant by

$$f \doteq \sum_{k=1}^{\infty} c_k u_k \,; \quad c_k = \langle u_k, f\rangle \qquad \forall\, f \in \mathcal{H} \,.$$

But there is more. The generalized Fourier series has a perspicuous property: it is a length and angle preserving (isometric) isomorphism. It is a one-to-one linear correspondence – let us call it $\mathcal{F}$, as we have done all along – between

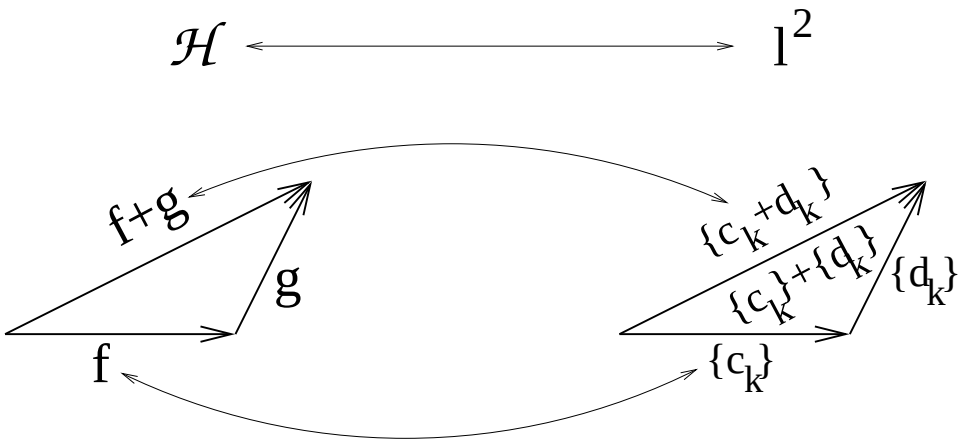

Figure 2.7: Linear map between $\mathcal{H} \subseteq L^2$, and l^2, the space of square summable sequences. This map is an isometry which is induced by the generalized Fourier series.

$f \in \mathcal{H} \subseteq L^2(a,b)$ and $\{c_1, c_2, \ldots, c_k, \ldots\} \in \ell^2 =$ the Hilbert space of square summable series ("functions on the integers"). The correspondence

$$\begin{aligned} f &\stackrel{\mathcal{F}}{\to} \mathcal{F}[f] = \{c_k\} \equiv \{c_1, c_2, \ldots, c_k, \ldots\} \\ g &\stackrel{\mathcal{F}}{\to} \mathcal{F}[g] = \{d_k\} \equiv \{d_1, d_2, \ldots, d_k, \ldots\} \\ \alpha f + \beta g &\stackrel{\mathcal{F}}{\to} \mathcal{F}[\alpha f + \beta g] = \alpha\{c_k\} + \beta\{d_k\} \end{aligned}$$

(i) is one-to-one and onto, which means it has an inverse:

$$\mathcal{F}^{-1}[\{c_k\}] = f \equiv \sum_{k=1}^{\infty} c_k u_k \stackrel{\mathcal{F}^{-1}}{\leftarrow} \{c_k\} \quad ;$$

(ii) is linear, which means it takes closed triangles in L^2 into closed triangles in ℓ^2:

$$\{c_k + d_k\} \quad = \quad \{c_k\} + \{d_k\}\,;$$

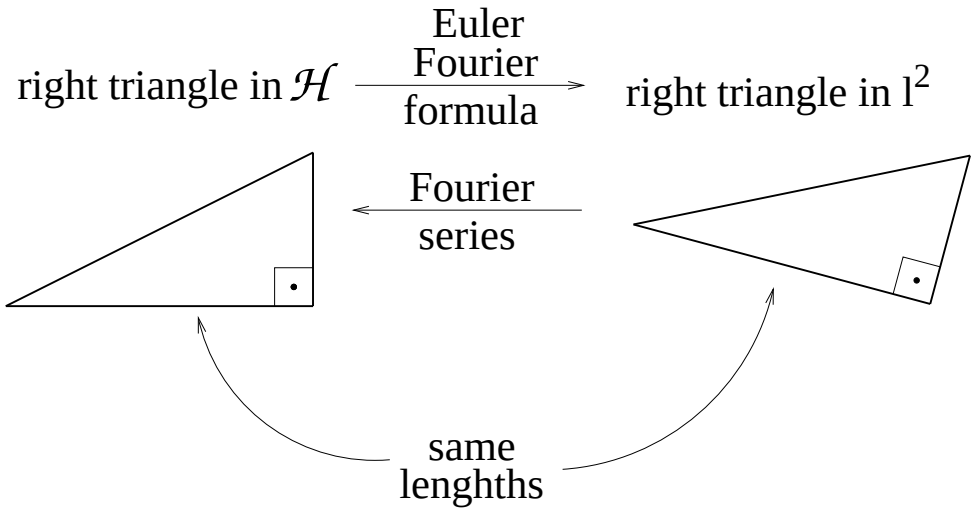

Figure 2.8: The isometry between $\mathcal{H} \subseteq L^2$, the space of square integrable functions, and l^2, the space of square summable sequences, preserves lengths and angles.

(iii) preserves lengths. Indeed,

$$\begin{aligned} f &= \sum_{k=1}^{\infty} c_k u_k \\ g &= \sum_{\ell=1}^{\infty} d_\ell u_\ell \end{aligned}$$

implies

$$\begin{aligned} \langle f-g, f-g\rangle &= \sum_{k=1}^{\infty} (\overline{c}_k - \overline{d}_k)(c_k - d_k) \\ \|f-g\|^2 &= \sum_{k=1}^{\infty} |c_k - d_k|^2 \qquad \forall\, f, g \in \mathcal{H}\,. \end{aligned}$$

It follows that

$$\begin{aligned} \langle f, g\rangle &= \sum_{k=1}^{\infty} \overline{c}_k d_k \\ \|f\|^2 &= \sum_{k=1}^{\infty} |c_k|^2 \quad . \end{aligned}$$

Consequently, $\mathcal{F}$ preserves lengths, inner products, and angles (if the Hilbert space is real).

Definition: A linear transformation which is one-to-one and onto is called an *isomorphism*.

Definition: A distance preserving transformation between two metric spaces is called an *isometric transformation*, or simply an *isometry*.

In that case, the two spaces are said to be *isometric spaces*. This means they look the same from the viewpoint of geometry.

To summarize, the striking feature of the completeness, i.e., Parseval's relation is that it establishes an *isometric isomorphism*, or more briefly an *isometry* between the two spaces.

Thus

$$\mathcal{H}\ (\subseteq L^2(a,b)) \text{ and } \ell^2 \text{ are isometric Hilbert spaces}\,.$$

They are geometrically the same (right triangles in one space correspond to right triangles in the other space).

Because one can establish a linear isometry between *any* Hilbert space and one and the same ℓ^2, the space of square summable series, one obtains the

Theorem: (Isomorphism theorem) Any two complex Hilbert spaces are isomorphic. In fact, so are any two real Hilbert spaces.

Lecture 17

Comment: The isometric isomorphism is a *unitary* transformation whose elements are $\{u_1(x), u_2(x), \dots\}$. Indeed, consider the equation

$$\sum_{k=1}^{\infty} u_k(x) c_k = f(x) \quad .$$

The coefficients c_k are the components of an infinite dimensional column vector in ℓ^2. The function f is an infinite dimensional column vector whose components $f(x)$ are labelled by the continuous index x. It follows that $\{u_k(x)\}$ are the entries of a matrix whose columns are labelled by k and whose rows are labelled by x. The orthogonality conditions

$$\langle u_i, u_j \rangle = \delta_{ij}$$

expresses the orthonormality of the colums of this matrix. The completeness relation

$$\sum_{k=1}^{\infty} u_k(x)\overline{u}_k(x) = \frac{\delta(x-x')}{\rho(x)}$$

expresses the orthonormality of the rows. It follows that $\{u_k(x)\}$ represents a unitary transformation which maps $\mathcal{H} \subseteq L^2(a,b)$ onto ℓ^2.

Exercise 2.5.2 (SQUARED LENGTHS AND INNER PRODUCTS)
An *isometry* between the Hilbert space $\mathcal{H}$ of square integrable functions f, and the Hilbert space ℓ^2 of square summable sequences $\{c_k\}_{k=1}^{\infty}$ is a linear one-to-one and onto transformation $f \to \{c_k\}$ with the property that it preserves squared lengths:

$$\langle f, f\rangle = \sum_{k=1}^{\infty} |c_k|^2 \ , \qquad \forall f \in \mathcal{H} \ .$$

SHOW that

$$\langle f, g\rangle = \sum_{k=1}^{\infty} \overline{c_k} d_k \ , \quad d_k = \langle u_k, g\rangle \quad \forall f, g \in \mathcal{H} \ .$$

where

$$f \to \{c_k\} \text{ and } g \to \{d_k\} \ .$$

Exercise 2.5.3
Let g be a fixed and given square integrable function, i.e.

$$0 < \int_{-\infty}^{\infty} \overline{g}(x)g(x)\ dx \equiv \|g\|^2 < \infty$$

One can think of g as a function whose non-zero values are concentrated in a small set around the origin $x = 0$.

Consider the concomitant "windowed" Fourier transform on $L^2(-\infty,\infty)$, the space of square integrable functions,

$$\begin{aligned} T: \quad L^2(-\infty,\infty) &\to \mathcal{R}(\mathcal{T}) \\ f &\to Tf(\omega,t) \equiv \int_{-\infty}^{\infty} \overline{g}(x-t)e^{-i\omega x} f(x)\ dx \end{aligned}$$

Let $h(\omega,t)$ be an element of the range space $\mathcal{R}(\mathcal{T})$. It is evident that

$$\langle h_1, h_2\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{h}_1(\omega,t)h_2(\omega,t)\ d\omega dt$$

is an inner product on $\mathcal{R}(\mathcal{T})$.

FIND a formula for $\langle Tf_1, Tf_2\rangle$ in terms of the inner product

$$(f_1, f_2) \equiv \int_{-\infty}^{\infty} \overline{f}_1(x) f_2(x)\, dx$$

on $L^2(-\infty, \infty)$.

Exercise 2.5.4 (SHANNON'S SAMPLING FUNCTIONS)
By

(a) starting with the orthonormality relation

$$\int_{-\pi}^{\pi} \delta_N\left(t - \frac{2\pi}{2N+1}k\right)\delta_N\left(t - \frac{2\pi}{2N+1}l\right)\, dt \;=\; \frac{2N+1}{2\pi}\delta_{kl}\,;$$
$$k, l = -N, \ldots, N$$

where

$$\delta_N(u) = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})u}{\sin\frac{u}{2}} \quad \left(= \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inu}\right),$$

(b) then rescaling the integration domain by introducing the variable

$$z = \frac{N+\frac{1}{2}}{2\pi w}\, t\,,$$

where $w > 0$ is a fixed constant (the "band width"),

(c) and finally going to the limits $N \to \infty$.

(i) Show that the set of functions

$$\left\{\frac{\sin\pi(2wz-k)}{\pi(2wz-k)} \equiv \sin c(2wz-k) : \quad k = 0, \pm1, \pm2, \ldots\right\}$$

is an orthogonal set satisfying

$$\int_{-\infty}^{\infty} \sin c(2wz-k)\sin c(2wz-l)dz = A\delta_{kl}\,.$$

What is A?

(ii) This set of functions

$$\left\{ u_k = \frac{1}{\sqrt{A}} \operatorname{sinc}(2wz - k) : \ k = 0, \pm 1, \cdots \right\}$$

is not complete on $L^2(-\infty, \infty)$, but it is complete on a certain subset $B \subset L^2(-\infty, \infty)$.

What is this subset? i.e. What property must a function $f(t)$ have in order that $f \in B$?

This question can be answered with the help of Parseval's ("completeness") relation as follows: Recall that completeness on B here means that $f \in B$ implies that one can write f as

$$f(z) = \sum_{k=-\infty}^{\infty} c_k u_k(z), \qquad u_k = \frac{1}{\sqrt{A}} \operatorname{sinc}(2wz - k)$$

with $c_k = \langle u_k, f \rangle$, which we know is equivalent to

$$\langle f, f \rangle = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad . \tag{2.35}$$

Thus, to answer the question, we must ask and answer: What property must f have in order to guarantee that Eq.(2.35) be satisfied? Therefore, to give a correct answer, one must (i) identify the property and (ii) then show that Parseval's relation is satisfied by every such function.

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Chapter 3

Fourier Theory

Lecture 18

A system is identified by its properties, and linearity, more often than not, plays a fundamental role in organizing them quantitatively. Thus, if ψ_1 and ψ_2 are two spatial amplitude profiles of a vibrating string, or two temporal histories of a simple harmonic oscillator, then so is the linear combination

$$\psi = c_1\psi_1 + c_2\psi_2 \ .$$

One says that the system obeys the *linear superposition principle*. Mathematically one says that the set of such ψ's forms a vector space.

By specifying functions such as $\psi(x), \psi_1(x)$, and $\psi_2(x)$ one directs attention to the fact that (i) the system has a specific domain and that (ii) the state (or the history) of the system consists of assigning to this domain numbers that characterize the state (or the history) in numerical terms.

There are linear systems whose intrinsic properties are independent of arbitrary translations in that domain. An infinite string having constant density and tension, or a simple harmonic oscillator with its time independent spring constant and mass, or a system of differential equations with constant coefficients is a case in point.

On the other hand, there are systems whose properties are independent only under discrete translations, such as a wave disturbance propagating through a discrete lattice or a simple harmonic oscillator whose spring constant varies periodically with time.

All such systems have the property that if $\psi(x)$ describes its state (or its history, in which case x is the time variable), so does $\psi(x + a)$. In other

words, a linear system which is invariant under space (or time) translation has the property that $\psi(x)$ and $T_a\psi(x) \equiv \psi(x+a)$ belong to the same vector space.

This immediately raises the algebraic question: What are the eigenvalues and eigenvectors of the translation operator T_a,

$$T_a\psi(x) = \lambda_a\psi(x) \ ?$$

i.e. which states (or histories) have the property that

$$\psi(x+a) = \lambda_a\psi(x) \ ? \tag{3.1}$$

First consider the case of arbitrary translation invariance. By differentiation w.r.t. a one finds that the solution is

$$\psi(x) = e^{cx},$$

where c is a constant. The requirement that this solution stay bounded in the whole domain $-\infty < x < \infty$ demands that the constant be purely imaginary:

$$\psi(x) = e^{ikx} \ . \tag{3.2}$$

These are the tranlation eigenfunctions whose eigenvalues are e^{ika}.

Fourier theory is based on introducing these functions to represent any state (or history) of the a linear translation invariant system. In brief, *Fourier theory is an expression of the translation invariance of a linear sustem.*

Next consider the case of discrete translation invariance. By rewriting the eigenvalue in Eq.(3.1) as

$$\lambda_a = e^{\rho a}$$

one finds that Eq.(3.1) becomes

$$e^{-\rho(x+a)}\psi(x+a) = e^{-\rho x}\psi(x) \equiv u_\rho(x) \ .$$

Thus $e^{-\rho x}\psi(x)$ is a function, say $u_\rho(x)$, with periodicity a. Consequently, an eigenfunction of discrete translation symmetry $x \to x+a$ has the form of an a-periodic function modulated by a (real or complex) exponential[1]

$$\psi_\rho(x) = e^{\rho x}u_\rho(x) \ ,$$

[1]This fact is known to engineers and to physicists as *Bloch's theorem*, and to mathematicians as *Floquet's theorem.*

while the eigenvalue has the form

$$\lambda_a = e^{\rho a} \ .$$

The periodic part of these eigenfunctions forms a portal from Fourier theory on the real line to Fourier series theory on the closed circle.

First of all, noticing the periodicity of each u_ρ, one isolates a specific and blindingly obvious characteristic of the points $x, x \pm a, x \pm 2a, \cdots$: the measurable value of the function u_ρ at these points. This value serves to unite these points into a higher order concept, namely, the equivalence class

$$\{x, x \pm a, x \pm 2a, \cdots\} \equiv \{x\} \ .$$

In other words, the function u_ρ renders these points as indistinguishable with respect to its measured value $u_\rho(x) = u_\rho(x \pm a) = u_\rho(x \pm 2a) = \cdots$.

Secondly, the continuity of the periodic functions u_ρ guarantees that the family of equivalent classes

$$\{\{x\} : 0 \leq x \leq a\}$$

forms a closed circle with circumference a.

Finally one asks, which of these periodic functions are eigenfunctions of the translation operation $x \to x + b$:

$$\phi(x + b) = \mu_b \phi(x) \ ?$$

The line of reasoning that led to Eq.(3.2) followed by the imposition of the periodicity condition on these functions leads to the answer that

$$\left.\begin{array}{rcl} \phi(x) & = & e^{ikx} \\ \mu_\rho & = & e^{ikb} \end{array}\right\} \ k = \frac{2\pi}{a} n \quad (n = \text{any integer}) \tag{3.3}$$

A linear combination of such eigenfunctions makes up a Fourier series. The conclusion is that *the theory of Fourier series is an expression of a linear system in a discrete translation invariant environment or of a system in a periodic state*. An electromagnetic wave propagating through a periodic lattice or the sound emitted by a ticking clock are examples of such systems.

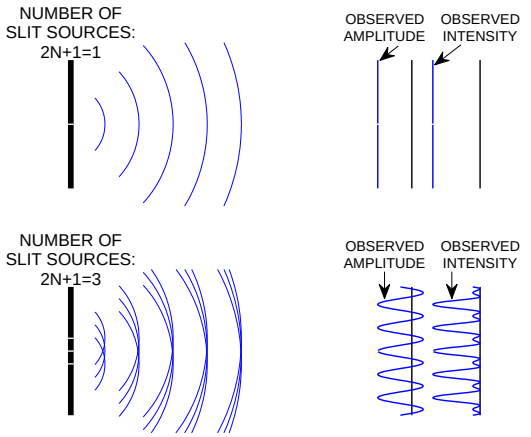


Figure 3.1: Dirichelet kernels of order $N = 0$ and $N = 1$.

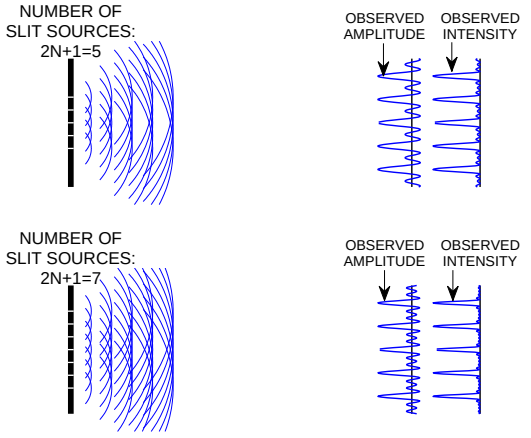


Figure 3.2: Dirichelet kernels of order $N = 2$ and $N = 3$. Each of these kernels is the observed far field amplitude profile of the radiation emitted coherently from an odd $(2N + 1)$ number of sources having equal strength. The observed intensity, which is perceived by the human eye, is proportional to the square of the amplitude. The separation between the source screen and the observation screen is much larger than is implied by the picture.

3.1 The Dirichlet Kernel

Consider a screen with equally spaced slits illuminated by a laser beam. Assuming the width of the slits is small compared to their separation, each

one of them acts as a source of radiation as depicted in Figures 3.1 and 3.2.

Observations show that at large distances from these sources – large compared to the laser wave length, and large compared to the slit separation – the emitted radiation forms a Fraunhofer diffraction which is characterized by the number of slit sources. For an odd number, say $2N+1 = 1, 3, 5, 7$ of them, the measured amplitude profiles and the observed intensity (= squared amplitude) profiles have forms which are shown in Figures 3.1-3.2

Each of the diffraction amplitude profiles is the interference pattern of radiation from an odd number of slit sources having equal amplitude and equal phase. Each pattern can be represented with mathematical precision as a finite Fourier sum with a number terms equal to the odd number of slit sources causing the pattern.

If the number of sources is $2N+1$, then the corresponding pattern is called a *Dirichelet kernel* of *integral order* N. Its mathematical form is

$$\delta_N(u) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})u}{\sin\frac{u}{2}} \ , \tag{3.4}$$

where u is the displacement along the screen where the pattern is observed. This kernel is a fundamental concept. Among others, it is the mathematical root of the Fourier Series theorem and the sampling theorem, applications which we shall develop as soon as we have defined this kernel mathematically.

Remark 1. Q: What can one say about diffraction patterns caused by an *even* number of slit sources?
A: The essential difference from an odd number lies in the observed amplitude. Whereas for an odd-numbered source the peak amplitude has always the same sign every period, for an even-numbered slit source the peak amplitude alternates between positive and negative from one period to the next. See Figure 3.3. However, such an amplitude is still given by Eq.(3.4), provided N assumes odd integer values, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$. Such an amplitude pattern is called a *Dirichelet kernel* of *odd half-integral order*.

Remark 2. Q: What happens to the diffraction pattern if each slit has a *finite* width, say w?
A: In that case the diffraction pattern gets modulated (i.e. multiplied) by the sinc function

$$\frac{\sin(u/w)}{u/w} \ .$$

This conclusion is validated by means of the Fourier Integral Theorem, which is developed in the next Section 3.3.1 on page 140.

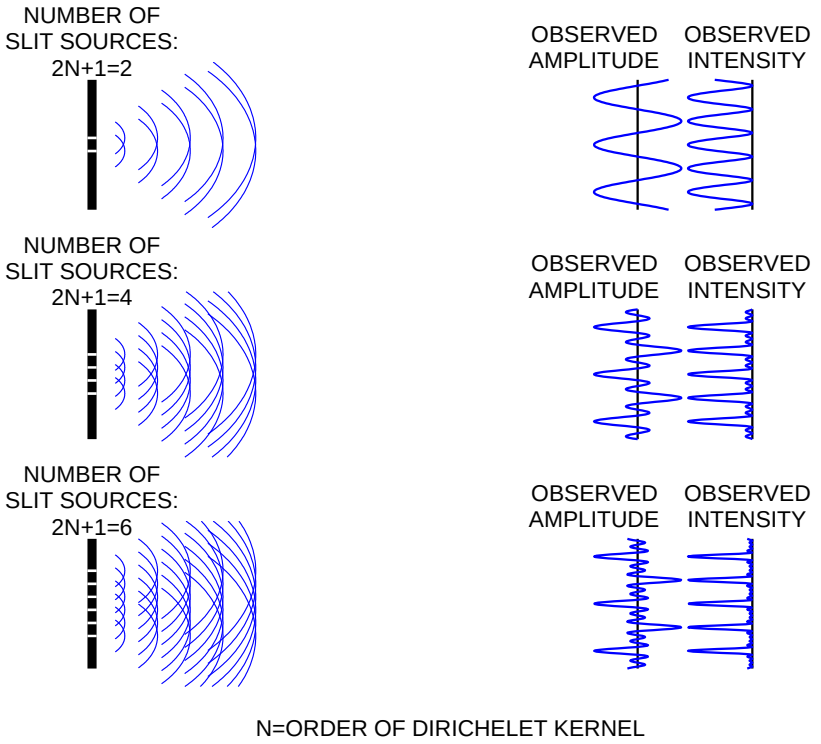


Figure 3.3: Dirichelet kernels of *odd* half-integral order $N = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. These kernels differ from those in Figures 3.1-3.2 in that here the number of sources is even $(2N+1)$. Furthermore, the observed amplitude has peaks that alternate in sign every period.

The Dirichlet kernel of integral order arises in the context of Fourier series whose orthonormal basis functions on $[0, 2\pi]$ are

$$\{u_k(x)\} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\} .$$

Consider the N^{th} partial sum S_N of f, a function integrable on the interval

$[0, 2\pi]$

$$\begin{aligned} S_N &= \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + \sum_{n=1}^{N} b_n \sin nx \\ S_N(x) &= \frac{1}{\pi} \int_0^{2\pi} \left[\frac{1}{2} + \sum_{n=1}^{N} \cos nx \cos nt + \sum_{n=1}^{N} \sin nx \sin nt \right] f(t) dt \\ &= \int_0^{2\pi} \frac{1}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{N} \cos n(x-t) \right] f(t) dt \\ &\equiv \int_0^{2\pi} \delta_N(x-t) f(t) dt \,. \end{aligned}$$

This is the *(optimal) least squares approximation* of f.
Definition: (Dirichlet kernel = "periodic finite impulse function") The function

$$\delta_N(u) = \frac{1}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{N} \cos nu \right] = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inu} \qquad \text{with} \quad u = x - t \tag{3.5}$$

is called the *Dirichlet kernel* and it is also given by

$$\frac{1}{2\pi} \frac{e^{-iNu} - e^{i(N+1)u}}{1 - e^{iu}} = \frac{1}{2\pi} \frac{\sin\left(N + \frac{1}{2}\right) u}{\sin \frac{u}{2}} = \delta_N(u) \,. \tag{3.6}$$

Lecture 19

3.1.1 Basic Properties

Property 1. $\delta_N(x-t)$ projects periodic functions onto their least squares approximation in the subspace spanned by $\{u_k(x)\}$.
Property 2. (The graph, Figure 3.4)
From Figure 3.4, Eq.(3.5), and Eq.(3.6) one sees that

- $\delta_N(u)$ is an even periodic function with period 2π:

$$\delta_N(u + 2\pi n) = \delta_N(u).$$

- $\delta_N(u)$ has $2N$ equally spaced zeros which are determined by

$$(N+\frac{1}{2})u \;\; = \;\; \pi, 2\pi, 3\pi, \ldots, 2N\pi \quad ,$$

and thus are given by

$$u \;\; = \;\; 1\frac{\pi}{N+\frac{1}{2}}, 2\frac{\pi}{N+\frac{1}{2}}, \ldots, 2N\frac{\pi}{N+\frac{1}{2}} \quad .$$

Note that the next term in this sequence is

$$(2N+1)\frac{\pi}{N+\frac{1}{2}} = 2\pi \quad ,$$

but $\delta_N(u)$ is not zero there. In fact, at all integral multiples of 2π, namely $u = 0, 2\pi, \ldots,$ $\delta_N(u)$ has the value

$$\delta_N(2\pi n) = \frac{1}{\pi}\left(N+\frac{1}{2}\right) ,$$

which is the maximum value of the function.

- $\delta_N(u)$ has N maxima and N minima per period.

3.1.2 Three Applications

Solution to Wave Equation via Dirichlet Kernel

In a subsequent chapter we shall study the inhomogeneous Helmholtz wave equation

$$(\nabla^2 + k^2)\psi = f(x,y,z) \quad .$$

It is amusing that the solution to this equation exhibits a property which is readily expressed in terms of a Dirichlet kernel and more generally in terms of a Fourier series. This property is so useful, physically fundamental, and deducible with so little effort that it is worthwhile to give a quick derivation. The property pertains to the field amplitude $\psi(x,y,z)$ when the inhomogeneity ("source") of the wave equation is concentrated at, say, $2N+1$ sources

$$(x_n, y_n, z_n) \quad n = 0, \pm 1, \ldots, \pm N \quad .$$

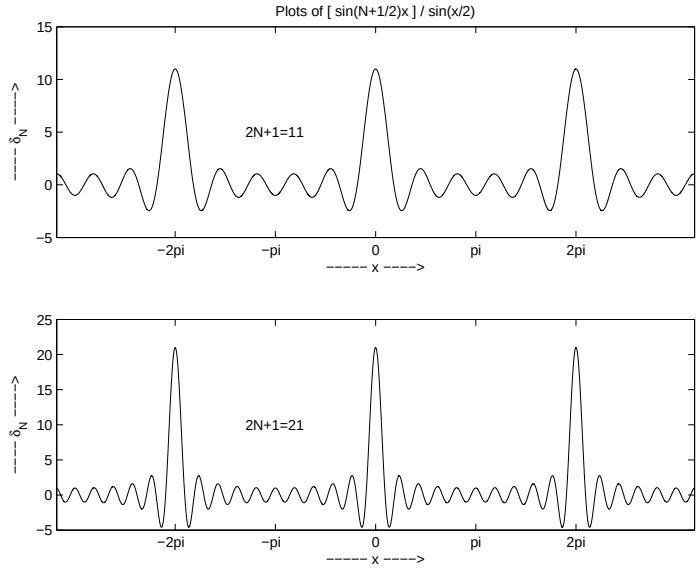


Figure 3.4: Dirichlet kernels for N=5 and N=10. There are N maxima and N minima per period. The number of zeroes is 2N and their spacing inside each periodic subinterval is $1/(2N+1)$.

In that case, the governing Helmholtz equation is

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right]\psi(x,y,z) = -\sum_{n=-N}^{N} A_n\ \delta(x-x_n)\delta(y-y_n)\delta(z-z_n) \tag{3.7}$$

One can readily show that the solution to this inhomogeneous wave equation is

$$\psi(x,y,z) = \frac{1}{4\pi}\sum_{n=-N}^{N} A_n \frac{e^{ikR_n}}{R_n} \quad .$$

Each term in this solution is proportional to the strength of each corresponding localized source of the wave equation. The quantity

$$R_n = \sqrt{(x-x_n)^2+(y-y_n)^2+(z-z_n)^2}$$

is the distance between (x,y,z), the point where the field is observed, and (x_n,y_n,z_n), the location of the nth source point.

We now consider the circumstance where this distance is large. More precisely, we assume that if the sources are distributed along, say, the x-axis,

$$(x_n, y_n, z_n) = (x_n, 0, 0) \quad n = 0, \pm 1, \cdots, \pm N$$

and the amplitude is observed at, say,

$$(x, y, z) = (x, 0, z)$$

so that the distance is

$$R_n = \sqrt{r^2 - 2xx_n + x_n^2} \quad \text{where} \quad r^2 = x^2 + z^2 \quad ,$$

then "large distance" means that r is so large that

$$\frac{x_n}{r} \ll 1 \quad n = 0, \pm 1, \cdots, \pm N \quad .$$

For such distances the solution has the form

$$\psi(x, y, z) = \frac{1}{4\pi} \sum_{n=-N}^{N} \frac{A_n}{R_n} \exp\{ikr - ikx\frac{x_n}{r} + ikx_n\frac{x_n}{2r} + (\text{negl. terms})\} \quad (3.8)$$

The long distance assumption can be strengthened by demanding that both

$$\frac{x_n}{r} \ll 1 \quad \text{and} \; kx_n\frac{x_n}{2r} \ll 1 \qquad (3.9)$$

be satisfied. This strengthened assumption is called the "Fraunhofer approximation". Under this approximation the third contribution to the phase in the exponential of the solution, Eq.(3.8), is so small that this contribution can also be neglected. As a consequence the solution assumes the perspicuous form

$$\psi(x, 0, z) = \frac{e^{ikr}}{4\pi r} \sum_{n=-N}^{N} A_n \exp\{-ikx_n\frac{x}{r}\} \quad . \qquad (3.10)$$

Suppose the $2N + 1$ sources are equally spaced and hence are located at

$$x_n = \triangle x \; n \quad n = 0, \pm 1, \cdots, \pm N \quad .$$

In that case the solution is a $(2N + 1)$-term Fourier series whose coefficients are the source strengths A_n in Eq.(3.7):

$$\psi(x, 0, z) = \frac{e^{ikr}}{4\pi r} \sum_{n=-N}^{N} A_n e^{-in\theta}; \quad \theta \equiv \frac{x}{r}(k\triangle x) \quad . \qquad (3.11)$$

We thus have proved the following fundamental

Theorem 3.1.1
(Fraunhofer-Kirchhoff) At sufficiently large distances expressed by Eq.(3.9), the solution to the inhomogeneous Helmholtz wave equation, Eq.(3.7), has the Fourier form Eq.(3.10) whose spectral coefficients are the strengths of the inhomogeneities in that wave equation. If these inhomogeneities are equally spaced, then the solution is a Fourier series, Eq.(3.11).

When all the sources have equal strength, say A, then the solution is proportional to the Dirichlet kernel,

$$\psi(x,0,z) = A\frac{e^{ikr}}{2r}\delta_N(\theta); \quad \theta \equiv \frac{x}{r}(k\triangle x) \quad , \tag{3.12}$$

which varies with θ in a way given in Fig. 3.4.

For the sake of completeness it is necessary to point out that the Fraunhofer approximation can always be satisfied by having the separation between the "observation" point and the finite source region be large enough. If it is not satisfied, i.e., if

$$\frac{x_n}{r} \ll 1 \quad \text{but } kx_n\frac{x_n}{2r} \approx 1 \text{ or } kx_n\frac{x_n}{2r} > 1 \quad , \tag{3.13}$$

then the third contribution to the phase of the solution, Eq.(3.8), cannot be neglected. This less stringent assumption is called the "Fresnel approximation".

Exercise 3.1.1 (SHIFTED INTEGRATION LIMITS)
Suppose that $f(x+2\pi) = f(x)$ is an integrable function of period 2π. Show that

$$\int_a^{2\pi+a} f(x)dx = \int_0^{2\pi} f(x)dx$$

where a is any real number.

Dirichlet Kernel: Fountainhead of All Subspace Vectors

Consider the space of functions which lie in the subspace

$$\begin{aligned} W_{2N+1} &= \text{span}\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nt, \frac{1}{\sqrt{\pi}}\sin nt\colon n = 1,\ldots,N\right\} \\ &= \text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{int}\colon n = 0,\pm1,\ldots,\pm N\right\} . \end{aligned}$$

One can say that each of these functions owes its existence to the Dirichlet kernel

$$\delta_N(t-x) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{in(t-x)}. \tag{3.14}$$

First, note that $\delta_N(t-x)$ is a vector in W_{2N+1} for every x. Second, note that this vector generates a set of orthonormal basis vectors for W_{2N+1}. They are generated by repeated shifts of the function $\delta_N(t)$ along the t-axis. Indeed the resulting vectors are

$$g_k(t) = \frac{2\pi}{2N+1}\delta_N(t-x_k)$$

where

$$x_k = \frac{\pi}{N+\frac{1}{2}}k\,, \qquad k = 0, 1, \ldots, 2N$$

is the amount by which the function $\delta_N(t)$ has been shifted to the right. The increment between successive shifts is evidently

$$\triangle t = \frac{\pi}{N+\frac{1}{2}} \quad ,$$

the separation between the successive zeroes of $\delta_N(t)$ in the interval $(0, 2\pi)$. This means that, to obtain the function $g_k(t)$, the maximum of $\delta_N(t)$ has been shifted to the location of its k^{th} zero. As a consequence, note that

$$\begin{aligned} g_k(t = x_\ell) &= \frac{2\pi}{2N+1}\delta_N(x_\ell - x_k) \quad \ell, k = 0, 1, \ldots, 2N \\ &= \begin{cases} 0 & \ell \neq k \\ 1 & \ell = k \end{cases} \quad . \end{aligned}$$

or more briefly,

$$\boxed{g_k(x_l) = \delta_{k\ell}}\,. \tag{3.15}$$

This is called the *sifting property* of the function g_k. What is the significance of this important property? To find out compare it with the fact that the functions are orthogonal relative to the given inner product; in particular

$$\frac{2N+1}{2\pi}\int_0^{2\pi} g_k(t)g_{k'}(t)\; dt = \delta_{kk'} \quad .$$

That is, except for a normalization factor, the set of elaments $\{g_k : k = 0, 1, \cdots, 2N\}$ forms an orthonormal basis for the subspace W_{2N+1}. Note that the property

$$g_k(x_\ell) = \delta_{k\ell}$$

does not depend on the inner product structure of the subspace W_{2N+1} at all. Instead, recall that this property is a manifestation of a universal property which all vector spaces have, regardless of what kind of inner product each one may or may not be endowed with. This universal property is, of course, the Duality Principle: *Every vector space, in our case W_{2N+1}, has a* dual *vector space*, which is designated by W^*_{2N+1} and which is the space of linear functionals on W_{2N+1}. In particular, this property expresses the duality between the given basis

$$\{g_k : k = 0, 1, \ldots, 2N\}$$

for W_{2N+1} and the dual basis

$$\{\omega^\ell : \ell = 0, 1, \ldots, 2N\}$$

for W^*_{2N+1}, the space of linear functionals on W_{2N+1}. A typical basis functional ("dual basis element")

$$\begin{array}{rccc} \omega^\ell : & W_{2N+1} & \to & \text{scalars} \\ & f & \rightsquigarrow & \omega^\ell(f) \equiv f(t = x_\ell) \end{array}$$

is the linear map ("evaluation" map) which assigns to the vector f the value of $f(t)$ (viewed as a function) at $t = x_\ell$.

By applying this linear functional to each of the basis vectors g_k in W_{2N+1} one finds that

$$\omega^\ell(g_k) \equiv g_k(x_\ell) = \delta_{kl} \quad .$$

This duality relationship between the two bases, we recall, verifies the duality between W_{2N+1} and W^*_{2N+1}.

The usefulness of this "evaluation" duality is that one can use it to solve the following *reconstruction problem*:
Given:

- a set of samples of the function $f \in W_{2N+1}$

$$\{\ (x_0, f(x_0)),\ (x_1, f(x_1)), \cdots, (x_k, f(x_k)), \cdots,\ (x_{2N}, f(x_{2N}))\ \}\ ;$$

- a basis $\{g_k\}$ for W_{2N+1} consisting of functions with the sifting property

$$g_k(x_\ell) = \delta_{k\ell} \quad .$$

Find: a set of coefficients $\{\alpha_k\}$ such that

$$\sum_{k=0}^{2N} \alpha_k g_k(t) = f(t)$$

whenever $t = x_0, x_1, \cdots, x_{2N}$.

This problem has an easy solution. Letting $t = x_0, x_1, \cdots, x_{2N}$, and using the duality relation, one finds

$$\alpha_k = f(x_k).$$

Consequently,

$$f(t) = \sum_{k=0}^{2N} f(x_k) g_k(t)\,. \tag{3.16}$$

The amazing thing about this equation is that it not only holds for the sampled values but also for any t in the interval $[0, 2\pi]$.

How is it, that one is able to reconstruct $f(t)$ with 100% precision on the *whole* interval $[0, 2\pi]$ by *only* knowing $f(t)$ at the points $\{x_k\}$?
Answer: we are *given* the fact that the function $f(t)$ is a *vector in* W_{2N+1}. We also are given that the functions $g_k(t), k = 0, 1, \cdots, 2N$, form a basis for W_{2N+1}, and that these functions have the same domain as $f(t)$. Equation (3.16) is a vector equation. Consequently, its reinterpretation as a function equation is also 100% accurate.

Exercise 3.1.2 (DIRICHELET BASIS)
Consider the $(2N+1)$-dimensional space $W_{2N+1} \subset L^2(0, 2\pi)$ which is spanned by the O.N. basis $\{\frac{1}{\sqrt{2\pi}} e^{ikt}, k = 0, \pm, \cdots, \pm N\}$:

$$W_{2N+1} = span\{\frac{1}{\sqrt{2\pi}} e^{ikt}\}_{k=-N}^{k=N}$$

Next consider the set of shifted Dirichelet kernel functions,

$$g_k(t) = \frac{2\pi}{2N+1} \delta_N(t - x_k) \equiv \frac{1}{2N+1} \sum_{n=-N}^{N} e^{in(t - k\pi/(N+\frac{1}{2}))} \quad .$$

Show that

$$\{g_k(t) : \ k = 0, \pm 1, \cdots, \pm N\} \equiv B$$

is a basis ("Dirichelet" basis) for W_{2N+1}. This, we recall, means that one must show that

(a) the set B is one which is linearly independent, and

(b) the set B spans W_{2N+1}, i.e. if f is an element of W_{2N+1}, then one must exhibit constants b_k such that

$$f(t) = \sum_{n=-N}^{N} b_k g_k(t) \ .$$

Whittaker-Shannon Sampling Theorem: The Finite Interval Version

To summarize: the reconstruction formula

$$f(t) = \sum_{k=0}^{2N} f(x_k) \frac{2\pi}{2N+1} \delta_N(t - x_k) \tag{3.17}$$

highlights the key role of the Dirichlet kernel in representing an arbitrary element of W_{2N+1} in terms of a finite set of sampled data. Start with the normalized Dirichlet kernel $\frac{2\pi}{2N+1}\delta_N(t)$, a vector in W_{2N+1}. By applying discrete shift operations generate a basis. Finally form the linear combination whose coordinates are the sampled values of the function. The resulting formula, Eq.(3.17) is also known as (a special case of) the *Whittaker-Shannon* sampling theorem and it constitutes the connecting link between the analogue world and the world of digital computers.

Lecture 20

Fourier Series of a Function

Consider a periodic function, $f(x) = f(x + 2\pi)$ and its Nth partial Fourier sum

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + \sum_{n=1}^{N} b_n \sin nx \tag{3.18}$$

$$= \int_{-\pi}^{\pi} \delta_N(t-x) f(t) dt \,. \tag{3.19}$$

Here $\delta_N(t-x)$ is the familiar Dirichlet kernel and the integration limits have been shifted downward without affecting the integral. This can always be done when integrating a periodic function. In fact, the shift can be any real amount:

$$\int_0^{2\pi} \begin{pmatrix} \text{periodic function} \\ \text{with period } 2\pi \end{pmatrix} dt = \int_a^{2\pi+a} \begin{pmatrix} \text{periodic function} \\ \text{with period } 2\pi \end{pmatrix} dt \,.$$

(*Verify* that this identity holds for any real a.)

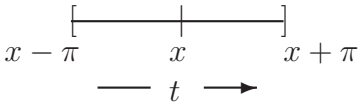
Question: What is $\lim_{N\to\infty} S_N(x)$?

Answer: $S_N(x) \to \frac{1}{2}[f(x^+) + f(x^-)]$ as $N \to \infty$.

One arrives at this answer by means of a four step argument.

1. Shift the integration limit by the amount $a = x$ and obtain

$$S_N(x) = \int_{x-\pi}^{x+\pi} \delta_N(t-x) f(t)\, dt \quad .$$



This places x at the center of the integration interval. Now break up the integral into two parts

$$S_N(x) = \underbrace{\int_{x-\pi}^{x} \delta_N(t-x) f(t) dt}_{J_N(x)} + \underbrace{\int_{x}^{x+\pi} \delta_N(t-x) f(t) dt}_{I_N(x)} \tag{3.20}$$

and show that

$$J_N(x) \to \frac{1}{2} f(x^-) \text{ and } I_N(x) \to \frac{1}{2} f(x^+) \text{ as } N \to \infty \quad .$$

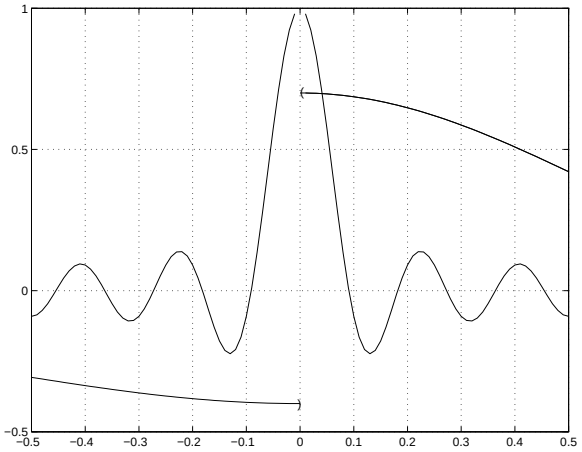


Figure 3.5: The graph of $\delta_N(u)$ and the function $f(x+u)$ around $u=0$, where it has the indicated jump discontinuity.

2. Look at each integral in turn. Let $u = t - x$ and obtain

$$I_N(x) = \int_0^\pi f(x+u)\delta_N(u)du \quad .$$

Figure 3.5 depicts the graphs of the two factors making up the integrand. Using

$$\delta_N(u) = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})u}{\sin\frac{u}{2}}$$

we obtain

$$\lim_{N\to\infty} I_N(x) = \frac{1}{2}f(x^+)\,.$$

The details are as follows:

$$I_N(x) = \frac{1}{2\pi}\int_0^\pi \underbrace{\frac{f(x+u)-f(x^+)}{\sin\frac{u}{2}}}_{G(u)} \sin\left[\left(N+\frac{1}{2}\right)u\right]\,du + f(x^+)\underbrace{\int_0^\pi \delta_N(u)du}_{\begin{array}{c}\| \\ \frac{1}{2} \\ \text{(indep.'t of } N)\end{array}} \quad .$$

Note that $G(u)$ is piecewise continuous. Why? Because, assuming that

f has one-sided derivatives $f'_L(x)$ and $f'_R(x)$ at x, we have

$$G(0^+) \equiv \lim_{u\to 0^+} \frac{f(x+u) - f(x^+)}{\sin\frac{u}{2}} = 2f'_R(x)\,.$$

Thus we see that the integrand is piecewise continuous throughout $[0,\pi]$, even at $u = 0$, where the denominator $\sin\frac{u}{2}$ vanishes. The total integral in question,

$$I_N(x) = \frac{1}{2\pi}\int_0^\pi G(u)\sin\left[\left(N+\frac{1}{2}\right)u\right]\,du + \frac{1}{2}f(x^+)\,,$$

is, therefore, well-defined.

As $N \to \infty$ the integrand is a function $G(u)$, piecewise continuous and finite on $[0,\pi]$, multiplied by a rapidly oscillating function.

Such an integral averages to zero as $N \to \infty$. The vanishing of such an integral is also known as the *Riemann-Lebesgue* lemma. See the ensuing exercise on page 125

Conclusion: $\lim_{N\to\infty} I_N(x) = 0 + \frac{1}{2}f(x^+)$.

3. Similarly, with $u = x - t$ and $\delta_N(-u) = \delta_N(u)$ one has

$$J_N(x) = \frac{1}{2\pi}\int_0^\pi f(x-u)\delta_N(u)\,du$$

and

$$\lim_{N\to\infty} J_N(x) = 0 + \frac{1}{2}f(x^-)\,.$$

4. Consequently, the limit of the partial sum, Eq.(3.20), is

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}[f(x^-) + f(x^+)]\,. \tag{3.21}$$

To summarize, one has the following

Theorem 3.1.2 (Fourier Series Theorem)

.

1. *Let $f(x)$ be a function which is piecewise continuous on $[-\pi, \pi]$.*

   Its Fourier series is given by

   $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(t)\cos n(t-x)\,dt = \frac{1}{2}[f(x^-) + f(x^+)]$$

   at each point $-\pi < x < \pi$ where the one sided derivatives $f'_R(x)$ and $f'_L(x)$ both exist.

2. *If f is continuous the result is*

   $$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} e^{in(x-t)}f(t)\,dt = \frac{1}{2}[f(x^-)+f(x^+)] = f(x) \quad \forall\, f \in C[-\pi,\pi]\,.$$

   or equivalently

   $$f(x) = \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{\sqrt{2\pi}}\; c_n$$

   with

   $$c_n = \int_{-\pi}^{\pi} \frac{e^{-int}}{\sqrt{2\pi}} f(t)\,dt$$

Exercise 3.1.3 (RIEMANN-LEBESGUE LEMMA)
The Riemann-Lebesgue lemma is a well-known fact among radio amateurs and electrical engineers. There it is the time average of an amplitude modulated signal,

$$G(u)\,\sin(N+\frac{1}{2})u\;,$$

whose carrier is $\sin(N+\frac{1}{2})u$, a rapidly oscillating function whose modulation amplitude is $G(u)$. The higher the carrier frequency $(N+\frac{1}{2})$, the more closely that average approaches zero.

Given that $G(u)$ is piecewise continuous on $[0,\pi]$ and has left and right hand derivatives at each point in $[0,\pi]$, show that

$$\lim_{N\to\infty}\int_0^{\pi} G(u)\,\sin(N+\frac{1}{2})u\,du \;=\; 0\;.$$

3.1.3 Poisson's summation formula

The periodicity of the Dirichlet kernel guarantees that Fourier's theorem holds also when x lies outside the interval $[-\pi, \pi]$, even if the function f is *not* periodic. Let us therefore apply the Fourier theorem to the new (continuous) function $f(t + 2\pi m)$:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{in(x-t)} f(t + 2\pi m)\, dt = f(x + 2\pi m) \quad .$$

Now

(i) shift the integration variable t by $2\pi m$,

$$f(x + 2\pi m) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi+2\pi m}^{\pi+2\pi m} e^{in(x-t'+2\pi m)} f(t')\, dt' \ ,$$

(ii) make use of the periodicity of the exponential,

$$f(x + 2\pi m) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi+2\pi m}^{\pi+2\pi m} e^{in(x-t')} f(t')\, dt' \ ,$$

(iii) sum over all the integral values of $m = 0, \pm 1, \pm 2, \cdots$,

$$\sum_{m=-\infty}^{\infty} f(x + 2\pi m) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{+\infty} \int_{-\pi+2\pi m}^{\pi+2\pi m} e^{in(x-t')} f(t')\, dt' \ ,$$

(iv) and use the fact that $\sum_{-\infty}^{\infty} \int_{-\pi+2\pi m}^{\pi+2\pi m} \cdots = \int_{-\infty}^{\infty} \cdots$.

The result is

$$\sum_{m=-\infty}^{\infty} f(x + 2\pi m) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{in(x-t)} f(t)\, dt, \tag{3.22}$$

or

$$\boxed{\sum_{m=-\infty}^{\infty} f(x + 2\pi m) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n) e^{inx}} \quad \text{"general Poisson sum formula"} \tag{3.23}$$

Here

$$F(n) = \int_{-\infty}^{\infty} e^{-int} f(t)\, dt \tag{3.24}$$

is the Fourier transform of the function f.

Setting $x = 0$, one obtains

$$\boxed{\sum_{n=-\infty}^{\infty} F(n) = 2\pi \sum_{m=-\infty}^{\infty} f(2\pi m),} \quad \text{"Poisson sum formula"} \tag{3.25}$$

An alternative equally useful form (see Eq.(3.32) on page 129) of Poisson's formula is

$$\boxed{\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} F(2\pi n)\ .} \tag{3.26}$$

Example (*Closed form via Poisson summation*)

Employ the Poisson summation formula to find the value of the sum

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty} \frac{1}{m^2 + a^2}$$

in terms of elementary functions.

The first step in finding this value is to determine the Fourier amplitude

$$F(k) \equiv \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{x^2 + a^2} dx$$

of the given function f. Using Cauchy's integral formula one finds that

$$F(k) = \frac{\pi}{a} e^{-|k|a}\ ,$$

where without loss of generality $a > 0$.

The second step consists of evaluating the right hand side of Eq.(3.26).

$$\begin{aligned} \sum_{n=0}^{\infty} F(-2\pi n) + \sum_{n=1}^{\infty} F(2\pi n) &= \frac{\pi}{a} \left[2 \sum_{n=1}^{\infty} e^{-2\pi a n} - 1 \right] \\ &= \frac{\pi}{a} \left[\frac{2}{1 - e^{-2\pi a}} - 1 \right] \\ &= \frac{\pi}{a} \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}} \\ &= \frac{\pi}{a} \coth \pi a\ . \end{aligned} \tag{3.27}$$

One has therefore the following result.

$$\sum_{m=-\infty}^{\infty} \frac{1}{m^2+a^2} = \frac{\pi}{a} \coth \pi a \ .$$

Exercise 3.1.4 (CLOSED FORM VIA GENERALIZED POISSON SUMMATION) Prove or disprove:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+b)^2+a^2} = \frac{\pi}{a} \frac{\cosh \pi a}{(\sinh^2 \pi a \ \cos^2 \pi b + \cosh^2 \pi a \ \sin^2 \pi b)} \tag{3.28}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+b)^2} = \frac{\pi^2}{\sin^2 \pi b} \tag{3.29}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(b+2\pi m)^2} = \frac{1}{4 \sin^2 \frac{b}{2}} \tag{3.30}$$

(Hint:

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{(x+b)^2+a^2} dx = \frac{\pi}{a} e^{-|k|a} e^{ikb} \quad \text{can be of help.)}$$

The simplicity of the Poisson summation formula is enhanced if one does not refer to the function f explicitly. Reexpressing the right hand side of Eq.(3.22) in terms of equally spaced Dirac delta functions,

$$\sum_{m=-\infty}^{\infty} f(x+2\pi m) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t-x-2\pi m) f(t) \ dt,$$

and observing the fact that Eq.(3.22) holds for all continuous functions f whose infinite sum of sampled values converges, we leave these individual functions unspecified and simply write Eq.(3.22) in the form

$$\begin{aligned} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inu} \equiv \lim_{N\to\infty} \delta_N(u) &= \sum_{m=-\infty}^{\infty} \delta(u-2\pi m) \\ &= \text{“comb}_{2\pi} u\text{”} \ ; u = t-x \ \ , \end{aligned} \tag{3.31}$$

or equivalently,

$$\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi}{a}nu} = |a| \sum_{m=-\infty}^{\infty} \delta(u - ma),$$

whenever a is positive. (Question: what happens when $a < 0$?) This is obviously an alternate form of the Poisson sum formula. It says that as the number of terms becomes very large, the Dirichlet kernel approaches a periodic train of delta function impulses. However, it needs to be emphasized that this relation is based on the tacit understanding that one first multiply by some appropriate function $f(t)$ and do the t-integration before one compares the sums on the two sides of this relation.

What happens if one first rescales the domain of the function $f(x)$ by a non-zero real factor before shifting it by the amount $2\pi m$? In that case one applies the Fourier theorem to the function

$$f\left(\frac{x + 2\pi m}{a}\right)$$

and the Poisson sum formula, Eq.(3.25), becomes

$$\sum_{n=-\infty}^{\infty} F(na) = \frac{2\pi}{a} \sum_{m=-\infty}^{\infty} f\left(\frac{2\pi m}{a}\right). \tag{3.32}$$

Exercise 3.1.5 (POISSON FORMULA AND ORTHONORMALITY) Stephane G. Mallat in his article "A Theory of Multiresolution Signal Decomposition: The Wavelet Theory" (IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 11, p. 674-692, 1989) makes the following claim in Appendix B of his article:

Let $\hat{\phi}(\omega)$ be the Fourier transform of $\phi(x)$

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-i\omega x} dx.$$

With the Poisson formula one can show that the family of functions

$$\{\phi(x - k) : k = 0, \pm 1, \pm 2, \cdots\}$$

is orthonormal if and only if

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 = const.$$

Prove or disprove this claim. If the claim is true, what would be the value of "*const*."

Exercise 3.1.6 (PHASE SHIFTED POISSON FORMULAS)
Using the Fourier transform notation

$$F(n) = \int_{-\infty}^{\infty} e^{-int} f(t)\, dt \ ,$$

prove or disprove that

$$\sum_{m=-\infty}^{\infty} f\left([2m+1]\pi\right) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(n)$$

$$\sum_{n=-\infty}^{\infty} F(n+\frac{1}{2}) = 2\pi \sum_{m=-\infty}^{\infty} (-1)^m f(2m\pi)$$

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inu} = \sum_{m=-\infty}^{\infty} \delta\left(u - [2m+1]\pi\right) \ ,$$

and more generally that

$$\sum_{m=-\infty}^{\infty} f\left(\frac{[2m+1]\pi}{a}\right) = \frac{a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(na)$$

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inua} = \sum_{m=-\infty}^{\infty} \frac{1}{|a|} \delta\left(u - \frac{[2m+1]\pi}{a}\right) \ .$$

3.1.4 Gibbs' Phenomenon

The Fourier series representation is a least squares fit of a linear combination of orthonormal functions to a given function, even if it has a finite number of discontinuities. However, something peculiar happens, if, using least squares partial sums, one attempts to compute the value of the function near one of its discontinuities. This peculiarity, now known as the Gibbs phenomenon, was highlighted in 1848 by an obscure mathematician, Henry Wilbraham. Fifty years later, unfamiliar with that work, the experimental physicist Albert A. Michelson, complained[2] about the fact that the value of the function

[2] A.A. Michelson, letter to the editor, *Nature* 58, 1898, p544-545. The driving force behind this letter came from observation. J.F. James explains: "Michelson and Stratton

represented by the bracketed sum in Eq.(3.37), when evaluated at $x = \frac{k\pi}{N}$ equals $k\pi$. The source of his dismay was that the graph of the partial sum is therefore

$$S_N(x) = \frac{1}{2} + \frac{2}{\pi} Nx,$$

which easily overshoots the values of $f(t)$ as given by Eq.(3.36) near $t = 0$.

Motivated by Michelson's observations, the physicist Willard Gibbs rediscovered Wilbraham's peculiarity and drew attention to the distinction between *the limit of the graphs of the partial sums* of the Fourier series (see Eq.(3.37) and Figure 3.7 below) as compared to *the graph of the function that is the limit of the partial sum* of the Fourier series (see Eq.(3.36) below).

Figure 3.7 shows the graph of an attempt to approximate the unit step function by means of a partial sum with $2N + 1 = 41$ terms. It is evident that that the partial sum, $S_N(x)$, in its attempt to approximate the function's discontinuity, overshoots its values by well defined amounts on both sides of the discontinuity[3]. This overshoot does not go away as N increases. However, its width decreases and makes no difference in the infinite limit of the least squares approximation.

Let us make these statements quantitative by calculating the magnitude of this overshoot.

---

designed a mechanical Fourier synthesizer, in which a pen position was controlled by 80 springs pulling together against a master-spring, each controlled by 80 gear-wheels which turned at relative rates of 1/80, 2/80, 3/80, $\cdots$ 79/80 and 80/80 turns per turn of a crank handle. The synthesizer could have the springs tensions set to represent 80 amplitudes of the Fourier coefficients and the pen position gave the sum of the series. As the operator turned the crank-handle a strip of paper moved uniformly beneath the pen and the pen drew the graph on it, reproducing to Michelson's mystification and dismay, not a square-wave as planned, but showing the Gibbs' phenomenon. Michelson assumed, wrongly, that mechanical short comings were the cause.

The machine itself [see the photograph in Figure 3.6, which can also be found in *A Student's Guide to Fourier Transforms with Applications in Physics and Engineering* by J.F. James (Cambridge University Press, 2002)], a marvel of its period, was constructed by Gaertner & Co. of Chicago in 1898. It now languishes in the archives of the South Kensington Science Museum".

[3]Such an overshoot resembles the diffracted amplitude profile in the shadow region of a sharp-edged screen illuminated by monochromatic radiation.

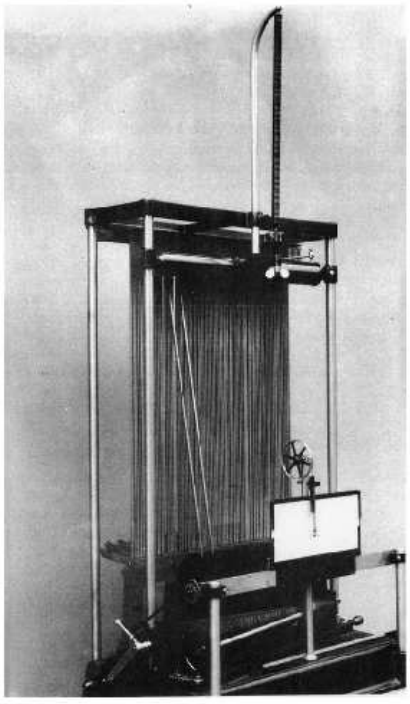


Figure 3.6: The original (1898) "Harmonic Integrator" designed by Michelson and Stratton. Its behaviour exhibited the Gibb's phenomenon in Figure 3.7 .

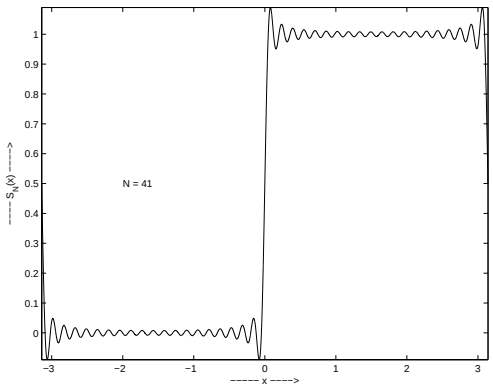


Figure 3.7: Graph of the N^{th} partial Fourier sum $S_N(x)$ for the unit step function.

We start with the shifted representation of the partial sum Eq.(3.19),

$$S_N(x) = \int_{-\pi}^{\pi} f(t)\delta_N(t-x)dt \ .$$

The overshoot, we suspect, happens where the function is discontinuous. Without loss of generality we assume that this happens at $t = 0$. Thus one must investigate how $S_N(x)$ behaves in the neighborhoods of this point: $-\pi < x < \pi$. By an appropriate change of integration variables one obtains

$$\begin{aligned} S_N(x) &= \int_{-\pi-x}^{0} f(u+x)\delta_N(u)du + \int_{0}^{\pi-x} f(u+x)\delta_N(u)du \\ &= \int_{0}^{\pi+x} f(x-u)\delta_N(u)du + \int_{0}^{\pi-x} f(u+x)\delta_N(u)du \ , \end{aligned}$$

where one uses $\delta_N(-u) = \delta_N(u)$ to obtain the second equality. The fact that $f(t)$ is discontinuous at $t = 0$ implies that its explicit form is

$$f(t) = \begin{cases} f_L(t) & -\pi < t < 0 \\ f_R(t) & 0 < t < \pi \end{cases}$$

This induces a corresponding decomposition of the integral representation for S_N:

$$S_N(x) = \underbrace{\int_{0}^{x} f_R(x-u)\delta_N(u)du}_{①} + \underbrace{\int_{x}^{\pi+x} f_L(x-u)\delta_N(u)du}_{②} + \underbrace{\int_{0}^{\pi-x} f_R(x+u)\delta_N(u)du}_{③}$$

Instead of keeping x fixed as $N \to \infty$, one now inquires about the value of $S_N(x)$ as $N \to \infty$, but lets $x \to 0$ simultaneously by setting

$$x = \frac{\pi}{2N+1}$$

while going to the limit.

The second integral is is

$$\begin{aligned}
\textcircled{2} &= \int_x^{\pi+x} f_L(x-u)\delta_N(u)du \\
&= \int_x^{\pi+x} \underbrace{\frac{f_L(x-u)-f_L(x^-)}{2\pi\sin\frac{u}{2}}}_{G_L(u)} \sin\left((N+\frac{1}{2})u\right) du \\
&\quad + f_L(x^-)\int_x^{\pi+x} \delta_N(u)du \ .
\end{aligned}$$

The function $G_L(u)$ vanishes at $u=0$ where $G_L(0) = f_L'(x^-)$, because f is assumed to have a left-sided derivative there. Consequently, upon setting $x = \frac{\pi}{2n+1}$, using the fact that $\int_0^\pi \delta_(u)du = \frac{1}{2}$, one finds with the help of the Riemann-Legesgue lemma that

$$\lim_{N\to\infty} \textcircled{2} = 0 + \frac{f_L(0^-)}{2} \ . \tag{3.33}$$

The same typen of reasoning yields

$$\lim_{N\to\infty} \textcircled{3} = 0 + \frac{f_R(0^+)}{2} \ . \tag{3.34}$$

The limit of the first integral ①is evaluated by introducing the integratioin variable

$$v = \left(N+\frac{1}{2}\right)u$$

Consequently,

$$\textcircled{1} = \int_0^{x\left(N+\frac{1}{2}\right)} f_R\left(x-\frac{v}{N+1/2}\right)\frac{\sin v}{2\pi\sin\frac{v}{2(N+1/2)}}\ \frac{dv}{N+1/2} \ .$$

Setting

$$x = \frac{\pi}{2(N+1/2)}$$

one finds

$$\begin{aligned}
\lim_{N\to\infty} \textcircled{1} &= \int_0^{\frac{\pi}{2}} f_R(0^+)\frac{v}{\pi v}dv \\
&= \frac{f_R(0^+)}{2} \times 1.179
\end{aligned} \tag{3.35}$$

Combining Eqs.(3.33), (3.34), and (3.35), one obtains

$$\lim_{N\to\infty} S_N\left(\frac{\pi}{2N+1}\right) = \lim_{N\to\infty}\{②+③+①\}$$
$$= \frac{f_L(0^-)+f_R(0^+)}{2} + \frac{f_R(0^+)}{2}\times 1.179$$

Thus, added onto the function value halfway along the discontinuity, the Fourier series representation adds an additional amount which is 118% of the remaining part of that discontinuity. Figure 3.7 illustrates the overshoot for the unit step function

$$f(t) = \begin{cases} f_L(t) = 0 & -\pi < t < 0 \\ f_R(t) = 1 & 0 < t < \pi \end{cases} \tag{3.36}$$

The graph of its $N = 41$ partial Fourier series (a pure sine series),

$$S_N(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \frac{\sin Nx}{N}\right) \tag{3.37}$$

is exhibited in this figure. There one can see the calculate value of Gibbs' 18 % overshoot numerically.

3.2 The Dirac Delta Function

Having already used the concept of a Dirac's "delta function" several times, we shall now introduce it in a way which lets us see how this concept fits into the familiar realm of functions and integrals.

Definition: Let $\delta_\alpha(x)$ be a set of α-parameterized functions with the following properties: for any "well-behaved" (in a sense which depends upon the context) function

1. $\lim_{\alpha\to 0}\int_{-\infty}^{\infty} f(x)\delta_\alpha(x)\,dx = f(0)$
2. (a) $\lim_{\alpha\to 0}\int_{-\infty}^{-\varepsilon} f(x)\delta_\alpha(x)\,dx = 0$
   (b) $\lim_{\alpha\to 0}\int_{\varepsilon}^{\infty} f(x)\delta_\alpha(x)\,dx = 0$

   for all $\varepsilon > 0$.

Then Dirac's "delta function" is a linear functional from the vector space of continuous functions into the reals,

$$\delta(x): \qquad C^0 \quad \longrightarrow \quad R\,,$$

defined by

$$\delta(x): \qquad f \quad \rightsquigarrow \quad \int_{-\infty}^{\infty} f(x)\delta(x)\,dx \equiv \lim_{\alpha\to 0}\int_{-\infty}^{\infty} f(x)\delta_\alpha(x)\,dx \quad (= f(0)) \tag{3.38}$$

or

$$\text{“}\delta(x) = \lim_{\alpha\to 0}\delta_\alpha(x)\text{”} \quad .$$

Remark 1. We have put the last equation in quotes because, strictly speaking, $\delta(x)$ is *only defined when integrated against a "well-behaved" function.* In other words, the frequently quoted definition "this function equals zero everywhere, except at $x = 0$ where it is infinite so that its integral is one" is in conflict with the classical definition of a function and integral. Indeed, $\delta(x) = 0\ \forall\, x \neq 0$ and $\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\delta(x)\,dx \neq 0$ violates the classical definition of a function.

Remark 2. Equations (2) in the definition do *not* imply that

$$\lim_{\alpha\to 0}\delta_\alpha(x'-x) = 0 \qquad \text{for} \quad x' \neq x\,.$$

Example 4 (below) is an instance.

It is easy to come up with examples of parametrized functions that give rise to $\delta(x)$.

1. The Impulse Function

$$\delta_\alpha(x) \equiv \begin{cases} \dfrac{1}{\alpha} & |x| \leq \dfrac{\alpha}{2} \\ \\ 0 & |x| > \dfrac{\alpha}{2} \end{cases}$$

2. The Gaussian function

$$\delta_\alpha(x) \equiv \frac{1}{\alpha\sqrt{\pi}}\, e^{-x^2/\alpha^2}$$

3. The "Lorentz line" function

$$\delta_\alpha(x) = \frac{1}{\pi}\ \frac{\alpha}{x^2+\alpha^2}$$

4. The sinc function

$$\delta_\alpha(x) = \frac{\sin\frac{\pi}{\alpha}x}{\pi x} \equiv \frac{1}{\alpha}\text{sinc}\frac{\pi}{\alpha}x \tag{3.39}$$

All of them satisfy

(a) $\int_{-\infty}^{\infty}\delta_\alpha(x)\,dx = 1\ \forall\,\alpha > 0.$

(b) $\lim_{\alpha\to 0}\int_{-\infty}^{\infty} f(x)\delta_\alpha(x)\,dx = f(0).$

(c) Examples 1-3 are characterized by $\lim_{\alpha\to 0}\delta_\alpha(x) = 0$ whenever $x \neq 0$. However, for example 4 one has $\lim_{\alpha\to 0}\delta_\alpha(x) \neq 0$ whenever $x \neq 0$.

Exercise 3.2.1 (DIRAC DELTA AS A LIMITING WAVE PACKET)
Show that

$$\lim_{w\to\infty}\frac{\sin 2\pi wx}{\pi x}, \quad w > 0$$

is a representation of the Dirac δ-function.
Discussion:

Let

$$\delta_w(x) = \frac{\sin 2\pi wx}{\pi x}$$

and let $f(x)$ be a function which is piecewise continuous on $[-a, a]$, in particular,

$$\lim_{x\to 0^+} f(x) = f(0^+)$$

$$\lim_{x\to 0^-} f(x) = f(0^-)$$

To show that

$$\lim_{w\to\infty}\delta_w(x) = \delta(x) \quad \text{(Dirac delta function)}$$

one must show that

$$\lim_{w\to\infty}\int_{-a}^{a}\delta_w(x)f(x)\,dx = \frac{1}{2}f(0^+) + \frac{1}{2}f(0^-)$$

One way of doing this is to follow the approach used to obtain an analogous result in the process of establishing the validity of the Fourier series theorem, and then use the result that

$$\int_0^\infty \frac{\sin y}{y}\, dy = \frac{\pi}{2}$$

Exercise 3.2.2 (DERIVATIVE OF THE DIRAC DELTA)
Consider the integral

$$I = \int_{-\infty}^{\infty} \delta(x+a) f(x)\, dx \ .$$

Assuming that $f(x)$ is nearly linear so that

$$f(-a) = f(0) - a f'(0) + (\text{neglegible terms}), \tag{3.40}$$

show that I can be evaluated by means of the formal equation

$$\boxed{\delta(x+a) = \delta(x) + a\,\delta'(x)}\ , \tag{3.41}$$

where $\delta(x)$ and $\delta(x+a)$ are defined by Eq.(3.38) and $\delta'(x)$ is defined by

$$\delta'(x): \quad \int_{-\infty}^{\infty} \delta'(x) f(x)\, dx \equiv \lim_{\alpha\to 0} \int_{-\infty}^{\infty} \delta'_\alpha(x) f(x)\, dx \ . \tag{3.42}$$

Comment: For obvious reasons it is invalid to claim

$$\delta_\alpha(x+a) = \delta_\alpha(x) + a\,\delta'_\alpha(x)$$

without referring to test functions that can be approximated by Eq.(3.40).

3.3 The Fourier Integral

Question: What aspect of nature is responsible for the pervasive importance of Fourier analysis?
Answer: Translation invariance. Suppose a linear system is invariant under time or space translations. Then that system's behaviour becomes particularly perspicuous, physically and mathematically, when it is described in terms of translation eigenfunctions, i.e., in terms of exponentials which oscillate under time or space translations. (*Nota bene*: real exponentials are also translation eigenfunctions, but they won't do because they blow up at $-\infty$ or $+\infty$.) In other words, it is the translation invariance in nature which makes Fourier analysis possible and profitable.

3.3.1 Transition from Fourier Series to Fourier Integral

We now extend the domain of definition of a linear system from a finite interval, say $(-c, c)$, to the infinite interval $(-\infty, \infty)$. We shall do this by means of a line of arguments which is *heuristic* ("serving to discover or stimulate investigation"). Even though it pretty much ignores issues of convergence, it has the advantage of being physically precise. It highlights the relation between the discrete Fourier spectrum of a finite system with finitely separated walls and its limiting form as the system becomes arbitrarily large where the walls have arbitrarily large separation. The process of arriving at this limit will be revisited in the more general context of the spectral representation of the unit impulse response ("Green's function" in Section 4.12.3 on page 291) for an infinite string as the limit of a finite one (Section 4.13 on page 298).

By contrast, the advantage of formulating Fourier analysis in mathematically more precise terms lies in that it highlights unambiguously the nature of the functions that lend themselves to being Fourier analyzed.

We start with the complete set of basis functions orthonormal on the interval $[-c, c]$,

$$\left\{ \frac{e^{in\pi x/c}}{\sqrt{2c}} : n = 0, \pm 1, \dots \right\} .$$

The Fourier series for $f \in L^2(-c, c)$ is

$$\frac{1}{2}[f(x^+) + f(x^-)] = \lim_{N\to\infty} \sum_{n=-N}^{N} \underbrace{\left[\int_{-c}^{c} f(t) \frac{e^{-in\pi t/c}}{\sqrt{2c}} dt \right]}_{c_n} \frac{e^{in\pi x/c}}{\sqrt{2c}} .$$

If f is continuous at x, then

$$\frac{1}{2}[f(x^+) + f(x^-)] = f(x) .$$

Second, we let

$$\Delta k = \frac{\pi}{c}$$

and, after rearranging some factors, obtain

$$f(x) \;=\; \sum_{n=-\infty}^{\infty} \Delta k \frac{e^{in\triangle kx}}{\sqrt{2\pi}} \int_{-c}^{c} \frac{e^{-in\triangle kt}}{\sqrt{2\pi}} f(t) dt \quad . \tag{3.43}$$

Third, by introducing the points

$$k_n = n\triangle k, \quad n = 0, \pm 1, \pm 2, \cdots \quad ,$$

we partition the real k-axis into equally spaced subintervals of size $\triangle k = \pi/c$. We introduce these points into the Fourier sum, Eq.(3.43),

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \Delta k \frac{e^{ik_n x}}{\sqrt{2\pi}} \int_{-c}^{c} \frac{e^{-ik_n t}}{\sqrt{2\pi}} f(t)dt \tag{3.44}$$

$$\equiv \sum_{n=-\infty}^{\infty} \Delta k g_c(k_n, x) \tag{3.45}$$

Note that this Fourier sum is, in fact, a Riemann sum, a precursor (i.e. approximation) to the definite integral

$$f(x) \approx \lim_{R\to\infty} \int_{-R}^{R} dk \frac{e^{ikx}}{\sqrt{2\pi}} \int_{-c}^{c} \frac{e^{-ikt}}{\sqrt{2\pi}} f(t)dt \tag{3.46}$$

over the limit of the interval $[-R, R]$ as

$$R \equiv k_N \equiv N\triangle k \to \infty \quad .$$

The fourth and final step is to let $c \to \infty$ in order to obtain the result

$$f(x) = \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikt}}{\sqrt{2\pi}} f(t)\ dt \quad . \tag{3.47}$$

This result is Fourier's Integral Theorem.

3.3.2 The Fourier Integral Theorem

The mathematically more precise statement of this theorem is as follows:

Theorem 3.3.1 (Fourier's Integral Theorem)

Given:(i) *f is function piecewise continuous on every bounded closed interval of the x-axis.*

(ii) *At each point x the function f has both a left derivative $f'_L(x)$ and a right derivative $f'_R(x)$,*

(iii) $f \in L^1(-\infty, \infty)$, *i.e.* f *is absolutely integrable along the* x*-axis:*

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$

Conclusion:

$$\underbrace{\lim_{K\to\infty} \int_{-K}^{K} dk \frac{e^{ikx}}{\sqrt{2\pi}}}_{\int_{-\infty}^{\infty} dk \frac{e^{ikx}}{\sqrt{2\pi}}} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-ikt}}{\sqrt{2\pi}} f(t) dt}_{\hat{f}(k)} = \frac{1}{2}\left[f(x^-) + f(x^+)\right] \tag{3.48}$$

Comments:

1. This result can be restated as a Fourier transform pair,

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{e^{-ikt}}{\sqrt{2\pi}} f(t) dt \tag{3.49}$$

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{\sqrt{2\pi}} \hat{f}(k) dk \tag{3.50}$$

   whenever f is continuous.

2. The exponentials in Eqs.(3.49) and (3.50) give rise to a generalized completeness relation. By interchanging integration order in Eq.(3.48) and letting $K = 1/\alpha$, one has

$$f(x) = \lim_{\alpha\to 0} \int_{-\infty}^{\infty} \left\{ \underbrace{\int_{-1/\alpha}^{1/\alpha} dk \frac{e^{ik(x-t)}}{2\pi}}_{\delta_\alpha(x-t)} \right\} f(t)\, dt \ . \tag{3.51}$$

   This equation holds for all continuous functions $f \in L^1(-\infty, \infty)$. Thus $\delta_\alpha(x-t)$ is another delta convergent sequence:

$$\delta(x-t) = \lim_{\alpha\to 0} \delta_\alpha(x-t) = \int_{-\infty}^{\infty} \frac{e^{ik(x-t)}}{2\pi} dk \ . \tag{3.52}$$

It is of course understood that one first do the integration over t before taking to the indicated limit.

Either one of the two equations, Eq.(3.51) or (3.52), is a generalized completeness relation for the set of "wave train" functions,

$$\left\{ \frac{e^{ikx}}{\sqrt{2\pi}} : \ -\infty < k < \infty \right\} \ .$$

However, these functions are not normalizable, i.e., they $\notin L^2(-\infty, \infty)$. Instead, as Eq.(3.52) implies, they are said to be "δ-function normalized".

Proof of the Fourier integral theorem:

The proof of the Fourier integral theorem presupposes that the Fourier amplitude $\hat{f}(k)$ is well-defined for each k. That this is indeed the case follows from the finiteness of $|\hat{f}(k)|$:

$$|\hat{f}(k)| \leq \int_{-\infty}^{\infty} |\frac{e^{-ikt}}{\sqrt{2\pi}} f(t)| dt \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(t)| dt < \infty \ .$$

The last inequality is an expression of the fact that $f \in L^1(-\infty, \infty)$. Thus $\hat{f}(k)$ is well-defined indeed.

The proof of the Fourier integral theorem runs parallel to the Fourier series theorem on page 124. We shall show that

$$\lim_{K \to \infty} S_K(x) = \frac{1}{2} \left[f(x^-) + f(x^+) \right] \ ,$$

where

$$\begin{aligned} S_K(x) &= \int_{-\infty}^{\infty} dt \, f(t) \int_{-K}^{K} \frac{e^{ik(x-t)}}{2\pi} dk \\ &= \int_{-\infty}^{\infty} dt \, f(t) \, \frac{e^{ik(x-t)}}{2\pi i (x-t)} \Bigg|_{-K}^{K} \\ &= \int_{-\infty}^{\infty} dt \, f(t) \frac{\sin K(x-t)}{\pi (x-t)} \\ &= \underbrace{\int_{-\infty}^{x} f(t) \frac{\sin K(x-t)}{\pi (x-t)} dt}_{J_K(x)} + \underbrace{\int_{x}^{\infty} f(t) \frac{\sin K(x-t)}{\pi (x-t)} dt}_{I_K(x)} \end{aligned}$$

The evaluation of the integrals is done by shifting the integration variable. For the second integral one obtains

$$\begin{aligned} I_K(x) &= \int_0^\infty f(x+u)\frac{\sin Ku}{\pi u}du \\ &= \int_0^\infty \frac{f(x+u)-f(x^+)}{\pi u}\sin Ku\,du + \frac{f(x^+)}{\pi}\int_0^\infty \frac{\sin Ku}{u}du\ . \end{aligned}$$

Using the fact that

$$\int_0^\infty \frac{\sin Ku}{u}du = \frac{\pi}{2}\ ,$$

and the fact that

$$\frac{f(x+u)-f(x^+)}{\pi u} \equiv G(u)$$

is piecewise continuous everywhere, including at $u = 0$, where

$$G(0) \equiv \lim_{u\to 0^+}\frac{f(x+u)-f(x^+)}{\pi u} = f'_R(x)$$

is the right hand derivative of f at x, one finds that

$$\begin{aligned} \lim_{K\to\infty} I_K(x) &= \lim_{K\to\infty}\left\{\int_0^\infty G(u)\sin Ku\,du + \frac{f(x^+)}{\pi}\frac{\pi}{2}\right\} \\ &= 0 + \frac{1}{2}f(x^+) \end{aligned}$$

with the help the Riemann-Lebesgue lemma.

A similar analysis yields

$$\lim_{K\to\infty} J_K(x) = \frac{1}{2}f(x^-)$$

The sum of the last two equations yields

$$\lim_{K\to\infty} S_(K) = \lim_{K\to\infty}[J_K(x)+I_K(x)] = \frac{1}{2}\left[f(x^-)+f(x^+)\right]\ ,$$

This validates Fourier's integral theorem.

3.3.3 The Fourier Transform as a Unitary Transformation

The Fourier integral theorem establishes a quantitative relation between all $L^1(-\infty,\infty)$ functions and their spectral properties. Question: Can this also be done for signals of finite energy, i.e. $L^2(-\infty,\infty)$ functions:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty \ ?$$

Given that the physical world is perfect, i.e. fully real and hence worthy of the most painstaking study, the Swiss mathematician Michel Plancherel rose to the challenge by mathematizing the causal relation between the temporal and the spectral properties of these finite energy signals. The result was his famous theorem, which is the L^2 version of Fourier's L^1 integral theorem:

Theorem 3.3.2 (Plancherel's Theorem)
Given:
A function $f(x)$ which is square integrable over $(-\infty,\infty)$: $f \in L^2(-\infty,\infty)$.
Conclusion:
There exists a function $\hat{f}(k)$ belonging to $L^2(\infty,\infty)$ such that

1. $$\lim_{\sigma\to\infty} \int_{-\infty}^{\infty} \left| \hat{f}(k) - \int_{-\sigma}^{\sigma} \frac{e^{-ikx}}{\sqrt{2\pi}} f(x)\, dx \right|^2 dk = 0$$

2. $$\int_{-\infty}^{\infty} \left| \hat{f}(k) \right|^2 dk = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

3. $$\lim_{\sigma\to\infty} \int_{-\infty}^{\infty} \left| f(x) - \int_{-\sigma}^{\sigma} \hat{f}(k) \frac{e^{ikx}}{\sqrt{2\pi}}\, dk \right|^2 dx = 0$$

   The function $\hat{f}(k)$ is called the Fourier transform *of $f(x)$. It is determined except over a set of measure zero.*

Furthermore, in case

$$h(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\sqrt{2\pi}} f(x)\, dx$$

exists, one has

$$\hat{f}(k) = h(k)$$

almost everywhere.

An important corollary of this theorem is

Theorem 3.3.3 (Parseval's Theorem)
Given:

1. *Let $f_1(x)$ and $f_2(x)$ be square integrable: $f_1, f_2 \in L^2(-\infty, \infty)$.*

2. *Let $\hat{f}_1(k)$ and $\hat{f}_2(k)$ be their respective Fourier transforms.*

Conclusion:

1.
$$\int_{-\infty}^{\infty} \hat{f}_1(k)\hat{f}_2(k)\,dk = \int_{-\infty}^{\infty} f_1(x)f_2(-x)\,dx, \tag{3.53}$$

   or equivalently

$$\int_{-\infty}^{\infty} \overline{\hat{f}_1}(k)\hat{f}_2(k)\,dk = \int_{-\infty}^{\infty} \overline{f_1}(x)f_2(x)\,dx \tag{3.54}$$

2. *In particular,*

$$\int_{-\infty}^{\infty} \hat{f}_1(k)\hat{f}_2(k)e^{ikx}dk = \int_{-\infty}^{\infty} f_1(y)f_2(x-y)\,dx.$$

   Thus, if $\hat{f}_1(k)\hat{f}_2(k)$ belongs to $L^2(-\infty, \infty)$ as well as both of its factors, it is the Fourier transform of

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(y)f_2(x-y)\,dx. \tag{3.55}$$

This will also be true whenever f_1, f_2 and (3.55) all belong to $L^2(-\infty, \infty)$.

Fourier's integral theorem expresses a linear transformation, say $\mathcal{F}$, when applied to the space of square integrable functions. From this perspective one has

$$L^2(-\infty,\infty) \xrightarrow{\mathcal{F}} L^2(-\infty,\infty)$$
$$f(t) \rightsquigarrow \mathcal{F}[f](k) = \int_{-\infty}^{\infty} \frac{e^{-ikt}}{\sqrt{2\pi}} f(t)dt \equiv \hat{f}(k) \ . \tag{3.56}$$

Furthermore, this transformation is one-to-one because Fourier's theorem says that its inverse is given by

$$L^2(-\infty,\infty) \xrightarrow{\mathcal{F}^{-1}} L^2(-\infty,\infty)$$
$$\hat{f}(k) \rightsquigarrow \mathcal{F}^{-1}[\hat{f}](x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{\sqrt{2\pi}} \hat{f}(k)dk \equiv f(x) \ . \tag{3.57}$$

That $\mathcal{F}$ maps square integrable functions into square integrable functions is verified by the following computation, which gives rise to Parseval's identity: For $f \in L^2(-\infty,\infty)$ we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2 dx &= \int_{-\infty}^{\infty} \overline{f}(x) f(x) dx \\
&= \int_{-\infty}^{\infty} \overline{f}(x) \left[\int_{-\infty}^{\infty} \hat{f}(k') \frac{e^{ik'x}}{\sqrt{2\pi}} dk' \right] dx \\
&= \int_{-\infty}^{\infty} \hat{f}(k') \overline{\left[\int_{-\infty}^{\infty} \frac{e^{-ik'x}}{\sqrt{2\pi}} f(x) dx \right]} dk' \\
&= \int_{-\infty}^{\infty} \hat{f}(k') \overline{\hat{f}}(k') dk' \\
&= \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk \ .
\end{aligned}$$

Thus we obtain Parseval's identity (= "completeness relation", see Eq.(2.17) on page 85). The only proviso is (a) that the function f be square-inegrable and (b) that its Fourier transform $\hat{f}$ be given by the Fourier transform integral.

Remark 1: The fact that the Fourier transform is a one-to-one linear transformation from the linear space $L^2(-\infty,\infty)$ to the linear space $L^2(-\infty,\infty)$ is summarized by saying that the Fourier transform is an "isomorphism".

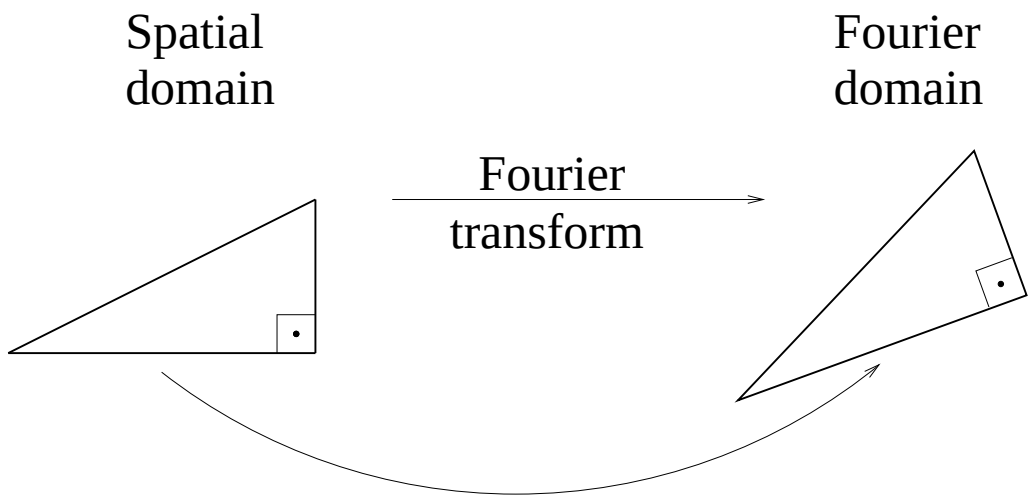


Figure 3.8: The Fourier transform is an isometry between $L^2(-\infty, \infty)$, the Hilbert space of square integrable functions on the spatial domain, and $L^2(-\infty, \infty)$, the Hilbert space of square integrable functions on the Fourier domain.

Remark 2: The line of reasoning leading to Parseval's identity also leads to

$$\langle f, g\rangle \equiv \int_{-\infty}^{\infty} \overline{f}(x)g(x)dx = \int_{-\infty}^{\infty} \overline{\hat{f}}(k)\hat{g}(k)dk \equiv \langle \hat{f}, \hat{g}\rangle$$

whenever $f, g \in L^2(-\infty, \infty)$.

Remark 3: The above two remarks imply that the Fourier transform is a *unitary* transformation in $L^2(-\infty, \infty)$. Unitary transformations are "isometries" because they preserve lengths and inner products. One says, therefore, that the space of functions defined on the spatial domain is "*isometric*" to the space of functions defined on the Fourier domain. Thus the Fourier transform operator is a linear isometric mapping. This fact is depicted by Figure 3.8

Note, however, that even though the Fourier transform and its inverse,

$$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x' - x)dx' = \int_{-\infty}^{\infty} \hat{f}(k)\frac{e^{ikx}}{\sqrt{2\pi}}dk \quad , \tag{3.58}$$

take square integrable functions into square integrable functions, the "basis elements" e^{ikx} are *not* square integrable. Instead, they are "Dirac delta

function" normalized, i.e.,

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{\sqrt{2\pi}} \frac{e^{-ik\overline{x}}}{\sqrt{2\pi}} dk = \delta(x - \overline{x}) \quad .$$

Thus they do not belong to $L^2(-\infty, \infty)$. Nevertheless linear combinations such as Eq.(3.58) are square integrable, and that is what counts.

Exercise 3.3.1 (THE FOURIER TRANSFORM: ITS EIGENVALUES)
The Fourier transform, call it $\mathcal{F}$, is a linear one-to-one operator from the space of square-integrable functions onto itself. Indeed,

$$\begin{array}{rccl} \mathcal{F}: & L^2(-\infty,\infty) & \to & L^2(-\infty,\infty) \\ & f(x) & \sim\to & \mathcal{F}[f](k) \equiv \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \equiv \hat{f}(k) \end{array}$$

Note that here x and k are viewed as points on the common domain $(-\infty, \infty)$ of f and $\hat{f}$.

(a) Consider the linear operator $\mathcal{F}^2$ and its eigenvalue equation

$$\mathcal{F}^2 f = \lambda f.$$

What are the eigenvalues and the eigenfunctions of $\mathcal{F}^2$?

(b) Identify the operator $\mathcal{F}^4$? What are its eigenvalues?

(c) What are the eigenvalues of $\mathcal{F}$?

Exercise 3.3.2 (THE FOURIER TRANSFORM: ITS EIGENVECTORS)
Recall that the Fourier transform $\mathcal{F}$ is a linear one-to-one transformation from $L^2(-\infty, \infty)$ onto itself.
Let ϕ be an element of $L^2(-\infty, \infty)$.
Let $\hat{\phi} = \mathcal{F}\phi$, the Fourier transform of ϕ, be defined by

$$\hat{\phi}(p) = \int_{-\infty}^{\infty} \frac{e^{-ipx}}{\sqrt{2\pi}} \phi(x) dx \quad p \in (-\infty, \infty).$$

It is clear that

$$\phi,\ \mathcal{F}\phi,\ \mathcal{F}^2\phi \equiv \mathcal{F}(\mathcal{F}\phi),\ \mathcal{F}^3\phi \equiv \mathcal{F}(\mathcal{F}^2\phi),\ \mathcal{F}^4\phi \equiv \mathcal{F}(\mathcal{F}^3\phi),\ \cdots$$

are square-integable functions, i.e. elements of $L^2(-\infty, \infty)$.
Consider the SUBSPACE $W \subset L^2(-\infty, \infty)$ spanned by these vectors, namely

$$W = \text{span}\{\phi, \mathcal{F}\phi, \mathcal{F}^2\phi, \mathcal{F}^3\phi, \mathcal{F}^4\phi, \cdots\} \subset L^2(-\infty, \infty).$$

(a) Show that W is finite dimensional.
What is $\dim W$?
(Hint: Compute $\mathcal{F}^2\phi(x), \mathcal{F}^3\phi(x)$, etc. in terms of $\phi(x), \hat{\phi}(x)$)

(b) Exhibit a basis for W.

(c) It is evident that $\mathcal{F}$ is a (unitary) transformation on W.
Find the representation matrix of $\mathcal{F}, [\mathcal{F}]_B$, relative to the basis B found in part b).

(d) Find the secular determinant, the eigenvalues and the corresponding eigenvectors of $[\mathcal{F}]_B$.

(e) For W, exhibit an alternative basis which consists entirely of eigenvectors of $\mathcal{F}$, each one labelled by its respective eigenvalue.

(f) What can you say about the eigenvalues of $\mathcal{F}$ viewed as a transformation on $L^2(-\infty, \infty)$ as compared to $[\mathcal{F}]_B$ which acts on a finite-dimensional vector space?

Exercise 3.3.3 (EQUIVALENT WIDTHS)
Suppose we define for a square-integrable function $f(t)$ and its Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f(t)\ dt$$

the *equivalent width* as

$$\Delta_t = \left| \frac{\int_{-\infty}^{\infty} f(t)\ dt}{f(0)} \right| ,$$

and the *equivalent Fourier width* as

$$\Delta_\omega = \left| \frac{\int_{-\infty}^{\infty} \hat{f}(\omega)\ d\omega}{\hat{f}(0)} \right| .$$

(a) Show that

$$\Delta_t \Delta_\omega = const.$$

is *independent* of the function f, and determine the value of this *const.*

(b) Determine the equivalent width and the equivalent Fourier width for the unnormalized Gaussian

$$f(t) = e^{-x^2/2b^2}$$

and compare them with its full width as defined by its inflection points.

Exercise 3.3.4 (AUTO CORRELATION SPECTRUM)
Consider the *auto-correlation* h [4]

$$h(y) = \int_{-\infty}^{\infty} f(x)f(x-y)dx \tag{3.59}$$

of the function f whose Fourier transform is

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\sqrt{2\pi}} f(x)dx.$$

Compute the Fourier transform of the auto correlation function and thereby show that it equals the "spectral intensity" (a.k.a. power spectrum) of f whenever f is a real-valued function. This equality is known as the *Wiener-Khintchine formula*.

Exercise 3.3.5 (MATCHED FILTER)
Consider a linear time-invariant system. Assume its response to a specific driving force, say $f_0(t)$, can be written as

$$\int_{-\infty}^{\infty} g(T-t)f_0(t)dt \equiv h(T).$$

Here $g(T-t)$, the "unit impulse response' (a.k.a. "Green's function", as developed in CHAPTER 4 and used in Section 4.4.1), is a function which characterizes the system completely. The system is said to be *matched* to the particular forcing function f_0 if

$$g(T) = \overline{f_0}(-T).$$

(Here the bar means complex conjugate.) In that case the system response to a generic forcing function $f(t)$ is

$$\int_{-\infty}^{\infty} \overline{f_0}(t-T)f(t)dt \equiv h(T).$$

A system characterized by such a unit impulse response is called a *matched filter* because its design is matched to the particular signal $f_0(t)$. The response $h(T)$ is called the *cross correlation* between f and f_0.

(a) Compute the total energy

$$\int_{-\infty}^{\infty} |h(T)|^2 dT$$

[4] Not to be confused with the convolution integral Eq.(3.68) on page 156

of the cross correlation $h(T)$ in terms of the Fourier amplitudes

$$\hat{f}_0(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f_0(t)dt$$

and

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f(t)dt.$$

(b) Consider the family of forcing functions

$$\{f_0(t), f_1(t), \cdots, f_N(t)\}$$

and the corresponding family of normalized cross correlations (i.e. the corresponding responses of the system)

$$h_k(T) = \frac{\int_{-\infty}^{\infty} \overline{f_0}(t-T) f_k(t)dt}{\left[\int_{-\infty}^{\infty} |f_k(t)|^2 dt\right]^{1/2}} \quad k = 0, 1, \cdots, N$$

Show that

(i) $h_0(T)$ is the peak intensity, i.e., that

$$|h_k(T)|^2 \leq |h_0(T)|^2 \quad k = 0, 1, \cdots$$

(*Nota bene:* The function $h_0(T)$ corresponding to $f_0(t)$ is called the auto correlation function of $f_0(t)$). Also show that

(ii) equality holds if the forcing function $f_k(t)$ has the form

$$f_k(t) = \kappa f_0(T) \quad \kappa = \text{constant}$$

Lecture 22

3.3.4 Fourier Transform via Parseval's Relation

The Fourier transform is so robust that it can also be applied to functions which are not square-integrable. In fact, it can even be applied to generalized functions ("distributions"), i.e. to entities which are defined by the fact that they are linear functionals on the linear space of the familiar locally integrable

functions. If $f(x)$ is such a function, then a generalized function, say $g(x)$, is defined by the fact that

$$\int_{-\infty}^{\infty} \overline{f}(x)g(x)dx \equiv \langle f, g\rangle < \infty$$

is finite. The Dirac delta function is an example of a generalized function because

$$\int_{-\infty}^{\infty} \overline{f}(x)\delta(x-x')dx \equiv \overline{f}(x') < \infty \quad .$$

Recall that whenever one has square-integrable functions f and g (whose Fourier transforms are $\hat{f}$ and $\hat{g}$) then the reasoning which lead to Parseval's identity leads to

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$$

One now turns this relation around and uses it to define the Fourier transform $\hat{g}$ of a given generalized function g. In other words, start with the set of locally integrable functions $\hat{f}(k)$ and their inverse Fourier transforms

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{\sqrt{2\pi}} \hat{f}(k)dk \quad .$$

Next define the Fourier transform of g as follows: Let it be that generalized function $\hat{g}$ which is determined by the compatibility (between functions and their transforms) requirement that

$$\langle \hat{f}, \hat{g}\rangle = \langle f, g\rangle \tag{3.60}$$

hold for *all* locally integrable functions $\hat{f}$. This equality is now our fundamental requirement. It determines $\hat{g}$ uniquely. Indeed, for every $\hat{f}$ one readily determines f and hence $\langle f, g\rangle$. This implies that Eq.(3.60) is the equation which defines the linear functional $\hat{g}$, the sought-after Fourier transform of g. This linear functional is unique and is denoted by

$$\hat{g}(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\sqrt{2\pi}} g(x)dx \quad ,$$

even though $g(x)$ may not be integrable in the standard sense.
Example 1(*Fourier transform of a "ticking clock" signal*)

Consider the generalized function

$$g(x) = \sum_{n=-\infty}^{\infty} \delta(x-n) \quad ,$$

the train of evenly spaced delta function impulses. What is its Fourier transform?

We know that for any continuous function $\hat{f} \in L^2(-\infty, \infty)$ one can determine its inverse Fourier transform f and hence

$$\langle f, g \rangle \equiv \int_{-\infty}^{\infty} \overline{f}(x) \sum_{n=-\infty}^{\infty} \delta(x-n)\, dx = \sum_{n=-\infty}^{\infty} \overline{f}(n) < \infty \quad . \tag{3.61}$$

The Fourier transform of g is determined by the requirement that for all appropriate $\hat{f}$

$$\begin{aligned} \int_{-\infty}^{\infty} \overline{\hat{f}}(k)\hat{g}(k)dk &\stackrel{1}{=} \langle f, g \rangle \\ &\stackrel{2}{=} \sum_{n=-\infty}^{\infty} \overline{f}(n) \\ &\stackrel{3}{=} \sqrt{2\pi} \sum_{m=-\infty}^{\infty} \overline{\hat{f}}(2\pi m) \\ &\stackrel{4}{=} \int_{-\infty}^{\infty} \overline{\hat{f}}(k) \sum_{m=-\infty}^{\infty} \sqrt{2\pi}\, \delta(k-2\pi m)\, dk \quad . \end{aligned}$$

Equality 1 is the fundamental consistency relation, Eq.(3.60); 2 holds because of Eq.(3.61); 3 holds because of Poisson's sum formula, Eq.(3.25) on page 127,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \sqrt{2\pi}\hat{f}(2\pi m)$$

with $\hat{f} = \sqrt{2\pi}F$:

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\sqrt{2\pi}} f(x)dx \quad ; \tag{3.62}$$

4 takes advantage of the sampling property of the Dirac delta function $\delta(k - 2\pi m)$. Thus one finds that

$$\int_{-\infty}^{\infty} \overline{\hat{f}}(k)\hat{g}(k)dk \;=\; \int_{-\infty}^{\infty} \overline{\hat{f}}(k) \sum_{m=-\infty}^{\infty} \sqrt{2\pi}\, \delta(k-2\pi m)\, dk$$

holds for all integrable functions $\hat{f}(k)$. This fact is reexpressed by the statement

$$\hat{g}(k) = \sum_{m=-\infty}^{\infty} \sqrt{2\pi}\ \delta(k - 2\pi m) \quad .$$

This is the desired result. It says that the Fourier transform of a periodic train of *infinitely sharp* pulses (with period $\Delta x = 1$) is another periodic train of pulses (with period $\Delta k = 2\pi$) in the Fourier domain.

Example 2 (*Fourier transform of a periodic function*) Consider a periodic function

$$g(x) = g(x + a)$$

whose Fourier series representation is

$$g(x) = \sum_{-\infty}^{\infty} c_n \frac{e^{i2\pi nx/a}}{\sqrt{a}} \ , \ \text{with} \ \ c_n = \int_0^a \frac{e^{-i2\pi nx/a}}{\sqrt{a}} g(x)\ dx \ .$$

What is its Fourier transform?

Note that for any integrable function f and its Fourier transform, Eq.(3.62), one has

$$\begin{aligned} \langle f, g\rangle &= \sum_{-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \overline{f}(x) \frac{e^{i2\pi nx/a}}{\sqrt{a}}\ dx \\ &= \sum_{-\infty}^{\infty} c_n \sqrt{\frac{2\pi}{a}}\ \overline{\hat{f}}\left(\frac{2\pi n}{a}\right) \\ &= \int_{-\infty}^{\infty} \overline{\hat{f}}(k) \sum_{-\infty}^{\infty} c_n \sqrt{\frac{2\pi}{a}} \delta(k - \frac{2\pi n}{a})\, dk \end{aligned}$$

Using the stipulated Parseval requirement,

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \equiv \int_{-\infty}^{\infty} \overline{\hat{f}}(k)\hat{g}(k)\ dk \ ,$$

which holds for all functions $\overline{\hat{f}}$, one sees that

$$\hat{g}(k) \equiv \mathcal{F}\left[g\right](k) = \sum_{n=-\infty}^{\infty} c_n \sqrt{\frac{2\pi}{a}} \delta(k - n\frac{2\pi}{a})$$

Thus we have the following result: *The Fourier transform of a function periodic on the given domain is a periodic train of Dirac delta functions on the Fourier domain, each one weighted by the respective Fourier coefficient.*

Conversely, *the Fourier transform of a periodic train of weighted Dirac delta functions is a periodic function.*

What happens if all the weight are equal? In that case the periodic function $g(x)$ turn out to be a generalized function, namely

$$g(x) = \sum_{n=-\infty}^{\infty} \frac{e^{i2\pi nx/a}}{\sqrt{a}} \tag{3.63}$$

$$= \sqrt{a} \sum_{m=-\infty}^{\infty} \delta(x - ma) \ . \tag{3.64}$$

The above Parseval-relation-based method for calculating the Fourier transforms applies to periodic and generalized functions as well. Consequently, one has

$$\hat{g}(k) \equiv \mathcal{F}[g](k) = \sum_{n=-\infty}^{\infty} \sqrt{\frac{2\pi}{a}} \delta(k - n\frac{2\pi}{a}) \ . \tag{3.65}$$

What happens if one takes the Fourier transform again? Without much ado one finds that the Fourier transform of the function $\hat{g}$ is

$$\mathcal{F}\left[\sum_{n=-\infty}^{\infty} \sqrt{\frac{2\pi}{a}} \delta(x - n\frac{2\pi}{a})\right](k) = \sum_{m=-\infty}^{\infty} \frac{e^{-i2\pi mk/a}}{\sqrt{a}} \tag{3.66}$$

$$= \sum_{n=-\infty}^{\infty} \sqrt{a}\ \delta(k - na) \ . \tag{3.67}$$

In other words, one recovers the original function g.

Exercise 3.3.6 (EIGENFUNCTIONS OF $\mathcal{F}^2$)
It is evident from Eqs.(3.67) and (3.64) that the function g is an eigenfunction of the Fourier transform taken twice, i.e. of the operator $\mathcal{F}^2$, with eigenvalue $\lambda = +1$. Are there any other such functions, and if so, characterize them by a simple criterion.

Exercise 3.3.7 (FOURIER TRANSFORM: BASIC PROPERTIES)
Let $\hat{g}(k) = \mathcal{F}[g(x)](k)$ and $H(k) = \mathcal{F}[h(x)](k)$ be the Fourier transforms of $g(x)$ and $h(x)$. Find

(i) $\mathcal{F}\left[\alpha g(x) + \beta h(x)\right](k)$, where α and β are constants.

(ii) $\mathcal{F}\left[g(x-\xi)\right](k)$

(iii) $\mathcal{F}\left[e^{ik_0 x} g(x)\right](k)$

(iv) $\mathcal{F}\left[g(ax)\right](k)$

(v) $\mathcal{F}\left[\dfrac{dg(x)}{dx}\right](k)$

(vi) $\mathcal{F}\left[xg(x)\right](k)$

in terms of $\hat{g}(k)$ and $\hat{f}(k)$.

3.3.5 Efficient Calculation: Fourier Transform via Convolution

Given the importance of the Fourier transforms of periodic functions, is there not a computationally more efficient way of finding these transforms? The answer is "yes", and it hinges on the remarkable properties of the convolution integral[5]

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} f(x-\xi) g(\xi)\, d\xi \\ &= \int_{-\infty}^{\infty} g(x-\xi) f(\xi)\, d\xi \end{aligned} \tag{3.68}$$

of the two functions f and g. Before identifying these properties we first describe the mental process which leads to the graph of this integral:

(i) Take the graph of the function $f(\xi)$ and flip it around the vertical axis $\xi = 0$. This yields the graph of the new function $f(-\xi)$.

(ii) Slide that flipped graph to the right by an amount x by letting $\xi \to \xi - x$, and thus obtain the graph of $f(x-\xi)$.

(iii) Multiply this graph by the graph of $g(\xi)$ to obtain the graph of the product function $f(x-\xi)g(\xi)$.

[5]Not to be confused with the auto-correlation integral, Eq.(3.59) on page 150.

(iv) Find the area under this product function.

As one slides the flipped graph to the right, this area generates the graph of $f * g(x)$.

Example 3 (*Periodic train of Gaussians via convolution*)

Consider the graph of the Gaussian

$$f(\xi) = e^{-(\xi-c)^2/2b^2} \tag{3.69}$$

having full width $2b$ centered around $\xi = c$, and let

$$g(\xi) = \sum_{n=-\infty}^{\infty} \delta(\xi - n) \tag{3.70}$$

be a periodic train of Dirac delta functions. To form the convolution $f * g(x)$, flip the function f to obtain

$$f(-\xi) = e^{-(-\xi-c)^2/2b^2} \ ,$$

which is centered around $\xi = -c$, shift it to the right by an amount x to obtain

$$f(x-\xi) = e^{-(x-\xi-c)^2/2b^2} \ ,$$

and finally do the integral

$$\begin{aligned} \int_{-\infty}^{\infty} f(x-\xi) \sum_{n=-\infty}^{\infty} \delta(\xi - n)\, d\xi &= \sum_{n=-\infty}^{\infty} f(x-n) \\ &= \sum_{n=-\infty}^{\infty} e^{-(x-n-c)^2/2b^2} \ . \end{aligned}$$

This is a periodic train of Gaussians, and the period is $\Delta x = 1$. This result also illustrates how a periodic function, in our case

$$h(x) = \sum_{n=-\infty}^{\infty} f(\xi - n) \equiv \sum_{n=-\infty}^{\infty} e^{-(x-n-c)^2/2b^2} \ ,$$

can be represented as the convolution

$$\boxed{\sum_{n=-\infty}^{\infty} f(x-n) = f * g(x)}$$

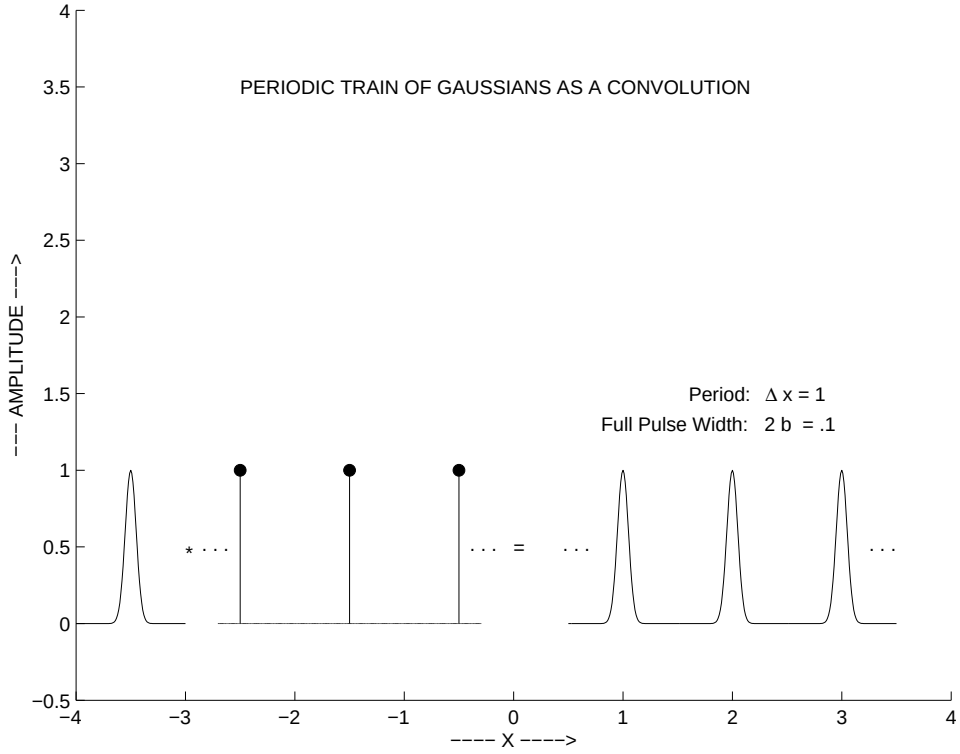


Figure 3.9: Periodic train of Gaussian pulses obtained by convolving a single Gaussian, on the very left, with a periodic train of Dirac delta functions, whose infinite amplitudes are represented in this figure by the heavy dots.The Fourier transform of the train of Gaussians is shown in Figure 3.10

where f and g are given by Eqs.(3.69) and (3.70). The graph of this convolution is the Gaussian train in Figure 3.9. Its Fourier transform, Figure 3.10 is calculated below using a fundamental property of the convolution integral.

Exercise 3.3.8 (PERIODIC FUNCTION AS A CONVOLUTION)
Show that any periodic function $f(\xi) = f(\xi + a)$ is the convolution of a nonperiodic function with a train of Dirac delta functions.

The convolution of two functions has several fundamental properties (commutativity, associativity, distributivity), but its most appealing property is that its Fourier transform is simply the product of the Fourier transforms of the respective functions,

$$\begin{aligned}\int_{-\infty}^{\infty} f * g(x) \frac{e^{-ikx}}{\sqrt{2\pi}}\, dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi) g(\xi)\, d\xi \frac{e^{-ikx}}{\sqrt{2\pi}}\, dx \\ &= \sqrt{2\pi}\hat{f}(k)\hat{g}(k) \end{aligned} \tag{3.71}$$

This result can be an enormous time saver. Let us apply it to the problem of finding the Fourier transform of $h(x)$, the periodic train of Gaussians considered in Example 3, but with $c = 0$, i.e. centered around the origin. The calculation yields

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty} e^{-\xi^2/2b^2} \frac{e^{-ik\xi}}{\sqrt{2\pi}} d\xi \\ &= be^{-b^2k^2/2} \end{aligned} \tag{3.72}$$

and

$$\begin{aligned}\hat{g}(k) &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(\xi - n) \frac{e^{-ik\xi}}{\sqrt{2\pi}} d\xi \\ &= \sum_{m=-\infty}^{\infty} \sqrt{2\pi}\delta(k - 2\pi m)\ . \end{aligned} \tag{3.73}$$

It follows that the Fourier transform of that train yields

$$\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-(x-n)^2/2b^2} \frac{e^{-ikx}}{2\pi} dx = be^{-b^2k^2/2} \sum_{m=-\infty}^{\infty} \sqrt{2\pi}\delta(k - 2\pi m)\ .$$

Figure 3.10 shows the real part of this transform. Study the relationship between this figure and Figure 3.9 carefully. They highlight the archetypical properties of the Fourier transform. To name a few:

- Local properties in the given domain get transformed into global properties in the Fourier domain.
- Jaggedness in the given domain gets transformed into broad spectral behaviour in the Fourier domain.
- Narrow pulses get transformed into wide envelopes, and vice versa.
- Periodicity in the given domain gets transformed into equally spaced (but in general nonperiodic) spectral lines in the Fourier domain.

And there are others.

The pulses that make up the periodic train of Gaussians, Fig. 3.9, have no internal structure. Thus the natural question is: What is the Fourier transform of a periodic train of pulses, each one made up of a finite number of oscillations as in Fig. 3.11? The next example addresses this question.

Example 4 (*Fourier transform of light from a mode-locked laser*)

A mode-locked laser generates light in the form of a periodic train of pulses of light. This periodicity is expressed in terms of the *separation* between successive pulses, and each pulse is characterized by three properties:

1. pulse *envelope*,
2. optical ("carrier") *frequency* and the
3. *phase* of the optical carrier vibrations relative to the pulse envelope.

The temporal amplitude profile of the the nth pulse is

$$f_n(t) = e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)} e^{i\delta_n}$$

The constant T is the separation between successive pulses. The first factor is the pulse envelope, which we take to be a Gaussian of full width $2b$ centered around time $t = nT$. The second factor expresses the oscillations of the optical carrier whose frequency is ω_0. The last factor expresses the phase

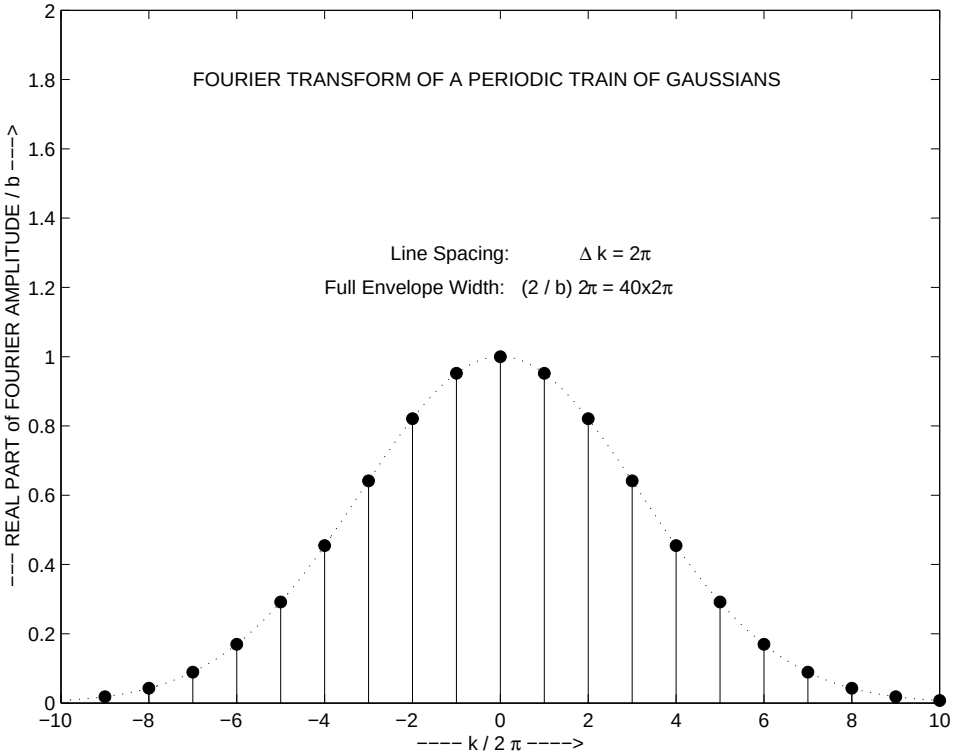


Figure 3.10: Set of equally spaced (here with $\Delta k = 2\pi$) spectral lines resulting from the Fourier transform of the periodic train of Gaussians in Fig. 3.9. The spectral envelope, here again a Gaussian, is the Fourier transform of one of the identical pulses which make up the train.

shift of the optical carrier relative to the pulse envelope. The optical pulse train is the sum

$$f(t) = \sum_{n=-\infty}^{\infty} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)} e^{i\delta_n} \ .$$

The width of the pulse envelope in lasers nowadays (2002) is less than 10 femtoseconds ($=10^{-14}$ sec.). This corresponds to light travelling a distance of less than three microns. Such a pulse width is achieved by the constructive interference of more than a million longitudinal laser modes phase-locked to oscillate coherently.

The pulse repetition rate for a phase-locked laser is determined by the round trip travelling time inside the laser cavity with a partially silvered

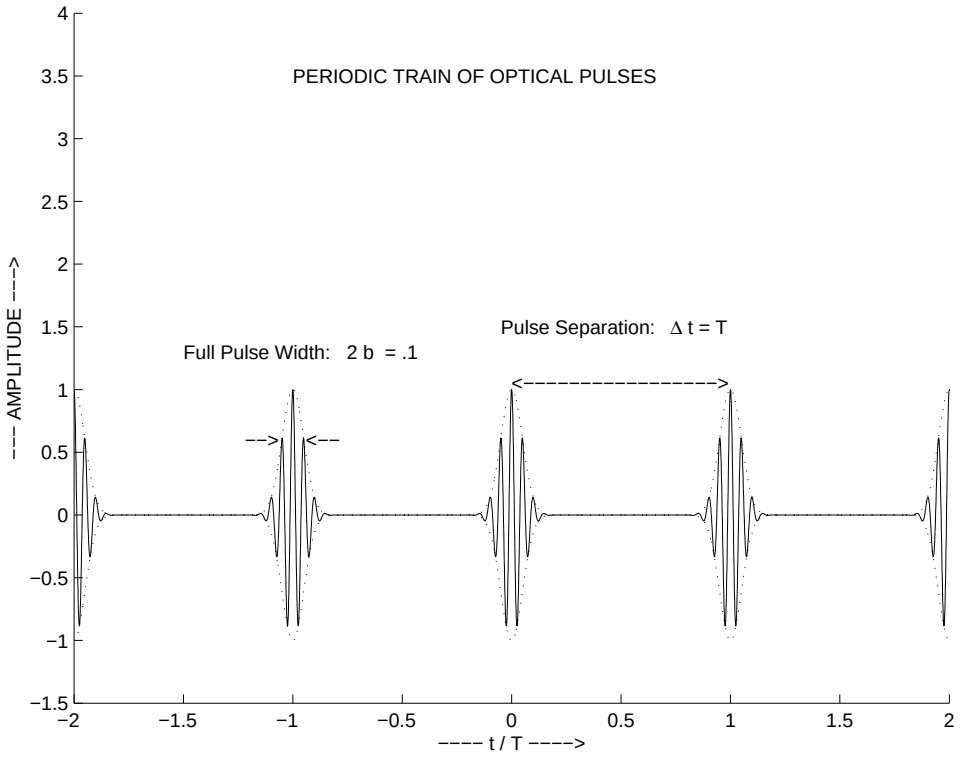


Figure 3.11: Periodic train of optical pulses emitted by a "mode-locked" laser. In this figure the pulse separation is highly understated. In an actual train the pulse separation is typically more than a million times the full width of each pulse. In spite of this, the optical phase (relative to the Gaussian envelope) shifts by a fixed and *controllable* amount from one pulse to the next. In this figure that phase shift is zero: the optical oscillation amplitude profile in each pulse is congruent to that in all the others.

mirror at one end. For a laser 1.5 meters long the pulses emerge therefore at one end at a rate of $1/T$=100 megaHertz, corresponding to a pulse separation of 3 meters of light travelling time between two pulses. In between two such pulses there is no light, no electromagnetic energy whatsoever. The destructive interference of the above-mentioned million laser modes guarantees it.

The pulses can therefore be pictured as micron-sized "light bullets" shot out by the laser. Because of their small size these bullets have an enormous amount of energy per unit volume, even for modestly powered lasers.

Ordinarily the phase δ_n varies randomly from one pulse to the next. In that case f is merely a train of pulses with incoherent phases. The Fourier transform of such a train would be a correspondingly irregular superposition of Fourier transforms. This superposition is exhibited in Figure 3.12

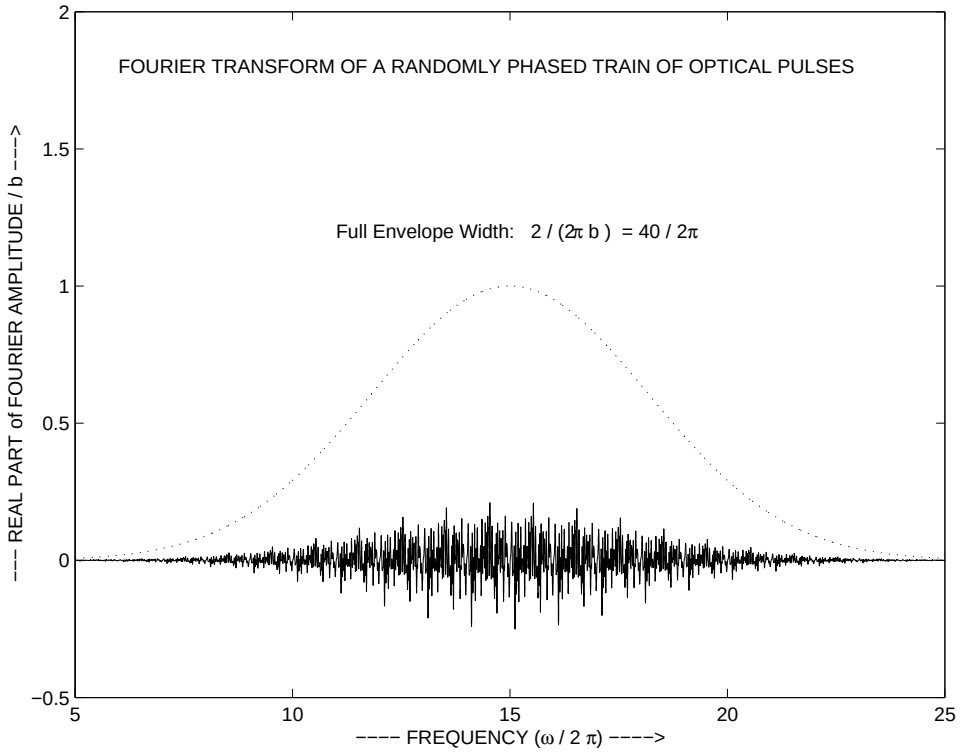


Figure 3.12: Average of the Fourier spectra of 81 pulses of a train like that in Fig. 3.11, but each pulser having random phase δ_n from one to the next. Compare the Fourier spectrum in this figure with the one in Fig. 3.14 whose pulse train is *coherently* phased (i.e. $\delta_n = 0$) and of infinite length.

However, a recent discovery shows that light generated by a laser operating in a "locked-mode" way can be made to produce pulses which are *phase coherent*, even though they are separated by as much as three meters. Indeed, experiments show that the phase δ_n increases by a *constant* amount from one pulse to the next. Evidently the amplifying medium in the laser must somehow "remember" the phase of the carrier oscillations from one

emitted pulse to the next. Thus

$$\delta_n = n\Delta\phi \ .$$

where $\Delta\phi$ is a constant as in Figure 3.13. In that case f is a periodic function,

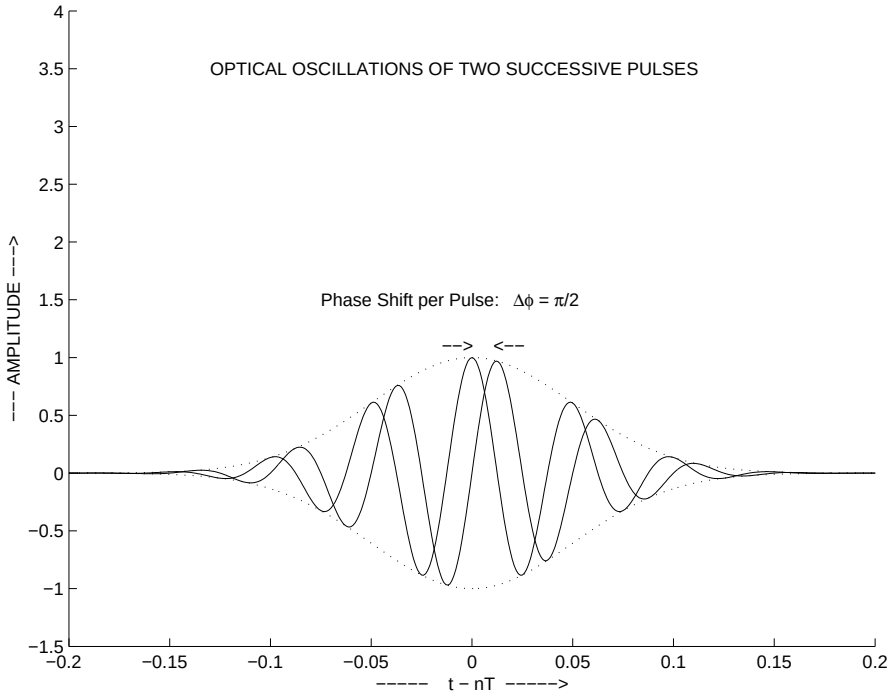


Figure 3.13: Overlay of two successive pulses with phase difference $\Delta\phi = \pi/2$.

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)} e^{in\Delta\phi} \\ &= \sum_{n=-\infty}^{\infty} e^{-(t-nT)^2/2b^2} e^{i\omega_0 t} e^{in(\Delta\phi-\omega_0 T)} \ . \end{aligned} \tag{3.74}$$

Here

$$\omega_0 T = (\#\text{ of optical carrier cycles between adjacent pulses}) \times 2\pi$$

What is the Fourier spectrum of such a periodic train? The result is depicted in Figure 3.14.

The line of reasoning leading to this result is as follows: Observe that the periodic train can be written as the convolution integral

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} e^{-(t-\xi)^2/2b^2} e^{i\omega_0(t-\xi)} \sum_{n=-\infty}^{\infty} e^{in\Delta\phi}\delta(\xi - nT)\, d\xi \\ &\equiv \text{pulse} * \text{comb}_T(t;\Delta\phi) \end{aligned}$$

where

$$\text{pulse}(t) \equiv e^{-t^2/2b^2} e^{i\omega_0 t}$$

is a carrier amplitude modulated by a Gaussian, and

$$\text{comb}_T(t;\Delta\phi) \equiv \sum_{n=-\infty}^{\infty} e^{in\Delta\phi}\delta(t - nT)$$

is a periodic train of linearly phased Dirac delta functions with fixed phase difference $\Delta\phi$ from one delta function to the next. The respective Fourier transforms are

$$\begin{aligned} \mathcal{F}[\text{pulse}](\omega) &= \int_{-\infty}^{\infty} e^{-t^2/2b^2} e^{i\omega_0 t} \frac{e^{-i\omega t}}{\sqrt{2\pi}} dt \\ &= be^{-(\omega-\omega_0)^2 b^2/2} \ , \end{aligned} \tag{3.75}$$

a Gaussian in frequency space centered around ω_0, and, with the help of Poisson's sum formula, Eq.(3.31),

$$\begin{aligned} \mathcal{F}[\text{comb}](\omega) &= \int_{-\infty}^{\infty} \text{comb}_T(t;\Delta\phi) \frac{e^{-i\omega t}}{\sqrt{2\pi}} dt \\ &= \sum_{n=-\infty}^{\infty} \frac{e^{i(\Delta\phi - \omega T)n}}{\sqrt{2\pi}} \\ &= \sum_{m=-\infty}^{\infty} \sqrt{2\pi}\delta(\omega T - 2\pi m - \Delta\phi) \ . \end{aligned} \tag{3.76}$$

This is a periodic set of Dirac delta functions in the frequency domain, but collectively shifted by the common amount $\Delta\phi$. The convolution theorem, Eq.(3.71), implies that the Fourier transform of the train of laser pulses,

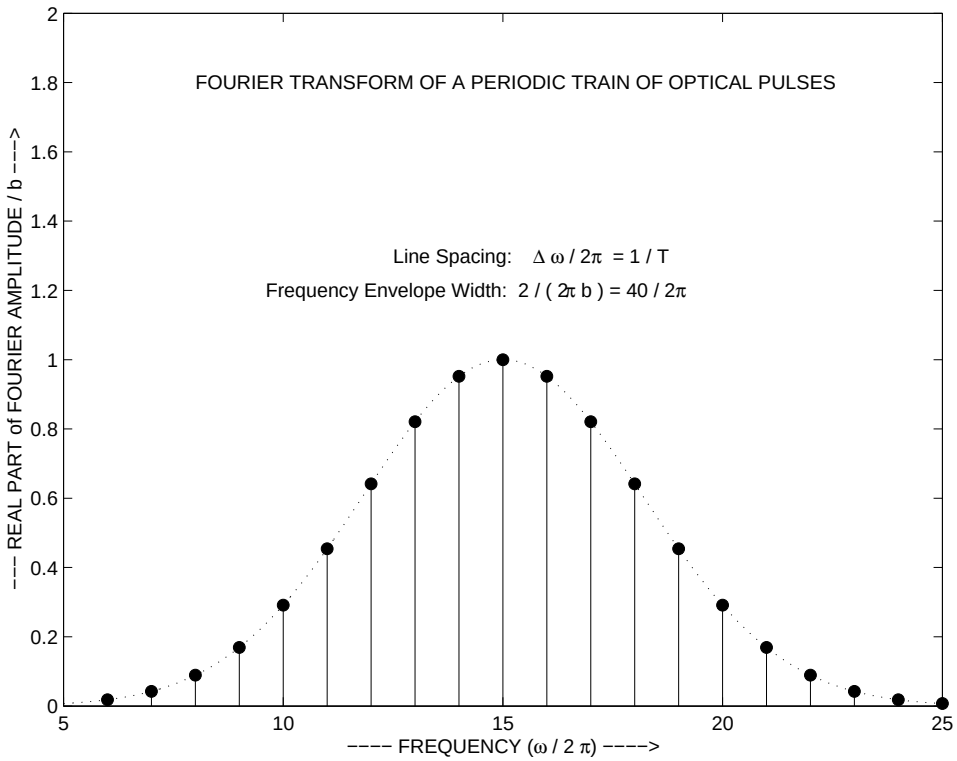


Figure 3.14: Set of equally spaced spectral lines resulting from the Fourier transform of the optical train of pulses in Fig. 3.11. The line spacings in the figure have the common value $\Delta\omega/2\pi = 1/T$ Hertz. For an actual laser generated train the typical value is $\Delta\omega/2\pi \sim 10^8$ Hertz, which is precisely the rate at which energy pulses back and forth in a laser cavity of length $\sim$1.5 meters. For a laser which generates 10 femtosecond pulses, the Gaussian spectral envelope encompasses $\sim 10^6$ spectral lines instead of only the 20 depicted in this figure.

Eq.(3.74) is simply the product of Eqs.(3.75) and (3.76):

$$\begin{aligned}
\mathcal{F}[f](\omega) &= \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f(t)dt \\
&= \underbrace{\sqrt{2\pi}be^{-(\omega-\omega_0)^2b^2/2}}_{\text{spectral envelope}} \underbrace{\sum_{m=-\infty}^{\infty} \delta(\omega T - \Delta\phi - 2\pi m)}_{\text{spectral lines}} \qquad (3.77) \\
&= \sqrt{2\pi}be^{-(\omega-\omega_0)^2b^2/2} \frac{1}{2\pi T} \sum_{m=-\infty}^{\infty} \delta\left(\frac{\omega}{2\pi} - m\frac{1}{T} - \frac{\Delta\phi}{2\pi}\frac{1}{T}\right) \qquad (3.78)
\end{aligned}$$

This is a discrete spectrum of equally spaced sharp spectral lines. The separation between them is

$$\frac{\Delta\omega}{2\pi} = \frac{1}{T} \ ,$$

which is the pulse repetition rate.

From one pulse to the next there is a change in the optical phase relative to the envelope. This phase change, $\Delta\phi$ (exhibited in Figure 3.13) results in all frequencies of the spectral lines under a pulse envelope being shifted by the common amount

$$\underbrace{\frac{\text{(fraction of a cycle)}}{\text{(pulse)}}}_{\frac{\text{phase shift}}{\text{pulse}} \times \frac{1}{2\pi}} \times \underbrace{\frac{\text{(pulses)}}{\text{(time)}}}_{\text{pulse repetition rate}} = \underbrace{\frac{\Delta\phi}{2\pi} \times \frac{1}{T}}_{\text{“frequency offset”}}$$

$$= \frac{\text{(cycles)}}{\text{(time)}}$$

This frequency offset does not apply to the spectral envelope, which remains fixed as one changes $\Delta\phi$. Instead, it applies only to the position of the spectral lines, which get shifted by this frequency offset. This is illustrated in Figure 3.15.

Finally note that, with light oscillating at its carrier frequency ω_0, the Gaussian envelope in Figure 3.14 is centered around the carrier frequency $\omega = \omega_0$ in the frequency domain. When $\omega_0 = 0$, Figs. 3.11 and 3.14 reduce to Figs. 3.9 and 3.10 of Example 3.

Exercise 3.3.9 (FINITE TRAIN OF PULSES)
Find the Fourier spectrum of a *finite* train of identical coherent ($\delta_n = 0$ for $n = 0, \pm 1, \cdots, \pm N$) pulses of the kind shown in Fig. 3.11. Describe the result in terms of a picture and a mathematical formula. Point out how the result differs from Figs. 3.12 and 3.14.

Exercise 3.3.10 (FOURIER SERIES OF A TRAIN OF GAUSSIANS)
Verify that

$$f(t) = \sum_{n=-\infty}^{\infty} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)}$$

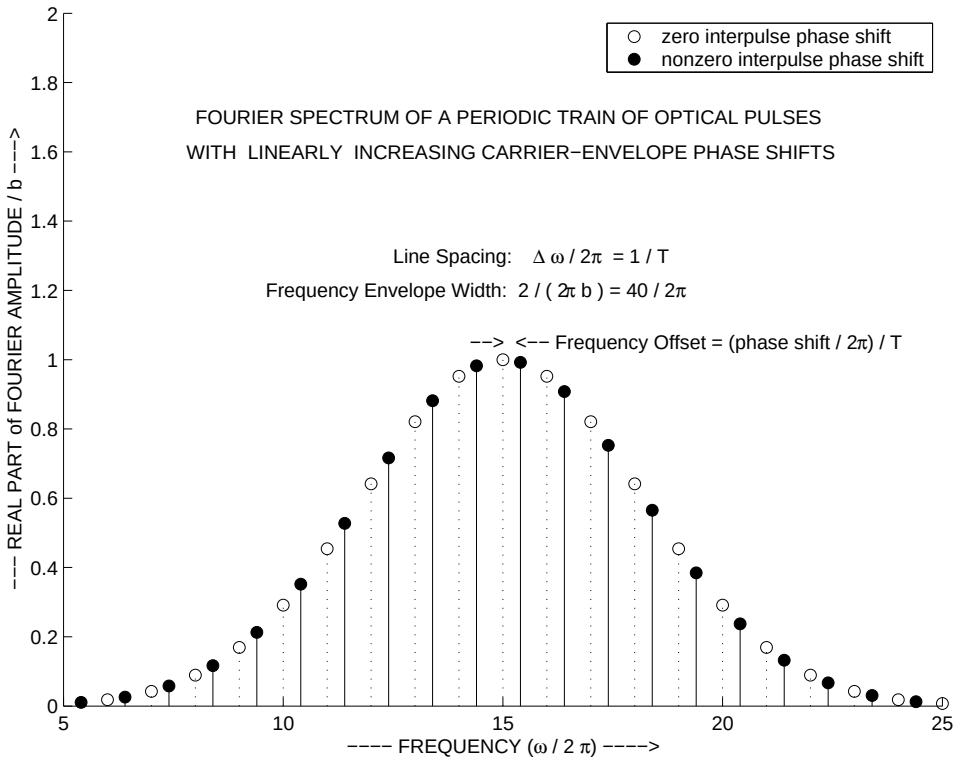


Figure 3.15: Two sets of equally spaced spectral lines resulting from the Fourier transform of two optical trains of pulses. The first (dotted) spectrum (which is identical to that of Fig. 3.14) is due to the train whose pulses have zero ($\Delta\phi = 0$) *interpulse carrier phase shift* of the optical carrier relative to the envelope. The second (solid) spectrum is due to pulses with nonzero ($\Delta\phi \neq 0$) interpulse carrier phase shift.

is a function periodic in t: $f(t+T) = f(t)$.
Find the Fourier series representation

$$f(t) = \sum_{m=-\infty}^{\infty} c_m e^{i\omega_m t}$$

of $f(t)$ by determining ω_m and c_m.

3.4 Orthonormal Wave Packet Representation

The Fourier representation of a square integrable function f ($\in L^2(-\infty,\infty)$) consists of the integral

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) \frac{e^{i\omega t}}{\sqrt{2\pi}} d\omega \,,$$

where

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f(t) dt \,.$$

The virtue of this representation is that the basis functions

$$u_\omega(t) = \frac{e^{i\omega t}}{\sqrt{2\pi}} \qquad\qquad -\infty < \omega < \infty$$

are *translation invariant*, i.e.,

$$u_\omega(t+a) = e^{i\omega a} u_\omega(t) \,.$$

It sould be noted that in reality translation invariance is a limiting feature, one that manifests itself *after* one has taken the limit of some parametrized family of integrable functions, for example,

$$u_\omega(t) \equiv \lim_{T\to\infty} u_\omega(t,T) = \lim_{T\to\infty} e^{-t^2/T^2} \frac{e^{i\omega t}}{\sqrt{2\pi}} \,.$$

Thus, although in the limit these basis functions are "Dirac delta function" orthonormalized,

$$\int_{-\infty}^{\infty} \overline{u}_{\omega'}(t) u_\omega(t) dt = \delta(\omega' - \omega) \,,$$

they are not square integrable ($\notin L^2(-\infty,\infty)$), i.e.,

$$\int_{-\infty}^{\infty} |u_\omega(t)|^2 dt = \infty \,.$$

This disadvantage can be overcome if one does not insist on the basis function being translation invariant, i.e. on going to the limit. The benefit accrued consists not only of the basis elements being square integrable, and hence orthonormal in the standard sense, but of the representation being in

the form of an infinite series instead of an infinite integral. This means that the Hilbert space of square integrable functions is discrete-dimensional: any element is a linear combination of a *countable* number of basis elements. A Hilbert space which has a basis which is countable is said to be a *separable* Hilbert space (with the implication that there are Hilbert spaces which are *nonseparable*, i.e. do not have a basis which is countable). A separable Hilbert space has the property that any of its elements can be approximated with arbitrary accuracy by a partial Fourier-type sum.

However, we shall find that the largest benefit of a discrete basis representation consists of the fact that it allows one to view the behaviour of a given function, say $f(t)$, and its Fourier transform $\hat{f}(\omega)$ from a single point of view: the basis elements reveal the structure of the given function *simultaneously* in the Fourier (ω) domain *and* in the time domain, or in the space domain, whichever the case may be. In practical terms this means that we shall resolve the given function $f(t)$ into a superposition of orthonormal wavepackets which are localized both in the frequency domain and in the time domain, i.e., they have a (mean) frequency and a (mean) location in time. Roughly speaking, each wave packet has the best of both arenas: one foot in the frequency domain and the other foot in the time domain.

By contrast, the Fourier integral representation consists of the given function being resolved into a superposition of infinite wave trains, each one having a definite frequency, but because of their infinite extent, having no definite location. This representation reveals the structure of the function in the Fourier domain, but not in the time domain.

3.4.1 Orthonormal Wave Packets: General Construction

There are many different complete sets of orthonormal wave packets. Each set is a countable basis for the Hilbert space $L^2(-\infty, \infty)$. The construction is basically the same for all these sets and it is illustrated by the following example.

Subdivide the real line, $-\infty < \omega < \infty$, of the Fourier domain into equal subintervals of length ε and consider a function, $P_{j\ell}(t)$, whose Fourier transform is zero everywhere except in one of these subintervals,

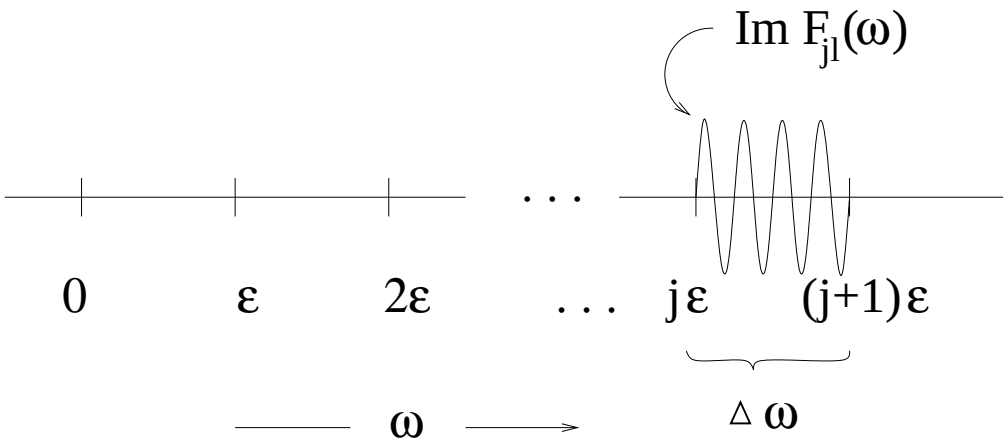


Figure 3.16: The imaginary part of the localized Fourier amplitude of the wave packet $P_{j\ell}(t)$. In this graph $\ell = 4$ and the mean frequency of the wave packet is $(j + \frac{1}{2})\varepsilon$.

$$F_{j\ell}(\omega) = \begin{cases} 0 & \omega \text{ not in } [j\varepsilon, (j+1)\varepsilon] \\ \dfrac{e^{-2\pi i \ell \omega/\varepsilon}}{\sqrt{\varepsilon}} & j\varepsilon \le \omega \le (j+1)\varepsilon \end{cases} . \tag{3.79}$$

We demand that $\ell = 0, \pm 1, \pm 2, \cdots$, is an integer so that $F_{j\ell}(\omega)$ can be pictured as a finite complex amplitude in the *jth frequency window* $j\varepsilon \le \omega \le (j+1)\varepsilon$. See Figure 3.16.

Note that

$$\int_{-\infty}^{\infty} \overline{F}_{j\ell}(\omega) F_{j'\ell'}(\omega) d\omega = \delta_{jj'}\delta_{\ell\ell'} ,$$

which can be easily verified. Such an orthonormality relation is the key to constructing complete sets of orthonormal wave packets. Simply invent a set of functions $F_{j\ell}(\omega)$ which satisfy such an orthonormality property. The example in Figure 3.16 illustrates this idea. Then use these functions to construct your own set of wave packets

$$P_{j\ell}(t) = \int_{-\infty}^{\infty} F_{j\ell}(\omega) \frac{e^{i\omega t}}{\sqrt{2\pi}} d\omega . \tag{3.80}$$

We see that the transformation

$$\begin{aligned} \mathcal{F}^{-1}: \quad L^2(-\infty,\infty) &\longrightarrow L^2(-\infty,\infty) \\ F_{j\ell}(\omega) &\rightsquigarrow \mathcal{F}^{-1}[F_{j\ell}] \equiv \int_{-\infty}^{\infty} F_{j\ell}(\omega)\frac{e^{i\omega t}}{\sqrt{2\pi}}d\omega \\ &\equiv P_{j\ell}(t)\ , \end{aligned}$$

which is represented by the "matrix",

$$\frac{e^{i\omega t}}{\sqrt{2\pi}}\ ,$$

is a *unitary transformation* because it preserves inner products between elements in $L^2(-\infty,\infty)$, or equivalently, because it preserves the inner products between the square integrable basis elements

$$\delta_{jj'}\delta_{\ell\ell'} = \langle F_{j\ell}, F_{j'\ell'}\rangle = \langle \mathcal{F}^{-1}[F_{j\ell}], \mathcal{F}^{-1}[F_{j'\ell'}]\rangle = \langle P_{j\ell}, P_{j'\ell'}\rangle\,.$$

3.4.2 Orthonormal Wave Packets: Definition and Properties

We shall now use the above construction idea to obtain a complete set of o.n. wave packets with Fourier domain windows of *constant width*. In the next subsection we shall do it for wave packets of variable width ("wavelets").

Definition

Applying Eq.(3.79) to Eq.(3.80) one finds that the typical wave packet is

$$P^{\varepsilon}_{j\ell}(t) = \int_{j\varepsilon}^{(j+1)\varepsilon} d\omega \underbrace{\frac{e^{i\omega t}}{\sqrt{2\pi}}}_{} \times \underbrace{\frac{e^{-2\pi i\ell\omega/\varepsilon}}{\sqrt{\varepsilon}}}_{} = \frac{1}{\sqrt{2\pi\varepsilon}}\ \frac{e^{i(t-\frac{2\pi\ell}{\varepsilon})(j+1)\varepsilon} - e^{i(t-\frac{2\pi\ell}{\varepsilon})j\varepsilon}}{i(t-\frac{2\pi\ell}{\varepsilon})}$$

$$\begin{matrix} \text{unitary} \\ \text{xformation} \end{matrix} \times \begin{matrix} \text{amplitude} \\ \text{in the} \\ \text{frequency} \\ \text{domain} \end{matrix} = \begin{matrix} \text{wave} \\ \text{packet} \end{matrix}$$

$$= \frac{2}{\sqrt{2\pi\varepsilon}}\frac{\sin\left[(t-\frac{2\pi\ell}{\varepsilon})\frac{\varepsilon}{2}\right]}{t-\frac{2\pi\ell}{\varepsilon}} \cdot \exp\{i(t-\frac{2\pi\ell}{\varepsilon})(j+\frac{1}{2})\varepsilon\}\ . \quad (3.81)$$

Notation: For the purpose of notational efficiency we shall suppress the superscript ε in the wavepacket name $P^{\varepsilon}_{j\ell}(t)$ throughout the remainder of Section 3.4.2. Thus we use simply $P_{j\ell}(t)$ instead. However, in the upcoming Sections 3.5 and 3.6 we shall always highlight ε by explicitly writing $P^{\varepsilon}_{j\ell}(t)$.

Four Properties

First of all, we observe that this wave packet consists of a real amplitude, a 'sinc' function, multiplied by an exponential phase factor, which is rapidly oscillating when the integer $|j|$ is large. From the viewpoint of engineering one says that the wave train $\exp\{i(t-2\pi\ell/\varepsilon)(j+\frac{1}{2})\varepsilon\}$ is getting modulated by the 'sinc' function. The resultant wave train amplitude has its maximum at $t=\frac{2\pi\ell}{\varepsilon}$. From the viewpoint of physics one says that the wave trains $\{\frac{e^{i\omega t}}{\sqrt{2\pi}}\colon j\varepsilon<\omega<(j+1)\varepsilon\}$ comprising the wave packet exhibit a beating phenomenon with the result that they *interfere constructively* at $t=\frac{2\pi\ell}{\varepsilon}$. From the viewpoint of mathematics one observes that the integral has a maximum value when the integrand does not oscillate, i.e. when $t=\frac{2\pi\ell}{\varepsilon}$.

Second, we observe that the spacing between successive zeroes is $\Delta t=\frac{2\pi}{\varepsilon}$. They are located at

$$t=\frac{2\pi}{\varepsilon}k\quad k=0,\pm1,\cdots\ \text{ but } k\neq\ell\ .$$

At $t=\frac{2\pi}{\varepsilon}\ell$ the wave packet $P_{j\ell}(t)$ has *maximum modulus* $\sqrt{\frac{\varepsilon}{2\pi}}$. These two properties are summarized by the *sifting property* of $P_{j\ell}(t)$:

$$P_{j\ell}\left(t=\frac{2\pi}{\varepsilon}k\right)=\sqrt{\frac{\varepsilon}{2\pi}}\delta_{k\ell}\,. \tag{3.82}$$

Consequently, the real and imaginary parts of the wave packets have profiles as depicted in Figure 3.17.

Third, it has *mean frequency* $(j+\frac{1}{2})\varepsilon$. Its *mean position* along the time axis is $\frac{2\pi}{\varepsilon}\ell$. Its *frequency spread* is the width of its frequency window in the Fourier domain

$$\Delta\omega=\varepsilon\,.$$

Its *temporal spread*,

$$\Delta t=\frac{2\pi}{\varepsilon}\quad ,$$

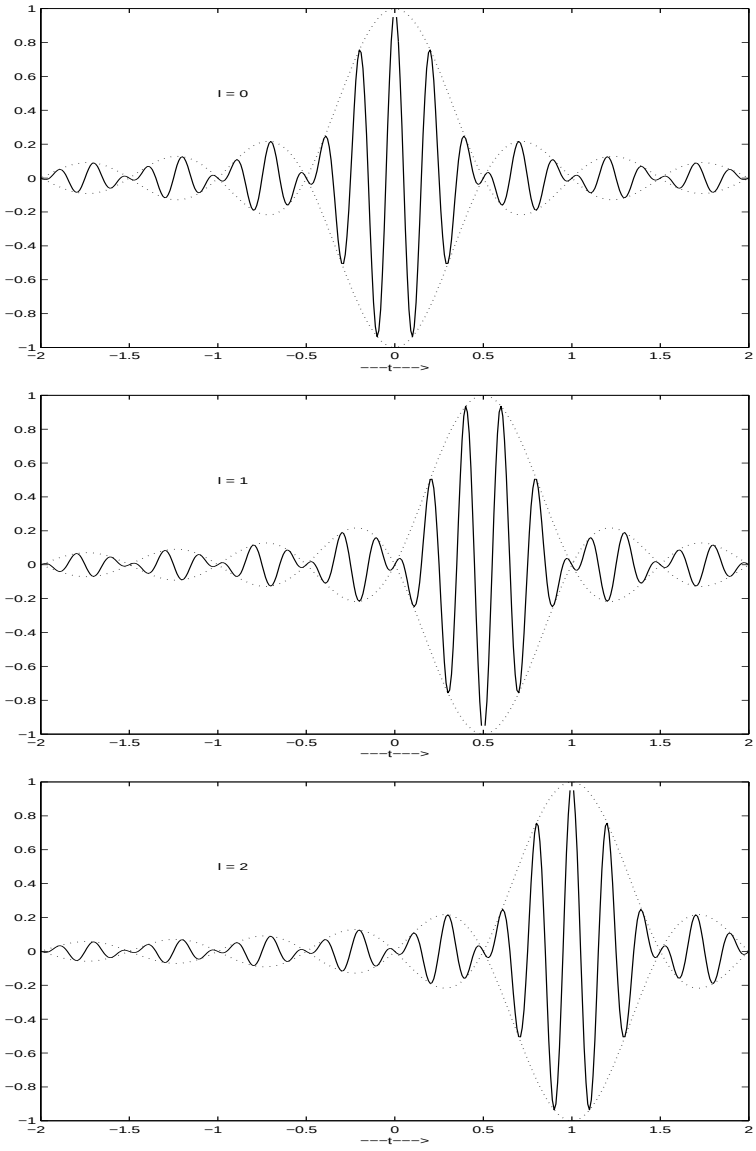


Figure 3.17: Real parts of the three wave packets $\sqrt{\pi}P_{j\ell}(2\pi t),\ \ell = 0, 1, 2$ as given by Eq.(3.81). The t-axis is in units of 2π, i.e. it is expressed in periods of some standard clock. The width of the Fourier window is taken to be $\varepsilon = 2$. The mean frequency of all three wave packets is $(j + \frac{1}{2})\varepsilon = (2 + \frac{1}{2})2 = 5$ oscillations per period.

is its half width centered around its maximum, which is located at $t = \frac{2\pi\ell}{\varepsilon}$. Consequently, the frequency spread times the temporal spread of each wave packet is

$$\Delta\omega\Delta t = 2\pi\,,$$

which is never zero. Thus, the only way one can increase the temporal resolution ($\Delta t \to 0$) is at the expense of the frequency resolution, i.e., by increasing ($\Delta\omega \to \infty$) the frequency bandwidth of each wave packet. Conversely, the only way to increase the frequency resolution is to increase the width of the wave packet.

The last property is expressed by the following exercise:

Exercise 3.4.1 (ORTHONORMALITY AND COMLETENESS)
Consider the set of functions ("wave packets")

$$\left\{ P_{jl}(t) = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{2\pi i l\omega/\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-i\omega t} d\omega\,; \qquad \begin{array}{l} j = 0, \pm1, \pm2, \cdots \\ l = 0, \pm1, \pm2, \cdots \end{array} \right\}$$

where ε is a fixed positive constant.

(a) SHOW that these wave packets are *orthonormal*:

$$\text{i.e.} \quad \int_{-\infty}^{\infty} P_{jl}(t)\bar{P}_{j'l'}(t)\,dt = \delta_{jj'}\delta_{ll'} \tag{3.83}$$

(b) SHOW that these wave packets form a *complete* set:

$$\text{i.e.} \quad \sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} P_{jl}(t)\bar{P}_{jl}(t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}\,d\omega \equiv \delta(t-t') \tag{3.84}$$

Note that the expression for a periodic train of delta functions, Poisson's sum formula, Eq.(3.31) on page 128, may be helpful here.

The completeness relation, Eq.(3.84) is equivalent to the statement that any square integrable function $f(t) \in L^2(-\infty, \infty)$ can be represented as a superposition of wave packets, namely

$$f(t) = \sum_{j=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \alpha_{j\ell} P_{j\ell}(t)\,, \qquad -\infty < t < \infty$$

where

$$\alpha_{j\ell} = \int_{-\infty}^{\infty} \overline{P}_{j\ell}(t') f(t')\, dt'$$

are the expansion coefficients.

Whittaker-Shannon Sampling Theorem: The Infinite Interval Version

Remark: Note that even though the expansion coefficients can be determined from these integrals, it is not necessary to do so. Instead, one can obtain the expansion coefficients $\alpha_{j\ell}$ from $f(t)$ directly. One need not evaluate the integral at all. The key to success lies in the sifting property, Eq.(3.82).

Suppose one knows that $f(t)$ has a Fourier transform which is non-zero only in the interval $j\varepsilon < \omega < (j+1)\varepsilon$. This is no severe restriction because f is square integrable, and one can set $j = 0$, provided we make ε large enough. This implies that

$$\alpha_{j\ell} = 0 \qquad \text{if } \; j \neq 0\,.$$

(Why?) Consequently, we have

$$f(t) = \sum_{\ell=-\infty}^{\infty} \alpha_{0\ell} P_{0\ell}(t) \qquad -\infty < t < \infty\,,$$

where the wave packets $P_{0\ell}$ are given by Eq.(3.86). It is easy to determine the expansion coefficients. Using the sifting property, Eq.(3.82), one obtains

$$f\left(\frac{2\pi}{\varepsilon}k\right) = \alpha_{0k}\sqrt{\frac{\varepsilon}{2\pi}} \qquad k = 0, \pm 1, \ldots\,.$$

This means that the expansion coefficients α_{0k} are determined from the values of f sampled at the equally spaced points $t = \frac{2\pi}{\varepsilon}k$. These sampled values of f determined its representation

$$f(t) = \sum_{\ell=-\infty}^{\infty} f\left(\frac{2\pi}{\varepsilon}\ell\right)\sqrt{\frac{2\pi}{\varepsilon}} P_{0\ell}(t)$$

in terms of the set of orthonormal wave packets. This representation of f in terms of its sampled values is 100% accurate. It is called the *Whittaker-Shannon sampling theorem*. It is a generalization of the special case, Eq.(3.17) mentioned on page 121.

Exercise 3.4.2 (WAVE PACKET TRAINS)
Consider the wave packet

$$Q_{j\ell}(t) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{(j-\frac{1}{2})\varepsilon}^{(j+\frac{1}{2})\varepsilon} e^{i\omega t} e^{-2\pi i \ell \omega/\varepsilon} d\omega.$$

Express the summed wave packets

(a) $\sum_{j=-\infty}^{\infty} Q_{j\ell}(t)$

(b) $\sum_{\ell=-\infty}^{\infty} Q_{j\ell}(t)$

(c) $\sum_{\ell=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} Q_{j\ell}(t)$

in terms of appropriate Dirac delta functions, if necessary.

Lecture 24

3.4.3 Phase Space Representation

Consider the two-dimensional space spanned by the time domain ($-\infty < t < \infty$) and the Fourier domain ($-\infty < \omega < \infty$) of the set of square-integrable function $f(t)$. The introduction of the set of orthonormal wave packets determines a partitioning of this space into elements of area whose shape, magnitude and location is determined by these o.n. wave packets. This partitioned two-dimensional space is called the *phase space* of the system whose state is described by the set of square-integrable functions.

The wave packet representation of the function

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \alpha_{j\ell} P_{j\ell}(t)$$

is represented geometrically as a set of complex amplitudes ($\alpha_{j\ell}$) assigned to their respective *elements of area* each one of size $\Delta\omega\Delta t = 2\pi$, which together comprise the *phase space* of the system. This phase space is two dimensional

and it is spanned by the time domain ($-\infty < t < \infty$) and the Fourier domain ($-\infty < \omega < \infty$) of the function $f(t)$.

The set of orthonormal wave packets determine a *partitioning* of this phase space into elements of *equal* area,

$$\Delta\omega\Delta t = 2\pi\,,$$

which are called *phase space cells*. The existence of this partitioning is guaranteed by the following

Theorem 3.4.1 (Wave Packet Representation Theorem)
Let $f(t)$ be a square-integrable function. Let $\hat{f}(\omega)$ be its Fourier transform,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} f(t)dt$$

and let

$$\hat{P}_{j\ell}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\sqrt{2\pi}} P_{j\ell}(t)dt \quad (= F_{j\ell}(\omega),\ \textit{which is given by Eq.(3.79)})$$

be the Fourier transform[6] *of the wave packet $P_{j\ell}(t)$. Then the Fourier transform pair*

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \alpha_{j\ell} P_{j\ell}(t)$$

[6]The form of this Fourier transform is generated by the action of the j^{th} power of the translation operation T_ϵ acting on $\hat{P}_{0\ell}(\omega)$. Indeed, introducing the rectangular function

$$\mathrm{rect}_{[0,\epsilon]}(\omega) = \begin{cases} 1 & 0 < \omega < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

one has

$$\begin{aligned} \hat{P}_{j\ell}(\omega) &= \frac{e^{-2\pi i\ell\omega/\epsilon}}{\sqrt{\epsilon}} \mathrm{rect}_{[j\epsilon,(j+1)\epsilon]}(\omega) \\ &= \frac{e^{-2\pi i\ell(\omega-j\epsilon)/\epsilon}}{\sqrt{\epsilon}} \mathrm{rect}_{[0,\epsilon]}(\omega - j\epsilon) \\ &= (T_\epsilon)^j\ \hat{P}_{0\ell}(\omega) \end{aligned} \tag{3.85}$$

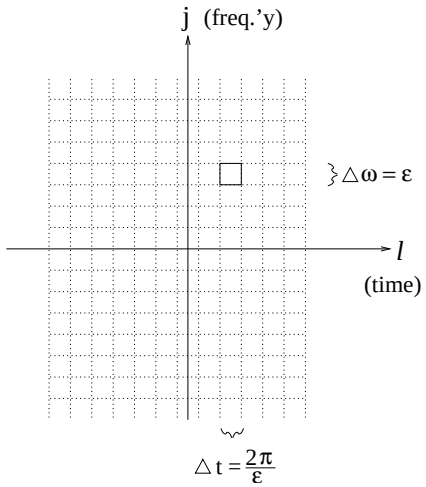


Figure 3.18: The smallest elements of phase space are the phase space cells. Each one, like the one depicted in this picture, has area 2π.

and

$$\hat{f}(\omega) = \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \alpha_{j\ell} \hat{P}_{j\ell}(t)$$

have the same *expansion components* $\alpha_{j\ell}$ *relative to the Fourier-related bases* $\{P_{j\ell}\}$ *and* $\{\hat{P}_{j\ell}\}$. *Both bases are orthonormal*

$$\begin{aligned} \langle P_{j\ell}, P_{j'\ell'} \rangle &\equiv \int_{-\infty}^{\infty} \overline{P}_{j\ell}(t) P_{j'\ell'}(t) dt = \delta_{jj'}\delta_{\ell\ell'} \\ \langle \hat{P}_{j\ell}, \hat{P}_{j'\ell'} \rangle &\equiv \int_{-\infty}^{\infty} \overline{\hat{P}}_{j\ell}(\omega) \hat{P}_{j'\ell'}(\omega) d\omega = \delta_{jj'}\delta_{\ell\ell'} \end{aligned}$$

and are complete

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} P_{j\ell}(t) \overline{P}_{j\ell}(t') &= \delta(t-t') \\ \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \hat{P}_{j\ell}(\omega) \overline{\hat{P}}_{j\ell}(\omega') &= \delta(\omega-\omega') \end{aligned}$$

This theorem implies that

1. the *location* of a typical phase space cell as determined by $P_{j\ell}(t)$ and $\hat{P}_{j\ell}(\omega)$ is given by

$$\begin{aligned} \frac{2\pi}{\varepsilon}\ell &= \text{mean temporal position} \\ \left(j+\frac{1}{2}\right)\varepsilon &= \text{mean frequency}\,. \end{aligned}$$

2. the *shape* of a typical phase space cell as determined by $P_{j\ell}(t)$ and $\hat{P}_{j\ell}(\omega)$ is given by

$$\begin{aligned} \Delta t &= \frac{2\pi}{\varepsilon} \\ \Delta\omega &= \varepsilon\,, \end{aligned}$$

   the temporal and the frequency spread of the wave packet.

3. the *area* of a typical phase space cell as determined by $P_{j\ell}(t)$ and $\hat{P}_{j\ell}(\omega)$ is

$$\Delta\omega\Delta t = 2\pi\,.$$

The wave packet representation

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \alpha_{j\ell} P_{j\ell}(t)$$

of a square integrable function determines the corresponding *phase space representation*. It consists of assigning the complex amplitude $\alpha_{j\ell}$ to the (j,ℓ)th phase space cell. Typically, the squared norm

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2\, dt$$

is proportional to the total "energy" of the signal represented by $f(t)$. If that is the case, then Parseval's identity

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} |\alpha_{j\ell}|^2$$

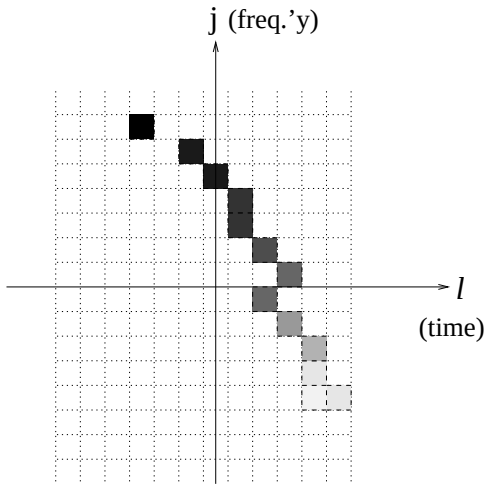


Figure 3.19: Phase space representation of a function.

implies that

$$|\alpha_{j\ell}|^2 \propto \quad \text{“energy” contained in the } (j,\ell)\text{th phase space cell}\,.$$

In other words,

$$\{|\alpha_{j\ell}|^2 \colon j,\ell = 0,\pm 1,\dots\}$$

is a decomposition of the energy of $f(t)$ into its most elementary *spectral* and *temporal* components relative to the chosen wave packet basis $\{P_{j\ell}(t)\}$. The wave packet representation of a signal $f(t)$ assigns to each phase space cell an intensity $|\alpha_{j\ell}|^2$. Each cell acquires a *level of grayness* $\propto |\alpha_{j\ell}|^2$.

Thus a signal gets represented by assigning a degree of darkness (“squared modulus”) and a phase factor of each phase space cell. This is shown in Figure 3.19.

An example of this geometrical phase space representation is the musical score of a piece of music. The *notes* represent the *phase space cells* in which there is a non-zero amount of energy. (Musicians ignore the phase factor which ordinarily would go with it.) A signal, say, Beethoven's Fifth Symphony, is therefore represented by a distribution of dots (of various gray levels) in phase space, with time running horizontally to the right, and pitch going up vertically.

A phase space representation relative to a chosen set of o.n. wave packets is, therefore, a highly refined and sophisticated version of a musician's

score. In fact, it constitutes the ultimate refinement. No better discrete representation is possible.

Final Remarks:

1. It is not necessary that the wave packets, Eq.(3.81) have their Fourier support centered around $(j+\frac{1}{2})\varepsilon$. Another possibility is that they be centered around $j\varepsilon$. In that case the resulting set of wave packets

$$Q^{\varepsilon}_{j\ell} \equiv \int_{(j-\frac{1}{2})\varepsilon}^{(j+\frac{1}{2})\varepsilon} \frac{e^{-2\pi i\ell\omega/\varepsilon}}{\sqrt{\varepsilon}} \frac{e^{i\omega t}}{\sqrt{2\pi}} d\omega \qquad \begin{array}{l} j = 0, \pm 1, \pm 2, \ldots \\ \ell = 0, \pm 1, \pm 2, \cdots \end{array}$$

   would still be orthonormal and complete.

2. Each wave packet of mean frequency zero, $Q^{\varepsilon}_{0\ell}(t), \ell = 0, \pm 1, \pm 2, \cdots$ is an integral representation of the sinc function, Eq.(3.39) on page 137, centered around $t = \frac{2\pi\ell}{\varepsilon}$,

$$\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{e^{-2\pi i\ell\omega/\varepsilon}}{\sqrt{\varepsilon}} \frac{e^{i\omega t}}{\sqrt{2\pi}} d\omega = \frac{2}{\sqrt{2\pi\varepsilon}} \frac{\sin\left[(t-\frac{2\pi\ell}{\varepsilon})\frac{\varepsilon}{2}\right]}{t-\frac{2\pi\ell}{\varepsilon}} = Q^{\varepsilon}_{0\ell}(t)\ . \qquad (3.86)$$

   In the limit as $\varepsilon \to \infty$ this tends towards an expression proportional to the Dirac delta function.

3.5 Orthonormal Wavelet Representation

The key property of the o.n. wave packets is that all phase space cell (i.e., wave packets) have the same shape

$$\begin{aligned} \Delta t &= \frac{2\pi}{\varepsilon} \\ \Delta\omega &= \varepsilon\,. \end{aligned}$$

Suppose, however, we must represent a signal which looks like the one in Figure 3.20. In other words, upon closer examination, the signal on a small scale is similar to the signal on the larger scale. In that case the large scale structure is represented "most economically" by a sequence of wide low frequency wave packets. The qualifier "most economically" means representing the signal with the fewest number of non-zero wave packet coefficients. The

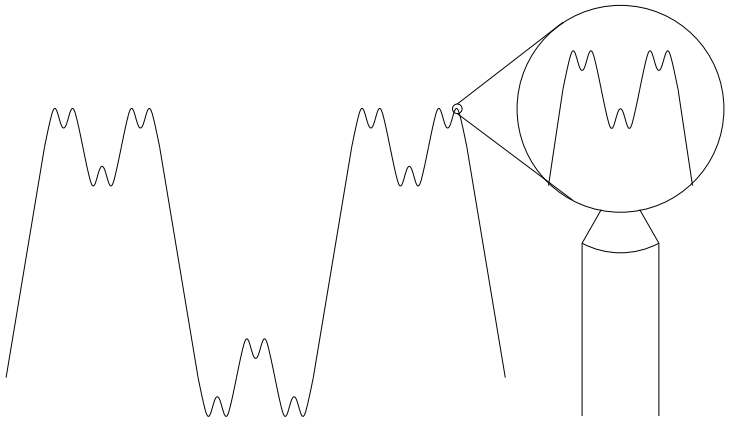


Figure 3.20: A function whose large scale and small scale structures are of equal importance.

existence of o.n. wave packets which are wide *and* narrow is the important new feature.

Let us apply the general idea of constructing o.n. wave packets in Section (3.4.1) to obtain o.n. wave packets with Fourier domain windows of *variable* width. In fact, we shall construct o.n. wave packets with adjacent, but non-overlapping, windows in the Fourier domain, with each window extending exactly over one *octave*. Thus each positive frequency window is twice as wide as its neighbor on the left.

In the time domain, all o.n. *wave packets* have the same half width, namely $\frac{2\pi}{\varepsilon}$. They are different in that they are related to one another by discrete equal shifts $\frac{2\pi}{\varepsilon}$ in time and also by equal shifts ε in frequency. If a signal changes "slowly" over time, i.e., does not change appreciably over a time interval less than the inter-wave packet spacing $\frac{2\pi}{\varepsilon}$, then the signal can be represented quite efficiently by a finite wave packet sum. If, however, the signal changes "abruptly", i.e., it changes appreciably over a time interval small compared to the width, and hence the spacing, of the wave packets, then the wave packets representation becomes less efficient. The wave packet sum must contain many high frequency packets that reinforce each other on one side where the abrupt change occurs and cancel each other on the other side of that change.

What is needed is an o.n. set of *variable* width wave packets. In effect, instead of having a uniform sampling rate, the sampling rate should be *variable* to *accomodate abrupt changes* in the signal. *Orthonormal wavelets*

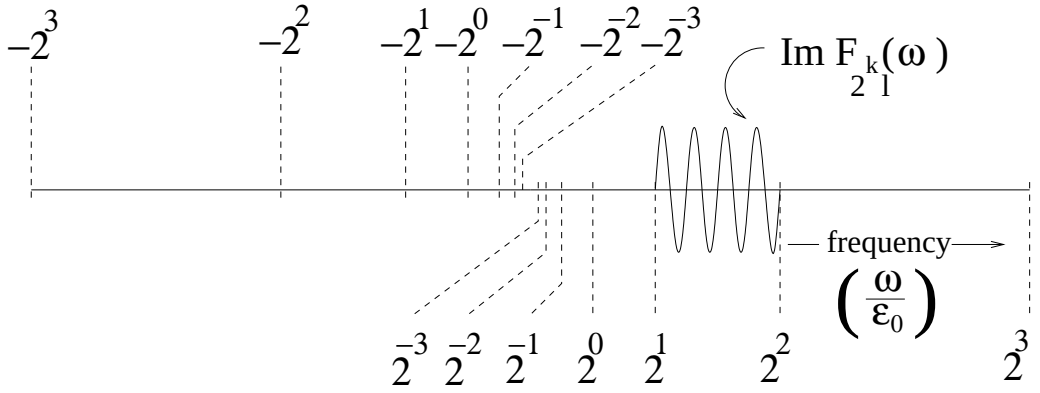


Figure 3.21: Unequal frequency windows of the set of o.n. wavelets. Each window is an *octave*.

fullfill this requirement.

Lecture 25

3.5.1 Construction and Properties

There are equally spaced large half width wavelets of low mean frequency. They enter into the representation of the low resolution, slowly varying features of the signal. There are also equally spaced small half width wavelets of corresponding higher spread in frequency. They enter into the representation of the high resolution, abruptly changing features of the signal.

Instead of the equally spaced frequency windows of the wave packets, the wavelets are synthesized over frequency windows whose width increases exponentially. Wave packets have *variable frequency* window $\varepsilon = 2^{-k}\varepsilon_0$. Inserting this into Eq. (3.79), we obtain the Fourier transform of a wavelet as the following windowed phase factor:

$$F_{2^{-k}\ell}(\omega) = \begin{cases} \sqrt{\dfrac{1}{2^{-k}\varepsilon_0}}\, e^{-2\pi i \ell \omega / 2^{-k}\varepsilon_0} & \omega \in [2^{-k}\varepsilon_0, 2^{1-k}\varepsilon_0] \\ 0 & \text{otherwise} \end{cases} .$$

Here $\varepsilon = 2^{-k}\varepsilon_0$ is the variable frequency window. Such a wavelet is a wave packet $P^{\varepsilon}_{j\ell}(t)$ (see Section 3.4.1) for which $j = 1$ and $\varepsilon = 2^{-k}\varepsilon_0$. The integer $k = 0, \pm 1, \pm 2, \ldots$ is the *octave number* of the wavelet. Let us designate this wavelet by $W^{+}_{k\ell}$. Its key properties are as follows:

1. Its explicit form is

$$W^+_{k\ell}(t) = \sqrt{\frac{1}{2^{-k}\varepsilon_0}} \int_{2^{-k}\varepsilon_0}^{2^{1-k}\varepsilon_0} e^{-2\pi i\ell\omega/2^{-k}\varepsilon_0} \frac{e^{i\omega t}}{\sqrt{2\pi}} d\omega \ , \tag{3.87}$$

and its Fourier transform is

$$\hat{W}^+_{k\ell}(\omega) = F_{2^{-k}\ell}(\omega) \tag{3.88}$$

2. Its *mean frequency* is

$$\overline{\omega} = \frac{1}{2}(2^{-k} + 2^{1-k})\varepsilon_0 \ .$$

3. Its *mean position* along the time axis is

$$\bar{t} = 2\pi\ell/2^{-k}\varepsilon_0 \ .$$

4. The wavelet has *half width*

$$\Delta t = 2\pi/2^{-k}\varepsilon_0 \ , \tag{3.89}$$

its *frequency spread* is

$$\Delta\omega = 2^{-k}\varepsilon_0 \ , \tag{3.90}$$

and its *phase space area* is

$$\Delta\omega\Delta t = 2\pi \ , \tag{3.91}$$

like that of any o.n. wave packet.

5. The wavelets, as well as their Fourier transforms, are *orthonormal*:

$$\int_{-\infty}^{\infty} \overline{W^+}_{k\ell}(t) W^+_{k'\ell'}(t) dt = \delta_{\ell\ell'}\delta_{kk'} \ ,$$
$$\int_{-\infty}^{\infty} \overline{\hat{W}^+}_{k\ell}(\omega) \hat{W}^+_{k'\ell'}(\omega) d\omega = \delta_{\ell\ell'}\delta_{kk'} \ .$$

6. They form a *complete set* in the given domain,

$$\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} W^{+}_{k\ell}(t)\overline{W^{+}}_{k\ell}(t') + W^{-}_{k\ell}(t)\overline{W^{-}}_{k\ell}(t') = \delta(t-t')\,,$$

as well as in the Fourier domain

$$\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \hat{W}^{+}_{k\ell}(\omega)\overline{\hat{W}^{+}}_{k\ell}(\omega') + \hat{W}^{-}_{k\ell}(\omega)\overline{\hat{W}^{-}}_{k\ell}(\omega') = \delta(\omega-\omega')\ .$$

Note that the negative frequency wavelets,

$$W^{-}_{k\ell}(t) = \sqrt{\frac{1}{2^{-k}\varepsilon_0}}\int_{-2^{1-k}\varepsilon_0}^{-2^{k}\varepsilon_0} e^{-2\pi i\ell\omega/2^{-k}\varepsilon_0}\frac{e^{i\omega t}}{\sqrt{2\pi}}d\omega\ , \tag{3.92}$$

must be included in order to form a complete set. These completeness relations imply that these wavelets as well as their Fourier transforms form bases for the vector space of square integrable functions $L^2(-\infty,\infty)$.

These six wavelet properties are summarized geometrically in terms of their phase space representatives. The set of o.n. wavelets induces a partitioning of phase space into cells of equal area

$$\Delta\omega\Delta t = 2\pi/2^{-k}\varepsilon_0\ 2^{-k}\varepsilon_0 = 2\pi\ ,$$

but unequal shape, Eq.(3.89)-(3.90). The orthogonality in the time and in the frequency domains implies that the areas of these cells should be pictured as nonoverlapping. The completeness relations imply that these cells cover the whole phase space without any gaps between them. In brief, the phase space is partitioned by the wavelets into *mutally exclusive and jointly exhaustive* cells of equal area, but different shapes as in Figure 3.22. This is different from Figure 3.18, which depicts the partitioning by the o.n. wave packets into cells. They also are mutually exclusive and jointly exhaustive and have equal area. But they have identical shape.

The variable ε_0 is a positive parameter which effects all wavelets at once. It therefore controls the way they partition phase space. What happens when one increases ε_0? Reference to propert 4. indicates that such an increase produces a global distortion which dilates all phase space cells along the vertical (frequency) direction while compressing them along the horizontal (time) direction.

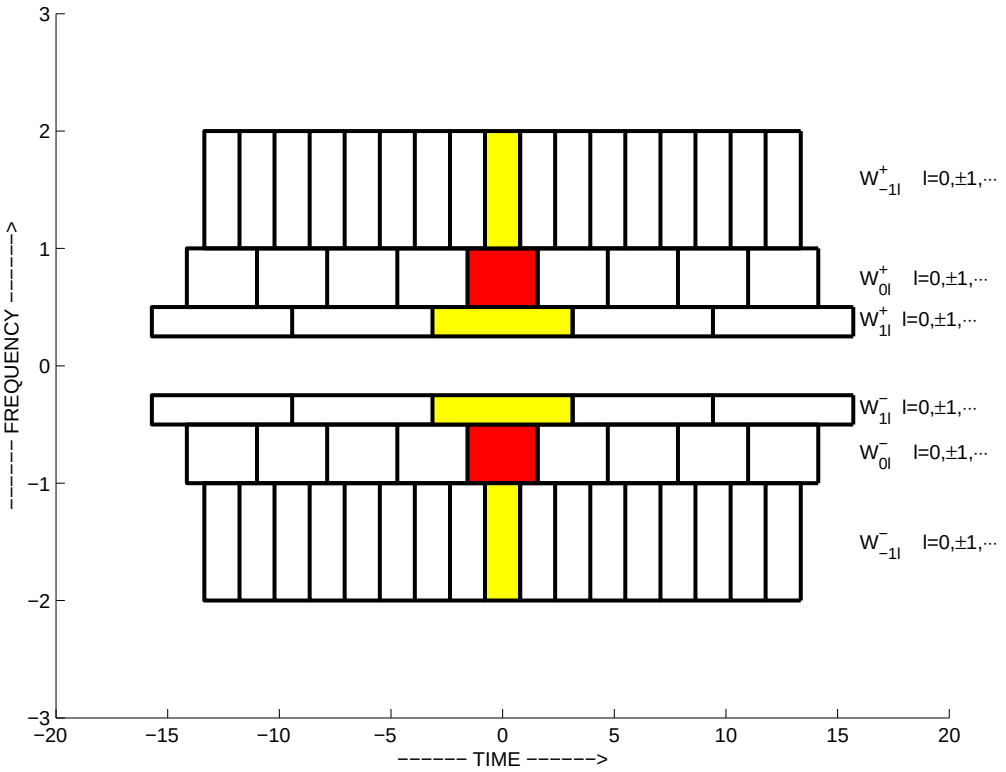


Figure 3.22: Partitioning of phase space by o.n. wavelets into cells of *equal* area ($\triangle t\ \triangle\omega = 2\pi$), but *unequal* shape ($\Delta t = 2\pi/2^{-k}\varepsilon_0,\ \Delta\omega = 2^{-k}\varepsilon_0$). For a given mean frequency the successive wavelets have equal width.

The distortion corresponds to that suffered by an incompressible fluid.

Once the parameter has doubled in value, the new partitioning is congruent to the old one. However, the integer octave label k gets shifted by one unit in the process: $k \to k+1$. More precicely, reference to Eq.(3.87) shows that one has

$$W^{\pm}_{k\ell}(t)\big|_{\varepsilon_0=\varepsilon_1} = W^{\pm}_{k+1\,\ell}(t)\big|_{\varepsilon_0=2\varepsilon_1}\ . \tag{3.93}$$

The set of o.n. wavelets is characterized by a seventh fundamental property:

7. All wavelets are derivable from a *single* standard wave packet. By *translating* and *compressing* this standard "mother wavelet", as it is known informally, one recovers any one of the o.n. wavelets. This recovery holds separately for the positive and negative frequency wavelets

and follows directly from their defining equations, Eq.(3.87) and (3.92): By changing the Fourier integration variable from ω to $\Omega = \frac{\omega}{2^{-k}}$, one obtains, for the typical wavelet, the alternate integral expression

$$\begin{aligned} W^{\pm}_{k\ell}(t) &= \sqrt{\frac{1}{2^{-k}\varepsilon_0}}(\pm)\int_{\pm 2^{-k}\varepsilon_0}^{\pm 2^{1-k}\varepsilon_0} e^{-2\pi i\ell\omega/2^{-k}\varepsilon_0}\frac{e^{i\omega t}}{\sqrt{2\pi}}d\omega \\ &= \sqrt{\frac{2^{-k}}{\varepsilon_0}}(\pm)\int_{\pm\varepsilon_0}^{\pm 2\varepsilon_0} e^{i\Omega(2^{-k}t-2\pi\ell/\varepsilon_0)}\frac{d\Omega}{\sqrt{2\pi}} \\ &\equiv \sqrt{2^{-k}}\psi^{\pm}(2^{-k}t-\frac{2\pi}{\varepsilon_0}\ell) \quad , \quad k,\ell = 0,\pm 1,\pm 2,\cdots \end{aligned} \tag{3.94}$$

The simplicity of this expression is striking. To obtain it, all one needs to do is apply a *translation*, *compression* and *amplification* to a *single* universal wave packet. Start with the *mother wavelet* (standard wave packet function),

$$\begin{aligned} W^{\pm}_{00}(t) \equiv \psi^{\pm}(t) &= \frac{1}{\sqrt{2\pi\varepsilon_0}}(\pm)\int_{\pm\varepsilon_0}^{\pm 2\varepsilon_0} e^{i\Omega t}d\Omega \\ &= \sqrt{\frac{\varepsilon_0}{2\pi}}e^{\pm 3it\varepsilon_0/2}\ \frac{\sin t\varepsilon_0/2}{t\varepsilon_0/2} \quad , \end{aligned}$$

and translate it along the t-axis by an amount $\frac{2\pi}{\varepsilon_0}\ell$ to obtain

$$\psi^{\pm}(t-\frac{2\pi}{\varepsilon_0}\ell)\ .$$

Next compress it uniformly along the t-axis by the compression factor 2^{-k} to obtain the compressed wave packet

$$\psi^{\pm}(2^{-k}t-\frac{2\pi}{\varepsilon_0}\ell)\ .$$

To preserve normalization amplify its amplitude by $\sqrt{2^{-k}}$ to obtain

$$\sqrt{2^{-k}}\psi^{\pm}(2^{-k}t-\frac{2\pi}{\varepsilon_0}\ell) \quad .$$

This three step process is sufficient to yields the generic wavelet, Eq.(3.94).

Note that the resulting set of orthonormal wavelets decomposes into different classes. Those wavelets belonging to the same class (k fixed) have the same mean frequency and the same temporal width, but are time translated relative to each other ($\ell = 0,\pm 1,\pm 2,\cdots$). By contrast, different classes are distinguished by different mean frequencies and hence different widths.

3.6 Multiresolution Analysis

A choice of basis for a vector space is a choice of the standard by which vectors are measured. Once chosen, this standard remains fixed. However, a question remains: Which basis does one pick? And, is the choice arbitrary or is there a principle that guides this choice?

It turns out that the latter is the case. This is illustrated by the following example involving two different bases for the space of square integrable functions.

3.6.1 Chirped Signals and the Principle of Unit-Economy

Consider chirped audio signals. Their frequency is a monotonic function of time. There are signals characterized by a down-chirp, like that of a bat using its sonar echolocation ability to track its prey. There are also signals

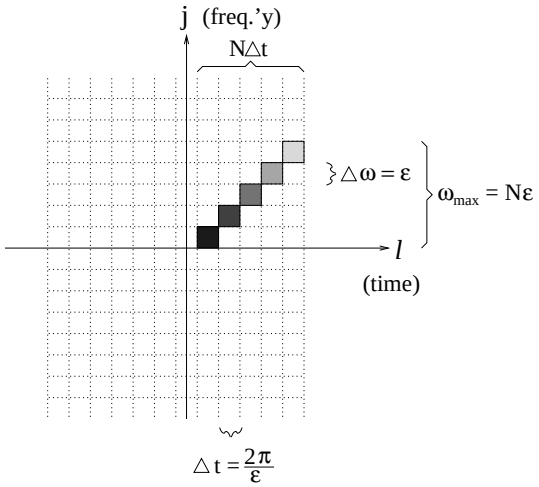


Figure 3.23: Phase space representation of a chirped signal occupying N phase space cells. The grey level of each cell expresses the intensity of the corresponding wave packet.

characterized by an up-chirp. The phase space representation of a typical example is depicted by the shaded phase space cells in Figure 3.23. An up-chirp signal starts at low frequency and stops at some maximum frequency, say

$$\omega_{max} = N\varepsilon \ .$$

Here N is the number of phase space cells which the signal occupies. For illustrative purposes consider a signal with a linear up-chirp,

$$f(t) = \sum_{n=1}^{N} \alpha_n P_{nn}^{\varepsilon}(t) \ . \tag{3.95}$$

This representation is based on the by-now-familiar orthonormal wave packet functions Eq.(3.81),

$$P_{jl}^{\varepsilon}(t) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{-2\pi i l\omega/\varepsilon} e^{i\omega t} d\omega \ .$$

Recall that the constant ε, which characterizes the shape

$$\begin{aligned} \Delta\omega &= \varepsilon \\ \Delta t &= \frac{2\pi}{\varepsilon} \end{aligned}$$

of each phase space cell, is a parameter which identifies this family of wave packets, $\{P_{jl}^{\varepsilon}(t) : \ j, \ell = 0, \pm 1, \cdots\}$.

However, there are also other families characterized by other parameter values. Consider another set of wave packets whose phase space cells have dimension

$$\begin{aligned} \Delta\omega &= 2\varepsilon \\ \Delta t &= \frac{2\pi}{2\varepsilon} \end{aligned}$$

These basis functions are obtained from $P_{jl}^{\varepsilon}(t)$ by making the replacement $\varepsilon \to 2\varepsilon$. This amplifies and compresses the wave packets in the time domain. Indeed, the defining integral expression for $P_{jl}^{2\varepsilon}(t)$ yields

$$P_{jl}^{2\varepsilon}(t) = \sqrt{2} P_{jl}^{\varepsilon}(2t) \ .$$

What is the representation of the given chirp signal with respect to this new basis? The answer is based on the transformation formula

$$P_{j'l'}^{\varepsilon}(t) = \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} P_{jl}^{2\varepsilon}(t) \, \langle P_{jl}^{2\varepsilon}, P_{j'l'}^{\varepsilon} \rangle \ .$$

It is worth while to do the calculation explicitly because the answer turns out to be fairly simple and informative. The simplicity starts to become evident when one splits the chirped signal into odd and even labelled terms. Assuming without loss of generality that the number of terms is even, $N = 2M$, one has

$$f(t) = \sum_{m=0}^{M-1} \alpha_{2m+1} P^{\varepsilon}_{(2m+1)\ (2m+1)}(t) + \sum_{m=1}^{M} \alpha_{2m} P^{\varepsilon}_{(2m)\ (2m)}(t)\ . \tag{3.96}$$

The odd and even transformation formulas are

$$P^{\varepsilon}_{(2m+1)\ \ell}(t) = \frac{1}{\sqrt{2}} P^{2\varepsilon}_{m\ 2\ell}(t) + \frac{1}{\pi\sqrt{2}} \sum_{k=0}^{\infty} \frac{2i}{2\ell - 2k - 1} P^{2\varepsilon}_{m\ (2k+1)}(t)$$

and

$$P^{\varepsilon}_{2m\ \ell}(t) = \frac{1}{\sqrt{2}} P^{2\varepsilon}_{m\ 2\ell}(t) - \frac{1}{\pi\sqrt{2}} \sum_{k=0}^{\infty} \frac{2i}{2\ell - 2k - 1} P^{2\varepsilon}_{m\ (2k+1)}(t)\ .$$

Consequently, the chirped signal is given by

$$\begin{aligned} f(t) = & \\ & \sum_{m=0}^{M-1} \left[\alpha_{2m+1} \frac{1}{\sqrt{2}} P^{2\varepsilon}_{m\ (4m+2)}(t) + \frac{i}{\pi\sqrt{2}} \sum_{k-\infty}^{\infty} \frac{2\alpha_{2m+1}}{2(2m+1) - 2k - 1} P^{2\varepsilon}_{m\ (2k+1)}(t) \right] \\ + & \sum_{m=1}^{M} \left[\alpha_{2m} \frac{1}{\sqrt{2}} P^{2\varepsilon}_{m\ 4m}(t) \qquad - \frac{i}{\pi\sqrt{2}} \sum_{k=-\infty}^{\infty} \frac{2\alpha_{2m}}{2(2m) - 2k - 1} P^{2\varepsilon}_{m\ (2k+1)}(t) \right]\ . \end{aligned} \tag{3.97}$$

Compare the two representations, Eq.(3.96) and Eq.(3.97), of the chirped signal. In Eq.(3.96) $f(t)$ is represented by a set of $N = 2M$ basis vectors (in physics and engineering also known as "degrees of freedom"),

$$\{P^{\varepsilon}_{nn}(t) :\ n = 1, \cdots, 2M\}. \tag{3.98}$$

In Eq.(3.97) by contrast, $f(t)$ is represented by the basis

$$\begin{aligned} & \{P^{2\varepsilon}_{m\ (4m+2)}(t) :\ m = 0, \cdots, M-1\} \cup \{P^{2\varepsilon}_{m\ (4m)}(t) :\ m = 1, \cdots, M\} \cup \\ & \{P^{2\varepsilon}_{m\ (2k+1)}(t) :\ m = 0, \cdots, M;\ k = 0, \pm 1, \pm 2, \cdots\}\ , \end{aligned} \tag{3.99}$$

which has a substantially larger number of elements.

Is the choice of basis vectors arbitrary? The principle of *unit-economy*[7] applied to this example demands that one pick the wave packet basis Eq.(3.98), whose coefficients in Eq.(3.95) express the essential properties of *the* degrees of freedom of the chirped signal. One should not pick the other basis, Eq.(3.99), whose amplitudes in Eq.(3.97) are *nonessential* because specifying them might lead to signals which are not chirped at all.

The mathematical implementation of the principle of unit-economy to signal processing consists of the requirement that one pick an *optimal* basis to represent the set of signals under consideration. This means that one pick a subspace of *minimal* dimension in order to accomodate these signals.

---

[7]The *principle of unit-economy* [1, 12], also known informally as the "crow epistemology", is the principle that stipulates the formation of a new concept

1. when the description of a set of elements becomes too complex,

2. when the elements of the set are used repeatedly, and

3. when the elements of the set require further study.

It is obvious that the last is the most important because that is the nature of cognition, pushing back the frontier of knowledge.

The principle of unit economy is implemented by a process of conceptualization, which is a method of expanding man's consciousness by reducing the number of its content's units – a systematic means to an unlimited integration of cognitive data.

The principle of unit-economy *forbids* the formation of a new concept if that formation is based on some nonessential property.

The principle of unit-economy is a statement not only about the structure of mathematics, but also more generally about why one forms concepts in the first place, be they first-order concepts based on perceptual data ("percepts"), or be they higher-level concepts based on already-formed concepts.

The principle of unit-economy is a guiding principle that leads us from an unlimited number of specific units (i.e. members of a class, in our example, signals characterized by a chirp) to a single new concept (in mathematics also known as an "equivalence class", in our example, the concept "chirped signal"). By repeatedly applying this principle to percepts, as well as to the product of such applications, one can reduce a vast amount of information to a minimal number of units. These one's consciousness can readily keep in the forefront of one's mind, digest them, assimilate them, manipulate them, and use them without any danger of information overload.

3.6.2 Irregular Signals and Variable Resolution Analysis

Is it possible to extend the optimal choice of a basis to signals which are much more irregular than those which are accomodated by a wave packet basis?

Consider the signals accomodated by a seismograph. Two of the most prominent signals are sudden bursts, such as explosions initiated for the purpose of locating petroleum reserves, or precursors to a vulcanic eruption, or earth quakes. Then there is the second type of signals, those which characterize the resonant vibrational or wave motions initiated by such bursts. It is obvious that the second type is most efficiently analyzed using Fourier

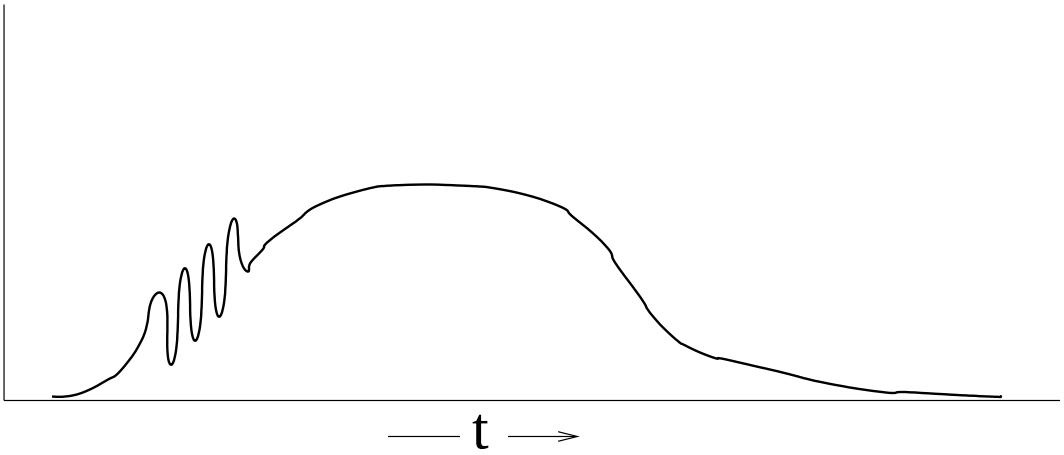


Figure 3.24: Amplitude profile of a structured pulse

or wave packet basis functions. However, a burst-like signal is characterized by variations localized in time. The signal has a finite time duration. It also has a starting edge with a finite temporal thickness which often contains rapid variations ("high frequency structure") as exemplified in Figure 3.24. Thus under low resolution one would simply measure the amplitude profile of the main body of a pulse of finite duration. But under higher resolution one would also measure the high frequency structure which in Figure 3.24 announces the beginning of that pulse.

Given such a signal, how does one represent it in the most efficient way i.e. in compliance with the priciple of unit-economy?

3.6.3 Multiresolution Analysis as Hierarchical

The answer is provided by multiresolution analysis (MRA). It provides a simple hiearchical framework for identifying the properties of a signal, i.e. for taking note of its existence, for measuring it, for representing it, and for even reproducing it. A key aspect of this framework is its Hierarchical structure.

At different resolutions the details of a signal generally characterize different physical aspects of an event. At coarse resolution these details correspond to the larger overall aspects, the *context* of an event. At fine resolution these details correspond to its distinguishing features. Such a course-to-fine strategy is typical in identifying an event.

The mathematical formulation of this resolution hiearchy is developed in four steps.

Central Approximation Space

First of all, construct a *central approximation space* $\mathbf{V}_0$, which is a subspace of the space of square-integrable functions $L^2(-\infty, \infty)$, which (a) is spanned by a translation-generated (a.k.a. "Riesz") basis

$$\mathbf{V}_0 = span\{\phi(t-l) : \ \ell = 0, \pm 1, \cdots\}$$

and (b) is orthonormal:

$$\int_{-\infty}^{\infty} \overline{\phi}(t-k)\phi(t-\ell)\, dt = \delta_{k\ell} \ . \tag{3.100}$$

The existence of such a basis is equivalent to the statement that $\mathbf{V}_0$ is closed under integral shifts of its elements, i.e.

$$f(t) \in \mathbf{V}_0 \Rightarrow f(t-\ell) \in \mathbf{V}_0 \text{ whenever } \ell = integer \ .$$

The function $\phi(t)$, known as a *scaling function* (a.k.a. "father wavelet") can be any square integrable function as long as it satisfies the integer-shifted orthonormality condition, Eq.(3.100).

A particular example of such a basis is the set of wave packets $\{Q^{\varepsilon}_{0\ell}(t)\}$, Eq.(3.86) on page 182:

$$\begin{aligned} Q^{\varepsilon}_{0\ell}(t) &= \sqrt{\frac{1}{2\pi\varepsilon}} \int_{-\varepsilon/2}^{\varepsilon/2} e^{-2\pi i\ell\omega/\varepsilon}\ e^{i\omega t} d\omega \\ &\equiv \phi\left(t - \frac{2\pi\ell}{\varepsilon}\right) \end{aligned} \tag{3.101}$$

For this basis the scaling function is obtained by setting $\varepsilon = 2\pi$ and letting $\ell = 0$:

$$\phi(t) = \frac{\sin \pi t}{\pi t} \ . \tag{3.102}$$

This scaling function happens to be one whose Fourier transform has compact support and is piecewise constant:

$$\hat{\phi}(\omega) = \begin{cases} \sqrt{\dfrac{1}{2\pi}} & \omega \in [-\pi, \pi] \\ 0 & \text{otherwise} \end{cases} \ .$$

The central approximation space $\mathbf{V}_0$ is spanned by the orthonormal basis

$$\phi(t-\ell) = \frac{\sin \pi(t-\ell)}{\pi(t-\ell)} \quad \ell = 0, \pm 1, \cdots \ . \tag{3.103}$$

It is the vector space of "band-limited" functions, i.e. functions whose Fourier transforms have compact support on the frequency interval $[-\pi, \pi]$. The basis for this space is generated by Eq.(3.102) and it is called the *Shannon basis*.

Translation Followed by Compression

Second, rescale the given domain of the integer-shifted basis elements $\phi(t-\ell)$. This rescaling yields a different basis for a different, but related, vector space. For the Shannon basis this is achieved by again using Eq.(3.101), but by first setting

$$\varepsilon = 2^{-k}\varepsilon_0 \quad k = integer$$

before letting $\varepsilon_0 = 2\pi$. The result is

$$\sqrt{2^{-k}}\, \phi(2^{-k}t - \ell) = \sqrt{2^{-k}}\, \frac{\sin \pi(2^{-k}t - \ell)}{\pi(2^{-k}t - \ell)} \ . \tag{3.104}$$

For each integer k these functions form an orthonormal basis for the space of those band-limited functions, whose Fourier domain is restricted to the frequency band $[-\pi 2^{-k}, \pi 2^{-k}]$. The orthonormality is guaranted by the fact that these functions are derived from the set of orthonormal wave packets $P_{0\ell}(t)$. The vector space

$$span \left\{ \sqrt{2^{-k}}\, \phi(2^{-k}t - \ell) : \ \ell = 0, \pm 1, \cdots \right\} \equiv \mathbf{V}_k \tag{3.105}$$

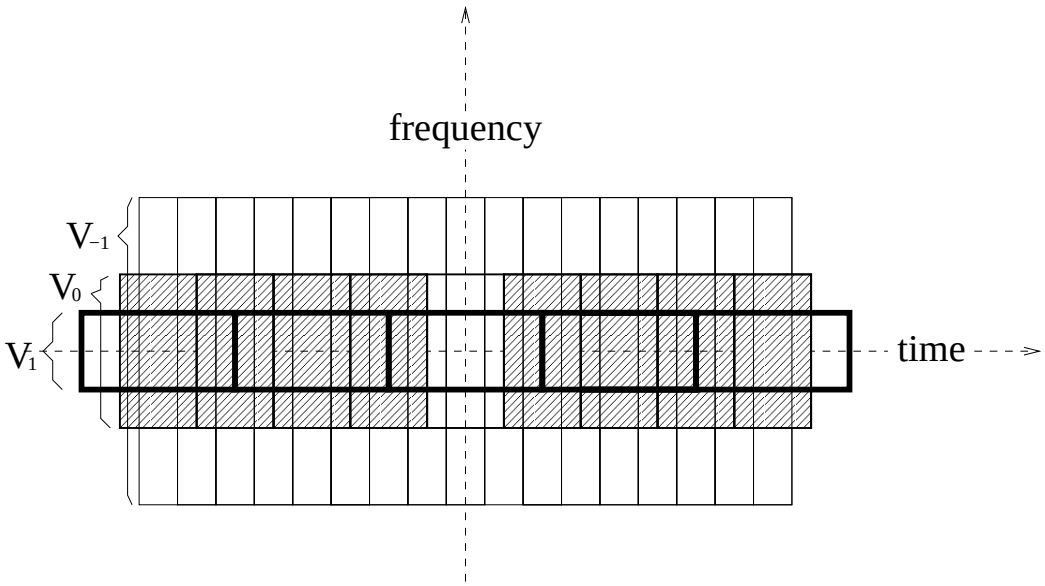


Figure 3.25: Partitioning of phase space by a collection hierarchical sets of band limited orthonormal basis functions. The heavy-lined rectangles are the phase space cells of low resolution wave packets; they span the $k = 1$st resolution vector space $\mathbf{V}_1$. The shaded rectangles are those of the next (i.e. more refined) resolution wave packets; they span the $k = 0$th resolution vector space $\mathbf{V}_0$. The thin and tall unshaded rectangles are those of the wave packets of still higher resolution. They span the $k = -1$st resolution vector space $\mathbf{V}_{-1}$. The unshaded rectangle in the middle is the phase space cell of the "father wavelet", the scaling function in Eq.(3.102). It yields (by compression and translation) all basis functions for all the vector spaces $\mathbf{V}_k$.

is called the kth *resolution space*. For fixed k these basis elements form the kth *resolution Shannon basis*, more simply the kth *Shannon basis*. They have the common phase space shape

$$\begin{aligned} \Delta t &= 2^k \\ \Delta\omega &= \frac{2\pi}{2^k} \\ \Delta t \Delta\omega &= 2\pi \ . \end{aligned}$$

These shapes are illustrated in Figure 3.25 for the vector spaces $\mathbf{V}_k$, $k = -1, 0, 1$. Relative to the phase space cells of $\mathbf{V}_0$, $k > 0$ implies that the phase

space cells get dilated in the time domain and compressed in the frequency domain in order to comply with $\Delta t \Delta\omega = 2\pi$.

Also note that increasing k designates increasing roughness, i.e, *lower* resolution. Thus increasing resolutions are labelled by *decreasing* integers. This labelling, which at first sight is backward, highlights the fact that the low resolution features of a signal are generally more significant than those of high resolution.

Resolution Spaces as Hierarchical

Third, take note of the hierarchical subspace structure of the resolution spaces $\mathbf{V}_k$, $k = \cdots, -1, 0, 1, \cdots$. The Fourier transform of the basis elements, Eq.(3.104), for $\mathbf{V}_k$ have compact support confined to $[-\pi 2^{-k}, \pi 2^{-k}]$. As was shown in Part (c) of Ex. 1.5.3 on page 103, these basis elements form a complete set. This means that $f \in \mathbf{V}_k$ if and only if its Fourier transform has support confined to $[-\pi 2^{-k}, \pi 2^{-k}]$. Next consider the vector space $\mathbf{V}_{k+1}$. The Fourier transform of its basis elements have support confined to

$$\left[-\pi 2^{-(k+1)}, \pi 2^{-(k+1)}\right] \subset \left[-\pi 2^{-k}, \pi 2^{-k}\right]$$

In fact, every element of $\mathbf{V}_{k+1}$ enjoys this property. This implies that such elements also belong to $\mathbf{V}_k$. Thus one has

$$f \in \mathbf{V}_{k+1} \Rightarrow f \in \mathbf{V}_k \ .$$

In other words, $\mathbf{V}_{k+1}$ is a subspace of $\mathbf{V}_k$:

$$\mathbf{V}_{k+1} \subset \mathbf{V}_k \ .$$

More explicitly, this inclusion property says that

$$\{0\} \subset \cdots \subset \mathbf{V}_2 \subset \mathbf{V}_1 \subset \mathbf{V}_0 \subset \mathbf{V}_{-1} \subset \mathbf{V}_{-2} \subset \cdots \subset L^2(-\infty, \infty) \ .$$

Such a hierarchy of increasing subspaces is called a *multiscale analysis* of the space of square-integrable functions. A multiscale analysis is always derived from (i.e. based on) a *scaling function* ϕ. In our illustrative example this scaling function ("father function") is Shannon's sinc function, Eq.(3.102).

Resolution Analysis as a Decomposition into Subspaces

Fourth, decompose each resolution subspace $\mathbf{V}_k$ into its subsequent resolution subspace $\mathbf{V}_{k+1}$ and its corresponding orthogonal complement, the subspace of details $\mathbf{O}_{k+1}$:

$$\mathbf{V}_k = \mathbf{V}_{k+1} \oplus \mathbf{O}_{k+1} \tag{3.106}$$

Given the fact that $\mathbf{V}_k$ is spanned by the kth resolution basis, Eq.(3.105), the meaning of such a decomposition consists of exhibiting an alternative o.n. basis part of whose elements span $\mathbf{V}_{k+1}$, while the remainder spans its orthogonal complement. This decomposition is achieved as follows:
Recall that every square integrable function $f(t)$ can be approximated as an optimal element in $\mathbf{V}_k$. This optimal approximation, which with Bessel's Theorem on page 78 in Section 2.5.2 was identified as the least squares approximation of $f(t)$ in the subspace $\mathbf{V}_k$, is uniquely expressed in terms of any orthonormal basis. Following Eq.(2.12), and using the o.n. basis, Eq.(3.104), one has the projection of $f(t)$ onto $\mathbf{V}_k$:

$$P_{\mathbf{V}_k} f(t) = \sum_{\ell=-\infty}^{\infty} 2^{-k} \phi(2^{-k}t - \ell) \, \langle \phi(2^{-k}u - \ell), f(u) \rangle \tag{3.107}$$

This is the least squares approximation of $f(t)$ based on the subspace $\mathbf{V}_k$, or more briefly the $\mathbf{V}_k$-*least squares approximation*. The next (less refined) approximation is the projection of $f(t)$ onto the subspace $\mathbf{V}_{k+1} \subset \mathbf{V}_k$:

$$P_{\mathbf{V}_{k+1}} f(t) = \sum_{\ell=-\infty}^{\infty} 2^{-(k+1)} \phi(2^{-(k+1)}t - \ell) \, \langle \phi(2^{-(k+1)}u - \ell), f(u) \rangle \ . \tag{3.108}$$

Here $P_{\mathbf{V}_k}$ and $P_{\mathbf{V}_{k+1}}$ are the projection operators onto $\mathbf{V}_k$ and $\mathbf{V}_{k+1}$ respectively.

Let us compare the two o.n. bases for the two resolution spaces $\mathbf{V}_k$ and $\mathbf{V}_{k+1}$. We shall presently see that they are two families of o.n. wave packets identified already on page 182 by Eq.(3.86):

$$Q^{\varepsilon}_{0\ell}(t) = \begin{cases} Q^{2\varepsilon'}_{0\ell}(t) & \varepsilon = 2\varepsilon' \\ \\ Q^{\varepsilon'}_{0\ell}(t) & \varepsilon = \varepsilon' \end{cases} \ .$$

Here nad throughout the ensuing development we always let

$$\varepsilon' = 2^{-k}, \quad k = 0, \pm 1, \pm 2, \cdots \ .$$

Using Eqs.(3.104) and (3.102) one finds that the $\mathbf{V}_k$-family members are

$$\begin{aligned}\mathbf{V}_k:\ \sqrt{2^{-k}}\,\phi(2^{-k}t-\ell) &= \sqrt{2^{-k}}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(2^{-k}t-\ell)}d\omega \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\varepsilon'}}\int_{-\varepsilon'}^{\varepsilon'} e^{-2\pi i\ell\omega/2\varepsilon'}e^{i\omega t}d\omega \\ &= Q_{0\ell}^{2\varepsilon'}(t)\ , \end{aligned} \tag{3.109}$$

and

$$\begin{aligned}\Delta t &= \frac{2\pi}{2\varepsilon'} = 2^k \\ \Delta\omega &= 2\varepsilon' = \frac{2\pi}{2^k}\ .\end{aligned}$$

By contrast, the $\mathbf{V}_{k+1}$-family members, which are twice as wide in the temporal domain and twice as narrow in the frequency domain, are

$$\begin{aligned}\mathbf{V}_{k+1}:\ \sqrt{2^{-(k+1)}}\,\phi(2^{-(k+1)}t-\ell) &= \sqrt{2^{-(k+1)}}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(t2^{-(k+1)}-\ell)}d\omega \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\varepsilon'}}\int_{-\varepsilon'/2}^{\varepsilon'/2} e^{-2\pi i\ell\omega/\varepsilon'}e^{i\omega t}d\omega \\ &= Q_{0\ell}^{\varepsilon'}(t) \end{aligned} \tag{3.110}$$

and

$$\begin{aligned}\Delta t &= \frac{2\pi}{\varepsilon'} = 2\times 2^k \\ \Delta\omega &= \varepsilon' = \frac{1}{2}\times\frac{2\pi}{2^k}\ .\end{aligned}$$

These two bases are represented by two overlapping arrays of phase space cells, as in Figure 3.26. The phase space cells referring to the $\mathbf{V}_k$-basis are taller and skinnier than those referring to $\mathbf{V}_{k+1}$. Furthermore, the phase space domain of the $\mathbf{V}_{k+1}$-basis is a horizontal strip which is contained entirely within that of the $\mathbf{V}_k$-basis. Consequently, the phase space domain of the $\mathbf{V}_k$-basis gets partitioned into three mutually exclusive and jointly exhaustive domains:

- the negative "band pass" frequency strip $-\varepsilon' < \omega < -\dfrac{\varepsilon'}{2}$,

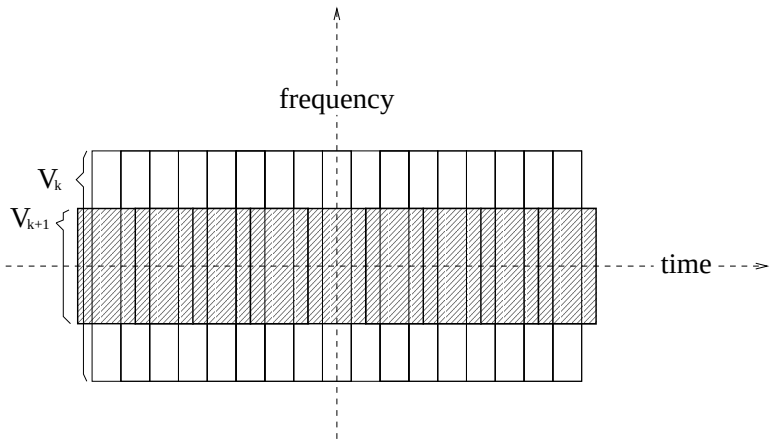


Figure 3.26: Partitioning of phase space by a nested set of band limited orthonormal basis functions. The tall thin rectangles and the shaded rectangles are the phasespace cells of the basis functions which span $\mathbf{V}_k$ and $\mathbf{V}_{k+1}$ respectively. These two sets of phase space cells are reproduced respectively on the l.h.s. and r.h.s. of Figure 3.27.

- the strip $-\frac{\varepsilon'}{2} < \omega < \frac{\varepsilon'}{2}$ generated by the $\mathbf{V}_{k+1}$-basis, and
- The positive "band pass" frequency strip $\frac{\varepsilon'}{2} < \omega < \varepsilon'$.

The mutual exclusivity of these three strips, together with the fact that their union equals the strip generated by the $\mathbf{V}_k$-basis, implies that $\mathbf{V}_k$ is spanned by *two* fundamental bases. Besides the one given by Eq.(3.109),

$$\mathbf{V}_k = span\{Q_{0\ell}^{2\varepsilon'}(t) : \ell = 0, \pm 1, \cdots\} ,$$

there also is

$$\mathbf{V}_k = span\ \{Q_{0\ell}^{\varepsilon'}(t) : \ell = 0, \pm 1, \cdots\} \cup$$

$$\{P_{1\ell}^{\varepsilon'/2}(t) : \ell = 0, \pm 1, \cdots\} \cup \{P_{-2\,\ell}^{\varepsilon'/2}(t) : \ell = 0, \pm 1, \cdots\} .$$

As one can see from Figure 3.27, this corresponds to the union of the three strips mentioned above. Here the P's are the familiar o.n. wave packets

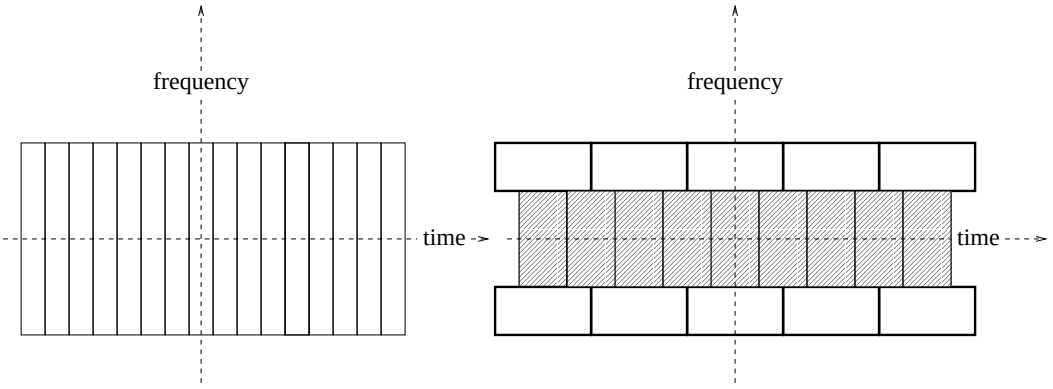


Figure 3.27: Two alternative partitionings of the same phase space domain of $\mathbf{V}_k$. The three different horizontal strips in the right hand partitioning refer to the three orthogonal subspaces $\mathbf{O}_{k+1}^{+}$, $\mathbf{V}_{k+1}$, and $\mathbf{O}_{k+1}^{-}$.

defined by Eq.(3.81):

$$P_{1\ell}^{\varepsilon'/2}(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\varepsilon'/2}} \int_{\varepsilon'/2}^{\varepsilon'} e^{-2\pi i \ell \omega/(\varepsilon'/2)} e^{i\omega t} d\omega \tag{3.111}$$

$$P_{-2\,\ell}^{\varepsilon'/2}(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\varepsilon'/2}} \int_{-\varepsilon'}^{-\varepsilon'/2} e^{-2\pi i \ell \omega/(\varepsilon'/2)} e^{i\omega t} d\omega \ , \tag{3.112}$$

and

$$\begin{aligned} \Delta t &= \frac{2\pi}{\varepsilon'/2} = 2 \times 2^{(k+1)} \\ \Delta\omega &= \varepsilon'/2 = \frac{1}{2} \times \frac{2\pi}{2^{(k+1)}} \end{aligned}$$

for both the positive and negative frequency wave packets. Due to the mutual orthogonality of all the P's and Q's combined, every band limited function $f \in \mathbf{V}_k$ is a unique linear combination of these elements. Thus we have identified *two* alternative bases of $\mathbf{V}_k$. The first one consists of the elements exhibited by Eq.(3.109). The second one consists of the elements exhibited by Eqs.(3.110)-(3.112).

This fact is reexpressed by the statement that $\mathbf{V}_k$ is the direct sum of the subspaces

$$\mathbf{O}^+_{k+1} \equiv span\{P^{\varepsilon'/2}_{1\ell}(t) : \ell = 0, \pm 1, \cdots\} \tag{3.113}$$

$$\mathbf{V}_{k+1} \equiv span\{Q^{\varepsilon'}_{0\ell}(t) : \ell = 0, \pm 1, \cdots\} \tag{3.114}$$

$$\mathbf{O}^-_{k+1} \equiv span\{P^{\varepsilon'/2}_{-2\,\ell}(t) : \ell = 0, \pm 1, \cdots\} \ , \tag{3.115}$$

or, symbolically, that

$$\mathbf{V}_k = \mathbf{V}_{k+1} \oplus \mathbf{O}^+_{k+1} \oplus \mathbf{O}^-_{k+1} \ ,$$

which is the same as Eq.(3.106), provided one sets

$$\mathbf{O}_{k+1} = \mathbf{O}^+_{k+1} \oplus \mathbf{O}^-_{k+1} \ ,$$

the direct sum of the positive and negative frequency subspaces othogonal to the $(k+1)$st resolution space $\mathbf{V}_{k+1}$.

3.6.4 Unit-Economy via the Two Parent Wavelets

It is quite evident that, by itself, the introduction of the translation-generated basis elements (the P's and the Q's) constitutes a proliferation of concepts: their sheer number prevents them from being automatically accessible for further study; one's mind run's the danger of being subjected to information overload. Such a state of affairs motivates an inquiry as to the applicability of the principle of unit-economy[8]. Can one, by introducing simplifying concepts, reduce this number by consolidating these P's and the Q's into one or two concepts?

An affirmative answer to this question is based on the introduction of two "mother wavelet" for all the P's and a "father wavelet" for all the Q's.

Recall that the Q's have already been consolidated by Eqs.(3.101), (3.102), and (3.110) into the single scaling function, the "father wavelet"

$$\phi(t) = \frac{\sin \pi t}{\pi t} \ . \tag{3.116}$$

Thus by applying to this wavelet a translation, a compression, and an amplification, one obtains

$$\sqrt{2^{-(k+1)}}\, \phi(2^{-(k+1)}t - \ell) = Q^{\varepsilon'}_{0\ell}(t) \quad \text{with } \varepsilon' = 2\pi\, 2^{-(k+1)} \ .$$

[8] As identified in the footnote on Page 192.

In other words,

$$\mathbf{V}_{k+1} = span\{\sqrt{2^{-(k+1)}}\,\phi(2^{-(k+1)}t-\ell):\ \ell=0,\pm1,\pm2,\cdots\}\ .$$

The successful application of the principle of unit-economy to the basis of $\mathbf{V}_{k+1}$ can be extended to the bases of $\mathbf{O}^{\pm}_{k+1}$ in an analogous manner. First of all, one observes that the basis elements of $\mathbf{O}^{+}_{k+1}$, Eq.(3.111), are precisely the positive frequency wavelets $W^{+}_{k+1\,\ell}(t)$, Eq.(3.87) with $k\to k+1$, provided one sets $\varepsilon_0=\pi$:

$$P^{\varepsilon'/2}_{1\ell}(t) = W^{+}_{k+1\,\ell}(t)\mid_{\varepsilon_0=\pi} \overset{1}{=} W^{+}_{k+2\,\ell}(t)\mid_{\varepsilon_0=2\pi} \overset{2}{=} \sqrt{2^{-(k+2)}}\,\psi^{+}(2^{-(k+2)}t-\ell)\ . \tag{3.117}$$

Similarly one finds that the basis elements of $\mathbf{O}^{-}_{k+1}$, Eq.(3.112), are precisely the negative frequency wavelets $W^{-}_{k+1\,\ell}(t)$, Eq.(3.92) with $k\to k+1$,

$$P^{\varepsilon'/2}_{-2\,\ell}(t) = W^{-}_{k+1\,\ell}(t)\mid_{\varepsilon_0=\pi} \overset{1}{=} W^{-}_{k+2\,\ell}(t)\mid_{\varepsilon_0=2\pi} \overset{2}{=} \sqrt{2^{-(k+2)}}\,\psi^{-}(2^{-(k+2)}t-\ell)\ . \tag{3.118}$$

Equality 1 follows from Eq.(3.93), while 2 follows from Eq.(3.94), and the positive (resp. negative) frequency mother wavelets ψ^{+} (resp. ψ^{-}) are given by

$$\psi^{\pm}(t) = e^{\pm 3i\pi t}\ \frac{\sin\pi t}{\pi t}. \tag{3.119}$$

They are merely complex conjugates of each other. Substitute Eqs.(3.117) and (3.118) into Eqs.(3.113) and (3.115). The result,

$$\begin{aligned}\mathbf{O}^{+}_{k+1} &= span\{\sqrt{2^{-(k+2)}}\,\psi^{+}(2^{-(k+2)}t-\ell):\ \ell=0,\pm1,\pm2,\cdots\}\\ \mathbf{O}^{-}_{k+1} &= span\{\sqrt{2^{-(k+2)}}\,\psi^{-}(2^{-(k+2)}t-\ell):\ \ell=0,\pm1,\pm2,\cdots\}\ ,\end{aligned}$$

highlights the fact that the two orthogonal subspaces $\mathbf{O}^{+}_{k+1}$ and $\mathbf{O}^{-}_{k+1}$ are spanned by basis vectors which are generated by the positive frequency mother wavelet and its complex conjugate respectively.

Thus one has the result that, for every integer k, each of the wavelets ψ^{+},ψ^{-} and ϕ procreates its respective vector space $\mathbf{O}^{+}_{k+1}$, $\mathbf{O}^{-}_{k+1}$, and $\mathbf{V}_{k+1}$. The application of this fact to their direct sum

$$\mathbf{V}_k = \mathbf{V}_{k+1}\oplus\mathbf{O}^{+}_{k+1}\oplus\mathbf{O}^{-}_{k+1}\ , \tag{3.120}$$

is as follows: Let f be any square-integrable function, and let $P_{\mathbf{V}_k} f$, Eq.(3.107), be its $\mathbf{V}_k$-least squares approximation. Then Eq.(3.120) expresses the fact that $P_{\mathbf{V}_k} f$ decomposes uniquely into three parts,

$$P_{\mathbf{V}_k} f = P_{\mathbf{V}_{k+1}} f + P_{\mathbf{O}^+_{k+1}} f + P_{\mathbf{O}^-_{k+1}} f \ . \tag{3.121}$$

They are

$$\begin{aligned} P_{\mathbf{V}_{k+1}} f(t) &= \sum_\ell 2^{-(k+1)} \phi(2^{-(k+1)}t - \ell) \langle \phi(2^{-(k+1)}u - \ell), f(u) \rangle \qquad (3.122) \\ P_{\mathbf{O}^+_{k+1}} f(t) &= \sum_\ell 2^{-(k+2)} \psi^+(2^{-(k+2)}t - \ell) \langle \psi^+(2^{-(k+1)}u - \ell), f(u) \rangle \\ P_{\mathbf{O}^-_{k+1}} f(t) &= \sum_\ell 2^{-(k+2)} \psi^-(2^{-(k+2)}t - \ell) \langle \psi^-(2^{-(k+1)}u - \ell), f(u) \rangle \ . \end{aligned}$$

They crystalize, *within the context of resolution* 2^{-k},

- the *essential* degrees of freedom of the space of square-integrable functions (a.k.a. signals) and
- the *detail* degrees of freedom relative to the next (lower) resolution $2^{-(k+1)}$.

The representations of f at resolutions 2^{-k} and $2^{-(k+1)}$ are given by

$$P_{\mathbf{V}_k} f(t) = \sum_\ell 2^{-k} \phi(2^{-k}t - \ell) \langle \phi(2^{-k}u - \ell), f(u) \rangle \tag{3.123}$$

and Eq.(3.122) respectively. The basis elements

$$\phi(2^{-k}t - \ell) \quad \ell = 0, \pm 1, \cdots$$

and

$$\phi(2^{-(k+1)}t - \ell) \quad \ell = 0, \pm 1, \cdots$$

are the *essential* degrees of freedom of f within the context of resolutions 2^{-k} and $2^{-(k+1)}$ respectively. The associated coefficients $\langle \cdots , \cdots \rangle$ are the corresponding amplitudes. We say that these degrees of freedom are *independent* – there is no redundancy – because the basis elements for each resolution space $\mathbf{V}_k$ are mutually orthogonal.

When one compares a function f at resolutions 2^{-k} and $2^{-(k+1)}$, then the difference

$$P_{\mathbf{V}_k} f - P_{\mathbf{V}_{k+1}} f = P_{\mathbf{O}^+_{k+1}} f + P_{\mathbf{O}^-_{k+1}} f \ ,$$

is called the *detail* of f relative to the next resolution $2^{-(k+1)}$. The basis elements

$$\psi(2^{-(k+1)}t - \ell) \quad \ell = 0, \pm 1, \cdots$$

are the corresponding *detail degrees of freedom*. They are independent of the essential degrees of freedom at resolution $2^{-(k+1)}$ but not so at resolution 2^{-k}. These detail degrees of freedom span the vector space

$$\mathbf{O}_{k+1} = \mathbf{O}^+_{k+1} \oplus \mathbf{O}^-_{k+1} \ ,$$

which is the orthogonal complement of $\mathbf{V}_{k+1}$ in $\mathbf{V}_k$. The relation between these subspaces is depicted in Figure 3.28 below.

$$\begin{array}{ccccccccc} \cdots & \mathbf{V}_{k-1} & \rightarrow & \mathbf{V}_k & \rightarrow & \mathbf{V}_{k+1} & \cdots \\ & & \searrow & \oplus & \searrow & \oplus & \\ & & & \mathbf{O}_k & & \mathbf{O}_{k+1} & \end{array}$$

Figure 3.28: Hierarchical relation between the resolution subspaces and their orthogonal detail subspaces. The arrows are orthogonal projections onto the subspaces.

It is difficult to overstate the importance of the principle of unit-economy. Its application is implicit and is taken for granted through out any theoretical development, ours in particular. However, there are situations where it is instructive to highlight particularly significant instances of its application. A case in point is the introduction of the scaling function, the father wavelet $\phi(t)$. By this process an entire aggregate of concepts has been condensed into a single new concept, a multiscale analysis (MSA), with a scaling function $\phi(t)$ residing at its core. The economy in the number of concepts achieved by this condensation is a tribute to this principle. It demands that any new concept be defined in terms of essential properties.

The gist of the last two pages consisted of the task of establishing the two alternative bases of the resolution space, Eq.(3.120), in terms of a single

scaling function, Eq.(3.116), and the two "mother wavelets", Eq.(3.119). Furthermore, the development was based on a scaling function having a rather specialized form, the Shannon wavelet $\sin \pi t/\pi t$. One therefore wonders whether the benefits to be gained from such a highly specialized activity are really worth the effort expended. That the answer is "yes" is due to the fact that the development identifies a wider principle constructively: *For every MSA there is a scaling function $\phi(t)$, and for every scaling function there exists a MSA*. The assumed specialized form, Eq.(3.119), is non-essential. The identification, MSA $\leftrightarrow$ $\phi(t)$, is captured by means of the following definition:

Definition (Multiscale Resolution Analysis)

An increasing sequence of Hilbert spaces

$$\{\mathbf{V}_k : \{0\} \subset \cdots \subset \mathbf{V}_2 \subset \mathbf{V}_1 \subset \mathbf{V}_0 \subset \mathbf{V}_{-1} \subset \mathbf{V}_{-2} \subset \cdots \subset L^2(-\infty, \infty)\} , \tag{3.124}$$

is said to be a multiscale analysis *of the space of square integrable functions $L^2(-\infty, \infty)$ if*

1. *(the Cauchy completion of) their union is that space of square integrable functions:*

$$\overline{\bigcup_{k=-\infty}^{\infty} \mathbf{V}_k} = L^2(-\infty, \infty) , \tag{3.125}$$

2. *their intersection is the zero function:*

$$\bigcap_{k=-\infty}^{\infty} \mathbf{V}_k = \{0\} , \tag{3.126}$$

3. *every resolution space $\mathbf{V}_k$ is related to a central (i.e. reference) space $\mathbf{V}_0$ by a dilation of its elements:*

$$f(t) \in \mathbf{V}_k \Longleftrightarrow f(2^k t) \in \mathbf{V}_0 , \tag{3.127}$$

4. *there exists a square-integrable function ϕ such that its integer translates form an orthonormal basis for the central approximation space $\mathbf{V}_0$:*

$$\mathbf{V}_0 = span\{\phi(t - \ell) : \ \ell = 0, \pm 1, \cdots\} \tag{3.128}$$

with

$$\int_{-\infty}^{\infty} \overline{\phi}(u-\ell)\phi(u-\ell')du = \delta_{\ell\ell'} \ . \tag{3.129}$$

Thus a multiscale analysis (MSA) is a type of hierarchy having properties 1-4. There are many other hierarchies that have only properties 1-2. But only MSA's are characterized by property 3.

This property is the essential (distinguishing) characteristic of a MSA. It says that, in order for a hierarchy of linear spaces to be a MSA, each one of these spaces must be a scaled version of a reference space $\mathbf{V}_0$, the *central approximation space*. By starting with a function in $\mathbf{V}_0$, and applying iteratively scaling operations, compression ($\times 2$) or dilation ($\times 2^{-1}$), to its argument, one moves up or down this hierarchy of *approximation* spaces.

The purpose of Property 4 is not to define what a MSA is. Instead, its role is to have the scaling function $\phi(t)$ serve as a unique identifier of the central approximation space $\mathbf{V}_0$, and hence, by Property 3, of a particular MSA. Thus Property 4 establishes a unique correspondence between the set of MSA's and the set of scaling functions.

The unique identification of $\mathbf{V}_0$ is achieved by having the discrete translates of $\phi(t)$ form an orthonormal basis of $\mathbf{V}_0$. That translation process is depicted in Figure 3.17 on page 174. The ablity of $\phi(t)$ to serve as a unique identifier for the whole MSA becomes evident when one applies Property 3 to this functions. One finds that the set of translated and scaled functions

$$\{\sqrt{2^{-k}}\,\phi(2^{-k}t-\ell) : \ l = 0, \pm 1, \cdots\}$$

form o.n. bases for the respective approximation spaces $\mathbf{V}_k$, and hence form a basis for the whole MSA. This means that every MSA is distinguished from every other MSA by means of its scaling function $\phi(t)$.

Consequently, the definition of a MSA by properties 1-4 not only defines what a MSA is, but also establishes a one-to-one correspondence between the set of MSA's and the set of scaling functions.

The translates of the scaling function need not be orthonormal. In that case the orthonormality condition, Eq.(3.129), gets replaced by the condition that $\{\phi(t-\ell)\}$ form a *Riesz* basis, i.e. that

$$A \sum_{\ell=-\infty}^{\infty} |c_\ell|^2 \leq \| \sum_{\ell=-\infty}^{\infty} c_\ell \phi(t-\ell)\|^2 \leq B \sum_{\ell=-\infty}^{\infty} |c_\ell|^2 \ .$$

Here $A, B > 0$ are positive constants, $\{c_\ell\}$ is a square-summable sequence, and $\|\cdots\|$ is the L^2-norm. In that circumstance there exists a theorem due to Mallat which guarantees that the Riesz basis can be orthonormalized by an appropriate renormalization procedure in the Fourier domain of $\phi(t)$.

Exercise 3.6.1 (IMPROVED FIDELITY BY AUGMENTATION)
SHOW that

$$\overline{\bigcup_{k=-\infty}^{\infty} \mathbf{V}_k} = L^2(-\infty, \infty) \Longleftrightarrow \lim_{k\to-\infty} \|P_{\mathbf{V}_k} f - f\| = 0 \ ,$$

where $P_{\mathbf{V}_k}$ is the orthogonal projection onto $\mathbf{V}_k$ and $\|\cdots\|$ is the L^2-norm.

Exercise 3.6.2 (LOSS OF FIDELITY BY CURTAILMENT)
SHOW that

$$\bigcap_{k=-\infty}^{\infty} \mathbf{V}_k = \{0\} \Longleftrightarrow \lim_{k\to\infty} \|P_{\mathbf{V}_k} f\| = 0 \ .$$

Exercise 3.6.3 (TRANSLATION INVARIANT FUNCTION SPACES)
(a) SHOW that $\mathbf{V}_0$ is *discrete translation invariant*, i.e. that

$$f(t) \in \mathbf{V}_0 \Longleftrightarrow f(t-\ell) \in \mathbf{V}_0 \quad \text{where } \ell \text{ is an integer.}$$

(b) SHOW that $\mathbf{V}_k$ is 2^k-shift invariant, in particular that

$$f(t) \in \mathbf{V}_k \Longleftrightarrow f(t - 2^k\ell) \in \mathbf{V}_k \ .$$

3.6.5 Multiscale Analysis as a Method of Measurement

Multiscale analysis introduces a breakthrough in the measurement of signals. It quantifies not only the location of characterisic features within a given signal (see Figure 3.24 on page 193), but, like a telescope with variable and calibrated zoom, it also quantifies their amplitudes in an optimally efficient way. Measuring rods capable of this dual capability are depicted schematically in Figure 3.29 on page 211.

Such an application of a multiscale analysis to any given signal always requires two steps:

1. specifying the scaling function, the standard of measurement and
2. measuring, and hence representing, the signal relative to the basis elements generated from that scaling function.

Let us consign the task of specifying the scaling function to the next subsection. Thus we assume that a choice of a scaling function has been made, and we endeavor to measure the given signal, say $f(t)$. This means that we find the coefficients which represent the kth approximation of f, i.e. the least squares approximation of f in the subspace $\mathbf{V}_k$, Eq.(3.123). This array of coefficients, the array of inner products

$$[P_{\mathbf{V}_k} f] = \{\langle \sqrt{2^{-k}}\phi(2^{-k}u - \ell), f(u)\rangle : \ \ell = 0, \pm 1, \cdots\} \ ,$$

is called the *discrete approximation of f at resolution* 2^{-k}, and it constitutes the result of the measuring process. It consists of the inner products

$$\begin{aligned}\langle \phi(2^{-k}u - \ell), f(u)\rangle &= \int_{-\infty}^{\infty} \overline{\phi}\left(-2^{-k}(2^k\ell - u)\right) f(u)du \\ &= \left[f(u) * \overline{\phi}(-2^{-k}u)\right](2^k\ell) \ ,\end{aligned}$$

which is the convolution integral evaluated at the equally spaced points $2^k\ell$. These values of the convolution integral are the output resulting from the signal $f(t)$ being passed through the filter $\overline{\phi}(-2^{-k}u)$. This is because in the Fourier domain the convolution integral is the product of two Fourier transforms. Thus one finds that the discrete approximation consists of the set of sampled values of the given signal after it has passed through a filter which is expressed by the Fourier integral

$$\int_{-\infty}^{\infty} \overline{\phi}(-2^{-k}u)\frac{e^{-i\omega u}}{\sqrt{2\pi}}du \ .$$

3.6.6 Multiscale Analysis vs Multiresolution Analysis: MSA or MRA?

The two names "multiscale analysis" (=MSA) and "multiresolution analysis"(=MRA) refer to the same concept. Both are characterized by the discrete set of powers of the number 2,

$$2^k, \quad k = 0, \pm 1, \pm 2, \cdots \ ,$$

and the corresponding set of orthonormal basis functions

$$\{\sqrt{2^{-k}}\,\phi(2^{-k}t-\ell):\ \ell=0,\pm1,\cdots\},\ k=0,\pm1,\pm2,\cdots\ .$$

The difference is that MRA and MSA highlight different aspects of the same thing. As k increases, the scale increases but the resolution decreases. This is like stepping away from a picture.

Consider the array of functions

$$\phi(2^{-k}t-\ell)=\phi\left(\frac{t-2^k\ell}{2^k}\right),\quad \ell=\cdots,-2,-1,0,1,2,\cdots\ .$$

This array is a set of identical localized graphs, each one displaced by the amount 2^k from its nearest neighbor. Thus each of these graphs serves as marker on the real line and 2^k is the distance between successive makers. In brief, the real line equipped with this set of markers constitutes a new kind of measuring rod for measuring signals. The integer k specifies the nature – the resolution – of this measuring rod. Every integral increase in k increases (decreases) the scale for performing these mesurements, and hence decreases (increases) the resolution of the measuring rod. Figure 3.29 depicts several such measuring rods.

3.6.7 The Pyramid Algorithm

The MSA measuring process starts with the acquisition of a signal as an element in the central ("fiducial", "reference") vector space $\mathbf{V}_0$. This means that a signal

$$f(t)=\sum_{\ell=-\infty}^{\infty}c_\ell^0\phi(t-\ell)\in\mathbf{V}_0$$

is acquired in the form of a square summable sequence of numbers

$$\{c_\ell^0\}=\{\langle\phi(u-\ell),f(u)\rangle\},\quad \ell=0,\pm1,\cdots$$

The problem is to determine the representation of the signal in each of the subsequent ("lower resolution") approximation spaces $\mathbf{V}_0\supset\mathbf{V}_1\supset\mathbf{V}_2\supset\cdots$, i.e. find

$$\{c_\ell^1\},\{c_\ell^2\},\cdots$$

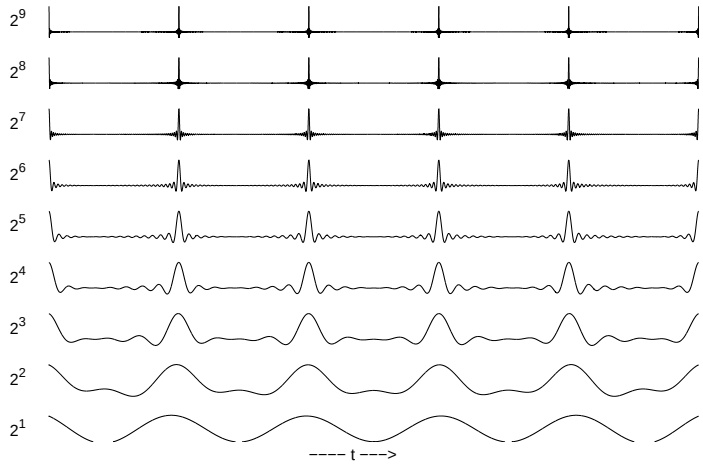


Figure 3.29: Nine different measuring rods. Each is graduated with its own set of markers, i.e. shifted scaling functions $\phi(2^{-k}(t-m))$, ranging from a set of very high resolution ($2^9 : k = -9$) markers, through a set of medium resolution ($2^5 : k = -5$), to the set of lowest resolution ($2^1 : k = -1$) markers. A high resolution measuring rod accomodates additional high resolution markers, which are, however, not depicted in this figure. The markers of each rod are uniformly spaced, as they must. The novelty of these rods is that each marker has the mathematically precise internal structure of a wavepacket. This novelty permits one to measure not only the locations of specific features in a given signal but also their amplitudes.

such that

$$\sum_{\ell=-\infty}^{\infty} c_\ell^k \phi(t-\ell) = P_{\mathbf{V}_k} f(t)$$

is the least squares projection of $f(t)$ onto $\mathbf{V}_k$ for $k = 1, 2, \cdots$. This turns out to be an iterative process which terminates after a finite number of steps.

The key observation which makes this process so powerful and appealing is that the relationship between two adjacent resolution spaces, say $\mathbf{V}_k$ and $\mathbf{V}_{k+1}$, is independent of the order k.

Given the fact that $\{\mathbf{V}_k : k = 0, \pm 1, \cdots\}$ is a MSA and that $\phi(t)$ is the corresponding scaling function, we know that

- for fixed integer k

$$\{\sqrt{2^{-k}}\,\phi(2^{-k}t-\ell):\ l=0,\pm1,\cdots\}$$

is an o.n. basis for $\mathbf{V}_k$, and that

- each of the basis elements $\sqrt{2^{-(k+1)}}\,\phi(2^{-(k+1)}t-\ell')$ for $\mathbf{V}_{k+1}$ also belongs to $\mathbf{V}_k$.

Consequently, each such basis element can be expanded uniquely in terms of the $\mathbf{V}_k$-basis

$$\phi(2^{-(k+1)}t-\ell')=\sum_{\ell=-\infty}^{\infty}\phi(2^{-k}t-\ell)\;2^{-k}\langle\phi(2^{-k}u-\ell),\phi(2^{-(k+1)}u-\ell')\rangle \quad (3.130)$$

By changing variables in the inner product integral one finds that

$$\begin{aligned}2^{-k}\langle\phi(2^{-k}u-\ell),\phi(2^{-(k+1)}u-\ell')\rangle &\equiv 2^{-k}\int_{-\infty}^{\infty}\overline{\phi}(2^{-k}u-\ell)\phi(2^{-(k+1)}u-\ell')du\\ &=\int_{-\infty}^{\infty}\overline{\phi}(u-\ell)\phi(2^{-1}u-\ell')du\\ &=\int_{-\infty}^{\infty}\overline{\phi}\,(u-(\ell-2\ell'))\,\phi(2^{-1}u)du\\ &\equiv\sqrt{2}\;h_{\ell-2\ell'}\end{aligned}$$

Exercise 3.6.4 (ORTHONORMALITY)
a) Point out why this inner product is the (ℓ,ℓ')th entry of the $\sqrt{2}$-multiple of a unitary matrix, which is independent of k.

b) Show that $\sum_{\ell=-\infty}^{\infty}\overline{h}_\ell h_{\ell-2\ell'}=\delta_{0\ell'}$.

When one computes the (complex) inner product of f with both sides of Eq.(3.130), one obtains

$$\underbrace{\langle\sqrt{2^{-(k+1)}}\phi(2^{-(k+1)}u-\ell'),f(u)\rangle}_{c_{\ell'}^{k+1}} = \sum_{\ell=-\infty}^{\infty}\langle\sqrt{2^{-(k+1)}}\phi(2^{-(k+1)}u-\ell'),\sqrt{2^{-k}}\phi(2^{-k}u-\ell)\rangle$$
$$\times\,\langle\sqrt{2^{-k}}\phi(2^{-k}u-\ell'),f(u)\rangle$$
$$= \sum_{\ell=-\infty}^{\infty}\underbrace{\langle\sqrt{2^{-k}}\phi(2^{-k}u-\ell),f(u)\rangle}_{c_{\ell}^{k}}\;\overline{h}_{\ell-2\ell'}$$

Thus, by setting $\overline{h}_{\ell-2\ell'}\equiv\tilde{h}_{2\ell'-\ell}$ one has

$$\boxed{c_{\ell'}^{k+1}=\sum_{\ell=-\infty}^{\infty}\tilde{h}_{2\ell'-\ell}\;c_{\ell}^{k}} \tag{3.131}$$

The sum on the r.h.s. is the discrete convolution of $\{\tilde{h}_\ell\}$ and $\{c_\ell^k\}$. It shows that the discrete approximation

$$\{c_\ell^{k+1}\}=\{\langle\sqrt{2^{-(k+1)}}\phi(2^{-(k+1)}u-\ell),f(u)\rangle:\;\ell=0,\pm1,\cdots\}$$

of f can be computed from $\{c_\ell^k\}$ by convolving it with $\{\tilde{h}_\ell\}$, and then keeping only every other sample from the result. Thus, if one starts out with a discrete approximation $\{c_\ell^k\}$ which represents f by means of a finite number of samples, then the next discrete approximation is represented by only half as many samples. After a sufficient number of such iterative steps the process stops because one has run out of samples. All successive discrete approximations of f are merely sequences of zeros. This iterative algorithm is known as the *pyramid algorithm* first introduced by Stephane Mallat [10]. It is a rather efficient algorithm because it terminates after only

$$\log_2(\#\text{ of sampled values of }f)\;.$$

iterations.

3.6.8 The Requirement of Commensurability

The problem of specifying a scaling function is the problem of choosing an appropriate standard.

A standard has to be *commensurable* with the things being measured. In fact, it has to be an element of the set of things being measured. Thus, if one wishes to measure the shape of functions which are, say, piecewise constant, then the scaling function should have the same property. If the functions to be measured have compact support, then the scaling function should also have compact support. If the functions to be measured have continuous derivatives, then the scaling function better have that property also. Thus the requirement of commensurability dictates the choice of an appropriate standard – an appropriate scaling function.

The Scaling Function as a MSA Identifier

Once a scaling function has been chosen and constructed, the corresponding MSA is uniquely determined. However, not all square-integrable functions qualify as scaling functions. In fact, to qualify, the definition of a MSA on page 206 implies that a scaling function must satisfy two key properties. They are (i) Eq.(3.129),

$$\int_{-\infty}^{\infty} \overline{\phi}(u-\ell')\phi(u-\ell'')du = \delta_{\ell'\ell''} \tag{3.132}$$

and (ii) Eq.(3.130) on page 212, or equivalently with $k=-1$

$$\boxed{\phi(t) = \sqrt{2}\sum_{-\infty}^{\infty} h_\ell \phi(2t-\ell)\ .} \tag{3.133}$$

where

$$h_\ell = \langle \phi(2t-\ell), \phi(u-\ell)\rangle\ .$$

The boxed is known as the *scaling equation* for the scaling function $\phi(t)$. Both Eqs.(3.132) and (3.133) put strong restrictions on the collection of square-integrable functions. The first constitutes a discrete infinitude of constraints and is equivalent to

$$\boxed{\int_{-\infty}^{\infty} \overline{\phi}(u)\phi(u-\ell)du = \delta_{0\ell}\ .}$$

The second is a statement about the dilation operator D,

$$\begin{aligned} L^2 &\xrightarrow{D} L^2 \\ f(t) &\rightsquigarrow Df(t) = \sqrt{2^{-1}} f(2^{-1}t) \ . \end{aligned}$$

That second constraint, Eq.(3.133), demands that, even though D changes the shape of the graph of f, the resulting function still lies in the subspace

$$\mathbf{V}_0 = span\{\phi(t-\ell) : \ \ell = 0, \pm 1, \cdots\}$$

spanned by the discretely tranlated functions $\phi(t-\ell)$. In other words, the function $\phi(t)$ is such that the subspace $\mathbf{V}_0$ generated from this function is invariant under D.

Both boxed equations put severe restrictions on the collection of square-integrable functions, but these restrictions are not strong enough to single out a unique function. Instead, they narrow the field of candidates to those L^2-functions which qualify as scaling functions for MSAs. The nature of these restrictions becomes more transparent if one expresses them in the Fourier domain, instead of the given domain. Thus by introducing the Fourier transform of $\phi(t)$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \phi(t)\, dt = \hat{\phi}(\omega) \ ,$$

one finds for the first equation that

$$\begin{aligned} \delta_{0\ell} &= \int_{-\infty}^{\infty} \overline{\phi}(t)\phi(t-\ell) dt \\ &= \int_{-\infty}^{\infty} \overline{\hat{\phi}(\omega)} \hat{\phi}(\omega) e^{i\omega\ell} d\omega \\ &= \int_{0}^{2\pi} \sum_{n=-\infty}^{\infty} \left| \hat{\phi}(\omega + 2\pi n \right|^2 e^{i\omega\ell} d\omega \end{aligned}$$

This equation says that the sum of the squared magnitude is a function all of whose Fourier coefficients vanish – except for the one corresponding to $\ell = 0$.

Such a function is a constant, namely

$$\boxed{\sum_{n=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi n)\right|^2 = \frac{1}{2\pi} \ .} \tag{3.134}$$

This condition on the Fourier transform of ϕ is Mallat's necessary and suffient condition for the discrete translates of $\phi(t)$ to form an o.n. basis for $\mathbf{V}_0$.

In the Fourier domain the second equation, the scaling equation (3.133), has a simple form. Taking the Fourier transform of the equivalent equation

$$\phi\left(\frac{t}{2}\right) = \sqrt{2}\sum_{-\infty}^{\infty} h_\ell \phi(t-\ell) \ ,$$

one finds

$$\boxed{\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega) \ ,} \tag{3.135}$$

where

$$H(\omega) = \frac{\sqrt{2}}{2}\sum_{\ell=-\infty}^{\infty} h_\ell e^{i\omega\ell}$$

is a 2π-periodic function of ω:

$$H(\omega + 2\pi) = H(\omega) \ .$$

Equation (3.135) is a linear equation. It expresses in the Fourier domain the relation between the input $\hat{\phi}(\omega)$ and the output $\hat{\phi}(2\omega)$ of a time-invariant linear system. In the theory of such systems the function $H(\omega)$ is therefore known as a *filtering function* or *filter* in brief. Its periodicity is an important but fairly mild restriction on $H(\omega)$. That condition can be strengthened considerably by incorporating the Fourier normalization condition, Eq.(3.134) into Eq.(3.135). One does this by inserting Eq.(3.135) into Eq.(3.134). The result is that $H(\omega)$ satisfy the additional normalization condition

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1 \ . \tag{3.136}$$

Thus the scaling equation is a linear equation, and to qualify as a scaling function, its Fourier transform must satisfy a simple linear equation, Eq.(3.135), having a normalized periodic coefficient. The nature of a particular scaling function, and hence the nature of the corresponding MSA, is controlled by the nature of that normalized coefficient function 2π-periodic on the Fourier domain.

Exercise 3.6.5 (FUNCTIONAL CONSTRAINT ON THE FILTER FUNCTION)
Verify the validity of the functional constraint, Eq.(3.136).

Exercise 3.6.6 (THE SCALING EQUATION SOLVED)
Consider a function $\phi(t)$ having the property

$$\left| \int_{-\infty}^{\infty} \phi(t)dt \right| \neq 0 \ .$$

Find the solution to the scaling equation, Eq.(3.135).

Answer: $\hat{\phi}(\omega) = \hat{\phi}(0) \prod_{k=1}^{\infty} H\left(\frac{\omega}{2^k}\right)$

Exercise 3.6.7 (TWO SOLUTIONS TO THE SCALING EQUATION)
Let $\phi^+(t)$ be a solution to the scaling equation

$$\phi(t) = \sqrt{2} \sum_{-\infty}^{\infty} h_\ell \phi(2t - \ell) \ .$$

a) Point out why

$$\hat{\phi}^-(\omega) = \begin{cases} \hat{\phi}^+(\omega) & \omega \geq 0 \\ -\hat{\phi}^+(\omega) & \omega < 0 \end{cases}$$

is the Fourier transform of a second independent solution to the above scaling equation.
b) Show that $\phi^+(t)$ and $\phi^-(t)$ are orthogonal,

$$\int_{-\infty}^{\infty} \overline{\phi}^+(t)\phi^-(t)dt = 0 \ ,$$

whenever (i) $\phi(t)$ is a real function or whenever (ii) its Fourier transform is an even function of ω.

3.6.9 Wavelet Analysis

The task of identifying the properties of an acquired signal starts with its given representation as an element in the reference (i.e. fiducial, central)

representation space $\mathbf{V}_0$. One singles out the large overall features by projecting it onto the next subspace $\mathbf{V}_1$. This projection process suppresses the finer details of the signal. They are no longer present when the signal is represented as an element of $\mathbf{V}_1$. Using the pyramid algorithm one repeats this process iteratively. In this process one moves from the resolution 2^{-k} of $\mathbf{V}_k$ to the lower resolution $2^{-(k+1)}$ of $\mathbf{V}_{k+1} \subset \mathbf{V}_k$.

To keep track of the finer details suppressed by this process, one introduces $\mathbf{O}_{k+1}$, the orthogonal complement of $\mathbf{V}_{k+1}$ in $\mathbf{V}_k$:

$$\mathbf{V}_{k+1} \perp \mathbf{O}_{k+1} \ .$$

Thus any signal f represented in $\mathbf{V}_k$ is the unique sum of the signal represented in $\mathbf{V}_{k+1}$ and its suppressed detail which lies in $\mathbf{O}_{k+1}$:

$$P_{\mathbf{V}_k} f = P_{\mathbf{V}_{k+1}} f + P_{\mathbf{O}_{k+1}} f \ .$$

In brief,

$$\mathbf{V}_k = \mathbf{V}_{k+1} \oplus \mathbf{O}_{k+1} \ .$$

The o.n. bases for $\mathbf{V}_k$ and $\mathbf{V}_{k+1}$, and hence the representations of the signal f in these spaces, are known and expressed in terms of the scaling function $\phi(t)$. These bases determine a unique basis for $\mathbf{O}_k$ whose purpose is to keep track of the suppressed $P_{\mathbf{O}_{k+1}} f$ of the signal f. The process of constructing the $\mathbf{O}_{k+1}$-basis resembles that for $\mathbf{V}_k$ and $\mathbf{V}_{k+1}$. One starts with a square-integrable function $\psi(t)$, the "mother wavelet". By applying translations and dilations to it, one obtains the desired o.n. basis for $\mathbf{O}_{k+1}$, the space of details at resolution $2^{-(k+1)}$. The crucial part of this endeavor is the construction of the mother wavelet from the scaling function of the MSA. The construction is done by means of the following theorem by Mallat:

Theorem 3.6.1
(Wavelet generation theorem)

1. *Let*

$$\cdots \supset \mathbf{V}_k \supset \mathbf{V}_{k+1} \supset \cdots$$

be the hierarchy of vector spaces which make up the MSA whose scaling function is $\phi(t)$ and whose corresponding pyramid algorithm is based on the filtering function

$$H(\omega) = \frac{\sqrt{2}}{2} \sum_{\ell=-\infty}^{\infty} h_\ell e^{i\omega\ell}$$

2. *Let $\psi(t)$ be a function whose Fourier transform is given by*

$$\hat{\psi}(\omega) = G\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)$$

where

$$G(\omega) = e^{i\omega}\overline{H(\omega+\pi)}\ ,$$

then

I.

$$\{\sqrt{2^{-k}}\psi(2^{-k}t-\ell):\ \ell=0,\pm1,\cdots\} \tag{3.137}$$

is an o.n. basis for $\mathbf{O}_k$ and

II.

$$\{\sqrt{2^{-k}}\psi(2^{-k}t-\ell):\ \ell,k=0,\pm1,\cdots\} \tag{3.138}$$

is an o.n. basis for $L^2(-\infty,\infty)$.

The validation of this theorem is a three step process.

1. First of all notice that the set of functions, Eq.(3.137), being orthogonal,

$$\begin{aligned}\delta_{\ell\ell'} &= \int_{-\infty}^{\infty}\sqrt{2^{-k}}\overline{\psi}(2^{-k}u-\ell)\sqrt{2^{-k}}\psi(2^{-k}u-\ell')du\\ &= \int_{-\infty}^{\infty}\overline{\psi}(u-\ell)\psi(u-\ell')du\ ,\end{aligned}$$

is equivalent to the statement that

$$\sum_{n=-\infty}^{\infty}\left|\hat{\psi}(\omega+2\pi n)\right|^2 = \frac{1}{2\pi}\quad -\infty<\omega<\infty\ . \tag{3.139}$$

The reasoning is identical to that leading to Eq.(3.134).

2. Secondly note that $\mathbf{V}_{k+1} \subset \mathbf{V}_k$, and hence

$$\mathbf{V}_k = \mathbf{V}_{k+1} \oplus \mathbf{O}_{k+1} \quad \text{where } \mathbf{V}_{k+1} \perp \mathbf{O}_{k+1} \ ,$$

with

$$\begin{aligned} \mathbf{V}_{k+1} &= span\{\sqrt{2^{-k}}\phi(2^{-k}t - \ell)\} \\ \mathbf{O}_{k+1} &= span\{\sqrt{2^{-k}}\psi(2^{-k}t - \ell)\} \ , \end{aligned}$$

implies that any basis element of $\mathbf{V}_{k+1}$ or of $\mathbf{O}_{k+1}$ is a linear combination of the basis elements of $\mathbf{V}_k$. Applying this fact to the case $k = -1$, one has

$$\begin{aligned} \phi\left(\frac{t}{2}\right) &= \sqrt{2}\sum_{-\infty}^{\infty} h_\ell \phi(t - \ell) \\ \psi\left(\frac{t}{2}\right) &= \sqrt{2}\sum_{-\infty}^{\infty} g_\ell \phi(t - \ell) \ . \end{aligned}$$

The corresponding Fourier transformed equations are

$$\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega) \quad \text{with } H(\omega) = \frac{\sqrt{2}}{2}\sum_{-\infty}^{\infty} h_\ell e^{i\omega\ell} \tag{3.140}$$

$$\hat{\psi}(2\omega) = G(\omega)\hat{\psi}(\omega) \quad \text{with } G(\omega) = \frac{\sqrt{2}}{2}\sum_{-\infty}^{\infty} g_\ell e^{i\omega\ell} \tag{3.141}$$

3. Thirdly note that the orthogonality condition, Eq.(3.139), when combined with Eq.(3.141), yields a normalization condition on $G(\omega)$ analogous to Eq.(3.136) on page 216,

$$|G(\omega)|^2 + |G(\omega + \pi)|^2 = 1 \ .$$

This is not the only constraint that G must satisfy. One must also take into account that $\mathbf{O}_{k+1}$ is the orthogonal complement of $\mathbf{V}_{k+1}$ in $\mathbf{V}_k$ This fact, which is expressed by

$$\int_{-\infty}^{\infty} \overline{\phi}(u - \ell)\psi(u - \ell')du = 0 \quad \text{for all integers } \ell \text{ and } \ell' \ ,$$

is equivalent to

$$\sum_{n=-\infty}^{\infty} \hat{\phi}(\omega + 2\pi n)\hat{\psi}(\omega + 2\pi n) = 0 \quad -\infty < \omega < \infty \tag{3.142}$$

Inserting Eqs.(3.140) and (3.141) into Eq.(3.142), using the fact that H and G are 2π-periodic,

$$H(\omega + 2\pi n) = H(\omega)$$
$$G(\omega + 2\pi n) = G(\omega)$$

and taking advantage of Eq.(3.134), one finds that the additional constraint on G is

$$\overline{H}\left(\frac{\omega}{2}\right) G\left(\frac{\omega}{2}\right) + \overline{H}\left(\frac{\omega}{2} + \pi\right) G\left(\frac{\omega}{2} + \pi\right) = 0 \ .$$

Thus the filter functions H and G satisfy

$$\overline{H}(\omega)G(\omega) + \overline{H}(\omega + \pi)G(\omega + \pi) = 0 \tag{3.143}$$

and

$$\overline{G}(\omega)G(\omega) + \overline{G}(\omega + \pi)G(\omega + \pi) = 1 \tag{3.144}$$
$$\overline{H}(\omega)H(\omega) + \overline{H}(\omega + \pi)H(\omega + \pi) = 1 \ . \tag{3.145}$$

A good way of remembering these constraints is that the matrix

$$\begin{bmatrix} H(\omega) & G(\omega) \\ H(\omega + \pi) & G(\omega + \pi) \end{bmatrix}$$

is unitary. These constraints are useful if for no other reasons than that they (i) place the two sequences of Fourier coefficients $\{h_\ell\}$ and $\{g_\ell\}$ on certain quadratic surfaces in the Hilbert space ℓ^2 and that they (ii) establish a tight relation between the h_ℓ's and the g_ℓ's. Indeed, from Eq.(3.143) one finds

$$G(\omega) = \overline{H}(\omega + \pi)e^{-i\omega} \ .$$

This relation is not unique. Other possibilities are

$$G(\omega) = \overline{H}(\omega + \pi)e^{-(2p+1)i\omega} \quad \text{where } p \text{ is an arbitrary integer} \ .$$

Each side of this equation is a Fourier series in powers of $e^{i\omega}$. Equating equal powers one finds

$$g_\ell = (-1)^\ell \; h_{2p+1-\ell} \; .$$

With both $G(\omega)$ and $H(\omega)$ at hand, one solves the two Eqs.(3.140) and (3.141). The solutions are

$$\hat{\phi}(\omega) = \hat{\phi}(0) \prod_{k=1}^{\infty} H\left(\frac{\omega}{2^k}\right)$$
$$\hat{\psi}(\omega) = \hat{\phi}(0) G(\omega) \prod_{k=1}^{\infty} H\left(\frac{\omega}{2^k}\right) \; .$$

The inverse Fourier transform of these solutions yields the sought after scaling function

$$\phi(t) = \hat{\phi}(0) \int_{-\infty}^{\infty} \prod_{k=1}^{\infty} H\left(\frac{\omega}{2^k}\right) \frac{e^{i\omega t}}{\sqrt{2\pi}} d\omega$$

and the mother wavelet

$$\psi(t) = \hat{\phi}(0) \int_{-\infty}^{\infty} G(\omega) \prod_{k=1}^{\infty} H\left(\frac{\omega}{2^k}\right) \frac{e^{i\omega t}}{\sqrt{2\pi}} d\omega \; .$$

Shifting and dilating this mother wavelet yields the o.n. basis functions, Eq.(3.137), for $\mathbf{O}_k$

Exercise 3.6.8 (ORTHOGONALITY OF THE DETAIL SPACES)
Validate conclusion # II. of the theorem on page 219, i.e. point out why, whenever $k \neq k'$, the functions in the space of details $\mathbf{O}_k$ are orthogonal to the functions in the space of details $\mathbf{O}_{k'}$.

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Chapter 4

Green's Function Theory

Lecture 26

We shall now direct our efforts towards finding what in linear algebra corresponds to the inverse of the linear operator $A - \lambda B$. This means that we are going to find a linear operator G which satisfies the equation

$$(A - \lambda B)G = I \quad . \tag{4.1}$$

Once we have found this inverse operator G, it is easy to solve the inhomogeneous problem

$$(A - \lambda B)\vec{x} = \vec{b} \tag{4.2}$$

for $\vec{x}$. This is so because the solution is simply

$$\vec{x} = G\vec{b} \quad . \tag{4.3}$$

If the vector space arena is an infinite-dimensional Hilbert space, the inverse operator

$$G_\lambda = (A - \lambda B)^{-1}$$

is usually called the *Green's function* of $A - \lambda B$, although in the context of integral equations the expression

$$G_\lambda = (A - \lambda I)^{-1}$$

is sometimes called the *resolvent* of A. Its singularities yield the eigenvalues of A, while integration in the complex λ-plane yields, as we shall see, the corresponding eigenvectors. It is therefore difficult to overstate the importance of the operator G_λ.

4.1 Cause and Effect Mathematized in Terms of Linear Operator and Its Adjoint

The matrix/operator G_λ is the fundamental bridge between the nature of the physical world and linear mathematics – more precisely, a particular instance of the connection between the law of causality ("The action of an entity follows from its nature." [1,2]) and its formulation in quantitative terms.

Being perfect, i.e. totally real, the physical world is worthy of the most painful study. A necessary and sufficient condition for success in this process consists of (i) seeking awareness of and identifying causal connections in the world and (ii) doing so in terms of the language of mathematics. Equations (4.1)-(4.2) and their consequence Eq.(4.3) comprise the prototypical mathematization[3] of both of these within the framework of linear mathematics. The vector $\vec{b}$ refers to a cause (e.g. force, source, signal, etc.) while $\vec{x}$ refers to the effect (e.g. movement, field, response, etc.). On the other hand, I refers to a type of standardized cause, while G refers to the corresponding effect. Once one has identified and mathematized this standard cause and effect connection, Eq. (4.3) yields with mathematical certainty the outcome of any linear cause and effect connection.

In other words, even though the study of the physical world might require considerable effort and perseverance, with linear systems such a study is comparatively straight forward. This is because, once one has found the solution to Eq.(4.1), solutions to Eq.(4.2) are easy to obtain by merely using Eq.(4.3).

The validity of this claim rests on the validity of three concepts, the first two of which we shall take for granted.

1. An *inner product*. This is the geometrical structure, a bi-linear scalar function, on the given inner product space $\mathcal{H}$,

$$\langle f, g\rangle = \overline{\langle g, f\rangle} \quad . \tag{4.4}$$

[1]More explicitly: "An entity of a certain kind necessarily acts in a certain way under a given set of circumstances."

[2]David Harriman, *THE LOGICAL LEAP: Induction In Physics*, pages 9, 21-22, 236-237. New American Library, Penguin Group, Inc., New York, 2010.

[3]The concept of "mathematization" refers to the process of expressing things in terms of the language of mathematics, i.e. in quantitative terms.

2. The *Hermitian adjoint*, T^H, of a given linear operator T. It is defined by the requirement that,

$$\langle f, Tg\rangle = \langle T^H f, g\rangle \quad \text{for all } f, g \in \mathcal{H} \ . \tag{4.5}$$

3. the *reciprocity relation* between (i) the solution to the standard cause and effect problem and (ii) that of the Hermitian adjoint of this cause and effect problem.

If the[4] solution to the inhomogeneous problem, Eq.(4.2) exists, then the method of solving it is a four step process. The Hermitian adjoint, $(A-\lambda B)^H$, of the given operator $T = A - \lambda B$ plays a key role. Indeed, the existence of that solution depends on the existence of F, the solution to the equation

$$(A - \lambda B)^H F = I \ . \tag{4.6}$$

Step I. Find the solution to Eq.(4.6), if it exists.
Step II. Note that F^H, the Hermitian adjoint of F, satisfies

$$F^H(A - \lambda B) = I \ . \tag{4.7}$$

Step III. Multiply the given Eq.(4.2) by F^H,

$$F^H(A - \lambda B)\vec{x} = F^H\vec{b} \ . \tag{4.8}$$

Step IV. Apply Eq.(4.7) and obtain the solution

$$\vec{x} = F^H\vec{b} \ . \tag{4.9}$$

At first sight this result is somewhat daunting because we need to find the Hermitian adjoint of $A - \lambda B$, solve the corresponding equation for F, and finally find the Hermitian adjoint of F.
However, the *reciprocity theorem* allows us to dispense with having to find all these Hermitian adjoints.
Theorem 1 *(Reciprocity) Let G be a solution to Eq.(4.1), and let F be a solution to Eq.(4.6), then*

$$F^H = G \ . \tag{4.10}$$

[4]We are assuming that λ is such that the homogeneous problem $(A - \lambda B)u = 0$ has no non-trivial solution. This means that if a solution to Eq. (4.2) exists, it is *the* solution.

The proof is very simple: Multiply Eq.(4.1) by F^H to obtain

$$F^H(A - \lambda B)G = F^H \ . \tag{4.11}$$

Applying Eq.(4.7) one obtains the *reciprocity relation*,

$$G = F^H \ . \quad \text{QED.} \tag{4.12}$$

In light of this relation, the solution to the inhomogeneous problem, Eq.(4.2), is given by

$$\boxed{\vec{x} = G\vec{b} \ .} \tag{4.13}$$

In physics and engineering there is a large class of linear systems which are *self-adjoint*, for example vibrating or oscillating linear systems without any friction. They are mathematized by the *self-adjointness* condition

$$(A - \lambda B)^H = T^H = T = A - \lambda B \ . \tag{4.14}$$

More precisely, we have the following
Definition. A linear operator T is said to be *self-adjoint* or *Hermitian* if it satisfies

$$\langle f, Tg\rangle = \langle Tf, g\rangle \quad \text{for all } f, g \in \mathcal{H} \ .$$

Thus, if the Hermitian adjoint of an operator equals the operator itself, then the operator is said to be Hermitian. In light of Eq.(4.6), it should come as no surprise that the corresponding Green's ("response") matrix/operator/function satisfies the same condition.

$$G^H = G. \tag{4.15}$$

4.1.1 Adjoint Boundary Conditions

The operators which are of immediate interest to us are differential operators. Although their actions consist of taking derivatives, their definition is more restrictive. The additional properties they have to satisfy is the consequence of the fact that an operator is a type of mapping. As such, one must always specify not only its *fomula* (or *rule*) but also its *domain*. The domain in our case is a subspace of the given Hilbert space. Thus, to specify uniquely a 2nd order differential operator, one must specify three things:

(i) The domain $\mathcal{H}$, the Hilbert space, which we shall take to be $L^2(a,b)$, the space of functions square-integrable on $[a,b]$.

(ii) The *homogeneous boundary conditions* to be satisfied by $u \in \mathcal{H}$.

(iii) $L = \alpha(x)\dfrac{d^2}{dx^2} + \beta(x)\dfrac{d}{dx} + \gamma(x)$, i.e. the "formula".

Items (i)-(iii) are referred to collectively as the "operator L". One also should note that (i) and (ii) define a linear subspace $\mathcal{S}$ of $\mathcal{H}$ as follows: Let L to be the linear map whose image Lu has a well defined inner product, i.e. $\langle v, Lu\rangle$ =finite, for any square-integrable u and v. For the set of continuously differentiable functions, $C^2(a,b)$, this means that

$$\begin{array}{rccc} L: & \mathcal{S}\cap C^2(a,b) & \longrightarrow & \mathcal{H} = L^2(a,b) \\ & u & \rightsquigarrow & Lu \quad . \end{array}$$

Here $\mathcal{S}$ is the domain of L, and it is

$$\mathcal{S} = \{\ u \in L^2(a,b):\ \langle v, Lu\rangle = \text{finite}\ \forall\ v \in L^2(a,b); u \text{ satisfies the given homogeneous boundary conditions at } a \text{ and } b\}$$

What if u is not a continuously differentiable function? Then its image Lu is not square-integrable, but the inner product $\langle v, Lu\rangle$ is still well-defined because it is finite. For example, if u is a function which has a kink, then Lu would not be defined at that point and Lu would not be square-integrable. Nevertheless, the integral of $\overline{v}Lu$ would be perfectly finite.

The *(Hermitian) adjoint* L^* of an operator such as L is defined by the requirement that

$$\langle v, Lu\rangle = \langle L^* v, u\rangle$$

for all $u \in \mathcal{S}$ and all v belonging to $\mathcal{S}^*$, the domain of L^*. This is illustrated in the examples below. In compliance with standard notation, we are using L^*, and not L^H to refer to the Hermitian adjoint of the differential operator L. In some physics text books one finds $L^\dagger$ instead.

Example 1. Let $L = \dfrac{d}{dx}$ have as its domain the subspace

$$\mathcal{S} = \{\ u \in L^2(a,b):\ u(a) = 2u(b)\ ; \langle v, Lu\rangle = \text{finite}\ \ \text{whenever } v \in L^2(a,b)\}\ , \tag{4.16}$$

and let the inner product be

$$\langle v, u\rangle = \int_a^b \overline{v}(x)u(x)\ dx \quad .$$

FIND the adjoint of this operator.
To do this, one integrates by parts in order to move the operator from the second factor to the first and thereby obtains

$$\begin{aligned}\langle v, Lu\rangle &= \int_a^b \overline{v}\frac{d}{dx}u\ dx \\ &= \underbrace{\overline{v}(b)u(b) - \overline{v}(a)u(a)}_{P(\overline{v},u)|_a^b} + \int_a^b \underbrace{\overline{\left(-\frac{d}{dx}v\right)}}_{L^*v} u\ dx \quad .\end{aligned}$$

The bilinear expression $P(\overline{v}, u)$ is called the *bilinear concomitant* or the *conjunct* of $\overline{v}$ and u. Thus we have

$$\boxed{\langle v, Lu\rangle - \langle L^*v, u\rangle \equiv P(\overline{v}, u)|_a^b} \tag{4.17}$$

This intergral identity relates the behaviour of $v(x)$ and $u(x)$ in the interior of $[a, b]$ to their values on the boundary, $x = a$ and $x = b$. The construction of L^* from L is based on the requirement that

$$\langle v, Lu\rangle - \langle L^*v, u\rangle = 0 \quad .$$

This means that the bilinear concomitant evaluated at the endpoints must vanish,

$$P(\overline{v}, u)|_a^b = 0\ . \qquad (\text{"compatibility condition"}) \quad \text{(I)}$$

This is a compatibility condition between the given boundary condition, Eq.(4.16), i.e.

$$\mathcal{S} = \{u \in L^2(a, b) :\ u(a) - 2u(b) = 0\} \quad (\text{"given boundary condition"}) \quad (II)$$

and the to-be-determined adjoint boundary condition,

$$\mathcal{S}^* = \{u \in L^2(a, b) :\ \alpha u(a) + \beta u(b) = 0\} \quad (\text{"adjoint boundary condition"}) \quad (III)$$

This means that any two of the three sets of conditions implies the third:

1. (II) and (III) imply (I).
2. (III) and (I) imply (II).
3. (I) and (II) imply (III).

Using the boundary condition, one obtains

$$\begin{aligned} \langle v, Lu\rangle &= [\overline{v}(b) - 2\overline{v}(a)]u(b) + \int_a^b \overline{(-)\frac{d}{dx}v}\; u\; dx \\ &\equiv \langle L^* v, u\rangle \quad . \end{aligned}$$

This determines L^*, *provided* the boundary term vanishes *for all* $u \in \mathcal{S}$. This implies that v must satisfy the *adjoint boundary condition*

$$\overline{v}(b) - 2\overline{v}(a) = 0 \quad . \tag{4.18}$$

The conclusion is this: the adjoint L^* of L consists of *two parts*,

(i)
$$L^* = -\frac{d}{dx} \qquad (\text{“}the\ formula''\text{)}$$

(ii) the adjoint boundary condition, Eq.(4.18), which determines the *domain*

$$\mathcal{S}^* = \{\ v \in L^2(a,b) :\ v(a) = \frac{1}{2}v(b) \quad ;\quad \langle L^* v, u\rangle < \infty\ \forall u \in L^2(a,b)\}$$
$$(\text{“}the\ domain''\text{)}$$

on which L^* operates.

The expression $L^* = -\frac{d}{dx}$ without the boundary condition is called the *formal adjoint* of L. If v, and hence $\overline{v}$, satisfies the adjoint boundary condition, then the “formal” adjoint becomes *the* adjoint of L. In this case one has

$$\langle v, Lu\rangle = \langle L^* v, u\rangle$$

for all $u \in \mathcal{S}$ and all $v \in \mathcal{S}^*$.
It is clear that L and its adjoint L^* are different operators: they differ not only in their domain but also in their formula.

Example 2. Consider $L = i\dfrac{d}{dx}$ whose domain is the subspace $\mathcal{S} = \{\, u :\ u(a) = u(b)\ \}$.
FIND its adjoint.
Following the familiar procedure, one obtains

$$\begin{aligned} \langle v, Lu\rangle &= i[\overline{v}(b) - \overline{v}(a)]u(b) + \int_a^b \overline{i\frac{d}{dx}v}\ u\ dx \\ &= \langle L^* v, u\rangle \quad . \end{aligned}$$

This holds for all $u \in \mathcal{S}$, provided $v(a) = v(b)$. It follows that

$$\begin{aligned} L^* &= i\frac{d}{dx} && (\text{“}formula''\text{)} \\ \mathcal{S}^* &= \{\, v :\ v(a) - v(b) = 0\ \} && (\text{“}domain''\text{)} \end{aligned}$$

One sees that both the formal adjoint (“the formula”) and its domain are the same as the given operator. This observation motivates the following
Definition. An operator said to be *self-adjoint*, if both its formula and its domain are the same, i.e.

$$L^* = L$$

and

$$\mathcal{S}^* = \mathcal{S}$$

Reminder: Sometimes we shall mean by L^* only the “formal” adjoint of L, at other times we shall mean by L^* *the* adjoint of L, which includes the boundary conditions. The context will make clear which is which.

Lecture 27

4.1.2 Second Order Operator and the Bilinear Concomitant

Let us extend our considerations from linear differential operators of first order to those of second order. To do this, let us find the adjoint of a second order operator. The given operator consists of

(i) the *differential operator*

$$\boxed{L = \alpha(x)\frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(x)}$$

(ii) the *domain* $\mathcal{S} \subset \mathcal{H} = L^2(a,b)$ on which it operates,

$$\boxed{\mathcal{S} = \{\ u:\ u \in L^2(a,b);\ Lu \in L^2(a,b);\ B_1(u) = 0;\ B_2(u) = 0\ \}}$$

where B_1 and B_2 are two *homogeneous* boundary conditions,

$$\begin{array}{rcl} 0 = B_1(u) & \equiv & \alpha_1 u(a) + \alpha_1' u'(a) + \beta_1 u(b) + \beta_1' u'(b) \\ 0 = B_2(u) & \equiv & \alpha_2 u(a) + \alpha_2' u'(a) + \beta_2 u(b) + \beta_2' u'(b) \end{array} \tag{4.19}$$

The $\alpha_i' s$ and $\beta_i' s$ are given constants not to be confused with the functions $\alpha(x)$ and $\beta(x)$. The task is to find the *adjoint* of the given operator, namely FIND

(i) L^*

(ii) $\mathcal{S}^* = \{\ v \in L^2(a,b):\ B_1^*(v) = 0;\ B_2^*(v) = 0\ \}$

such that

$$\langle v, Lu\rangle = \langle L^* v, u\rangle$$

for all $u \in \mathcal{S}$ and all $v \in \mathcal{S}^*$. The left-hand side of this equation is given, and it is

$$\langle v, Lu\rangle = \int_a^b \left(\alpha\overline{v}\frac{d^2u}{dx^2} + \beta\overline{v}\frac{du}{dx} + \overline{v}\gamma u\right)\ dx\ \ .$$

In order to have the derivatives act on the function v, one does an integration by parts twice on the first term, and once on the second term. The result is

$$\begin{array}{rcl} \langle v, Lu\rangle & = & \underbrace{\int_a^b \overline{\left(\frac{d^2}{dx^2}\overline{\alpha}v - \frac{d}{dx}\overline{\beta}v + \overline{\gamma}v\right)} u\ dx}_{L^* v} \\ & + & \underbrace{[\overline{v}\alpha u' - (\overline{v}\alpha)' u + \overline{v}\beta u]_a^b}_{P(\overline{v},u)|_a^b} \end{array}$$

The bilinear expression $P(\overline{v},u)$ is called the *bilinear concomitant* or the *conjunct* of $\overline{v}$ and u. Thus we have

$$\boxed{\langle v, Lu\rangle - \langle L^* v, u\rangle \equiv P(\overline{v},u)|_a^b} \tag{4.20}$$

This important integral identity is the one-dimensional version of *Green's identity*. Indeed, it relates the behavior of $v(x)$ and $u(x)$ in the *interior*

of $[a,b]$ to their values on the *boundary*, here $x=a$ and $x=b$. It is an extension of the integrated Lagrange identity, Eq.(1.16), from formally self-adjoint second order operators to generic second order operators. Observe that when

$$\beta = \alpha' \quad ,$$

L becomes formally self-adjoint whenever the coefficient functions α, β, and γ are real. In this circumstance L is the Sturm-Liouville operator and the bilinear concomitant reduces to

$$P(\overline{v},u) = \alpha[\overline{v}u' - \overline{v}'u] \quad ,$$

which is proportional to the Wronskian determinant of $\overline{v}$ and u. The construction of L^* from L is based on the requirement that

$$\langle v, Lu\rangle - \langle L^*v, u\rangle = 0 \quad .$$

This means that the bilinear concomitant evaluated at the endpoints must vanish,

$$P(\overline{v},u)\big|_a^b = 0 \ . \qquad (\text{"compatibility condition"}) \qquad \text{(I)}$$

This is a compatibility condition between the given boundary conditions, Eq.(4.19),

$$\begin{array}{rcl} B_1(u) & = & 0 \\ B_2(u) & = & 0 \end{array} \quad , \qquad (\text{"given boundary conditions"}) \qquad \text{(II)}$$

and the adjoint boundary conditions,

$$\begin{array}{rcl} B_1^*(v) & = & 0 \\ B_2^*(v) & = & 0 \end{array} \quad . \qquad (\text{"adjoint boundary conditions"}) \qquad \text{(III)}$$

This means that any two of the three sets of conditions implies the third:

1. (II) and (III) imply (I).
2. (III) and (I) imply (II).
3. (I) and (II) imply (III).

The problem of obtaining the adjoint boundary conditions in explicit form,

$$\begin{array}{rcl} 0 = B_1^*(v) & \equiv & \alpha_1^* v(a) + \alpha_1^{*\prime} v'(a) + \beta_1^* v(b) + \beta_1^{*\prime} v'(b) \\ 0 = B_2^*(v) & \equiv & \alpha_2^* v(a) + \alpha_2^{*\prime} v'(a) + \beta_2^* v(b) + \beta_2^{*\prime} v'(b) \end{array} \qquad (4.21)$$

is a problem in linear algebra. One must combine the given boundary conditions, Eq.(4.19) with the compatibility condition (I) to obtain the coefficients $\alpha_i^*, \alpha_i^{*\prime}, \beta_i^*, \beta_i^{*\prime}$ in Eq.(4.21).

4.2 Green's Function and Its Adjoint

Presently our task is to solve what in linear algebra corresponds to

$$(A - \lambda B)\vec{u} = \vec{b} \ . \tag{4.22}$$

We shall find that the execution of this task is a straight forward calculation, provided there exists a unique answer. The existence and uniqueness questions are readily understood for any finite-dimensional matrix. Such understanding also applies to a second order operator, even though its domain and target space (="codomain") is infinite-dimensional and thus has its own mathematical subtleties.

An understanding of the existence and uniqueness questions is not only the difference between success and failure in solving Eq.(4.22) and its extension to a second order differential equation, but it is also the basis for the spectral applications in Section 4.10. In light of this we shall interrpt the development to give a rapid review of the linear algebra behind the Green's function concept.

4.3 A Linear Algebra Review: Existence and Uniqueness

In dealing with a linear system $A-\lambda B$ we are dealing with with the following causal relation

$$\begin{array}{rcl} \text{source } \vec{b} & & \text{response } \vec{u} \\ \longrightarrow & \boxed{A-\lambda B} & \longrightarrow \\ \text{input} & & \text{output} \end{array} \tag{4.23}$$

The properties of this causal connection are captured by means of the following equation,

$$(A - \lambda B)\vec{u} = \vec{b} \ . \tag{4.24}$$

The associated mathematical problem is this:
Given: (a) The linear mapping $A - \lambda B : \ U \to V$; (b) any vector $\vec{b} \in V$
Find: $\vec{u} \in U$ such that Eq.(4.24) is satisfied
Discussion: This linear algebra problem, it turns out, decomposes into the task of answering two questions:

1. Can one find a G such that

$$(A - \lambda B)G = I \text{ on } V \quad ?$$

2. Can one find an H such that

$$(A - \lambda B)^* H = I \text{ on } U \quad ?$$

If $A - \lambda B$ has the right properties, the answer is 'yes' to both questions.

4.4 The Inhomogeneous Problem

In the framework of calculus this means that we must solve the inhomogeneous boundary value problem

$$\begin{array}{rcll} Lu(x) & = & -f(x) & a < x < b \\ B_1(u) & = & 0 & \\ B_2(u) & = & 0 & \end{array} , \tag{4.25}$$

or, more generally, solve the problem

$$\begin{array}{rcll} Lv(x) & = & -f(x) & a < x < b \\ B_1(v) & = & d & \qquad (4.26) \\ B_2(v) & = & e \ , & \qquad (4.27) \end{array}$$

Notation: the minus sign on the right hand sides is a convention which complies with the interpretation of f as the force density on a simple string as discussed in the next section.
The operator L is a second order linear differential operator, while d and e are constants. We shall first deal with the first problem where the boundary conditions are homogeneous ($d = e = 0$). Once we have solved it, the solution to the second one is simply equal to the first solution augmented by that solution to the homogeneous differential equation which satisfies

$$\begin{array}{rcll} Lv_h(x) & = & 0 & a < x < b \\ B_1(v_h) & = & d & \\ B_2(v_h) & = & e & \end{array} .$$

Thus,

$$\begin{aligned} v(x) &= u(x) + v_h(x) \\ &= u(x) + c_1 v_1(x) + c_2 v_2(x) \quad . \end{aligned}$$

Here v_1 and v_2 are any two independent solutions to the homogeneous differential equation, and the constants c_1 and c_2 are adjusted so that the two linear endpoint conditions, Eqs.(4.26) and (4.27), are satisfied. This means that c_1 and c_2 are determined by

$$\begin{aligned} c_1 B_1(v_1) + c_2 B_1(v_2) &= d \\ c_1 B_2(v_1) + c_2 B_2(v_2) &= e \quad . \end{aligned}$$

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The solution to the inhomogeneous problem, Eqs.(4.25), is based on the corresponding Green's function. In the engineering sciences it is also known as the *unit impulse response*. It is given by the following

Definition. (*Green's function and its adjoint*)
Let $G(x;\xi)$ be a function with the property

$$\begin{aligned} LG(x;\xi) &= -\delta(x-\xi) \qquad a < x, \xi < b \\ B_1(G) &= 0 \\ B_2(G) &= 0 \end{aligned} \quad . \tag{4.28}$$

Such a function is the *Green's function* for the boundary value problem. The corresponding *adjoint Green's function* $H(x;\xi)$ is the function with the property

$$\begin{aligned} L^* H(x;\xi) &= -\delta(x-\xi) \qquad a < x, \xi < b \\ B_1^*(H) &= 0 \\ B_2^*(H) &= 0 \end{aligned} \quad , \tag{4.29}$$

where L^* is the formal *adjoint* of the differential operator L and $B_1^*(H) = 0$ and $B_2^*(H) = 0$ are the boundary conditions *adjoint* to $B_1(G) = 0$ and $B_2(G) = 0$.
The adjoint Green's function is very useful because it allows us to solve the inhomogeneous boundary value problem, Eqs.(4.25). The solution is obtained with the help of Green's identity, Eq.(4.20),

$$\langle H, Lu\rangle - \langle L^* H, u\rangle = \int_a^b (\overline{H} Lu - \overline{L^* H} u)\, dx = P(\overline{H}, u)\big|_a^b \quad .$$

Indeed, using the fact that the adjoint boundary conditions

$$\begin{aligned} B_1^*(H) &= 0 \\ B_2^*(H) &= 0 \end{aligned}$$

have been constructed so as to guarantee that

$$P(\overline{H}, u)|_a^b = 0 \ ,$$

we obtain with the help of the given Eq.(4.25), $Lu = -f$, and with (4.29), $L^*H = -\delta(x-\xi)$, the result

$$\int_a^b f(x)\ \overline{H}(x;\xi)\ dx = \int_a^b \delta(x-\xi)\ u(x)\ dx \quad ,$$

which yields the solution

$$\boxed{u(\xi) = \int_a^b \overline{H}(x;\xi)\ f(x)\ dx}$$

It turns out that the beauty of this result is that we *don't* even have to use the adjoint Green's function $H(x;\xi)$. Instead, one may use the original Green's function $G(x;\xi)$. This is based on the following

Theorem (*Green's function and its adjoint*)

$$\boxed{\overline{H}(x;\xi) = G(\xi;x)}$$

The proof of this equation is given below.

Remark 1. This result says that in order to obtain the adjoint Green's function,

$$H(x;\xi) = \overline{G(\xi;x)} \quad ,$$

simply interchange the arguments x and ξ and then take the complex conjugate of the Green's function, the solution to Eq.(4.28). With the help of this result the solution to the inhomogeneous problem becomes simply

$$\boxed{u(\xi) = \int_a^b G(\xi;x)\ f(x)\ dx} \tag{4.30}$$

The advantage is clear: don't bother solving Eq.(4.29). It is enough to find only the Green's function, i.e. the solution to Eq.(4.28).

Remark 2. The other noteworthy feature is algebraic. The process of interchanging the arguments x and ξ and then taking the complex conjugate is precisely the infinite-dimensional version of taking the Hermitian adjoint of a matrix. Moreover, the integration in Eq.(4.30) corresponds to the summation when a matrix acts on a vector and thereby yields a new vector.
Remark 3. If the boundary value problem is *self-adjoint*, i.e. $L = L^*$, together with $B_1^* = B_1$ and $B_2^* = B_2$, then $H(x;\xi) = G(x;\xi)$ and we have the result

$$\boxed{\overline{G(\xi;x)} = G(x;\xi)} \quad .$$

This is generally known as the *reciprocity relation.* It says that $G(x;\xi)$ is what in linear algebra corresponds to a "Hermitian matrix".
Proof: (In three steps)

(i) Again use Green's identity

$$\int_a^b (\overline{H}LG - \overline{L^*H}G)\ dx = P(\overline{H}, G)|_a^b$$

(ii) The boundary conditions of the two boundary value problems (4.28) and (4.29) guarantee that the linear concomitant vanishes at the endpoints,

$$P(\overline{H}, G)|_a^b = 0 \quad .$$

(iii) Inserting the two respective differential equations of (4.28) and (4.29) into the above Green's identity, one obtains

$$\int_a^b \overline{H}(x;\xi')\delta(x-\xi)\ dx = \int_a^b G(x;\xi)\delta(x-\xi')\ dx$$

or

$$\overline{H}(\xi;\xi') = G(\xi';\xi) \quad ,$$

which is what had to be shown.

4.4.1 Translation Invariant Systems

It is difficult to overstate the power and versatility of the Green's function method. From the viewpoint of mathematics it allows one to generate solutions to any inhomogeneous linear differential equation with boundary conditions. From the viewpoint of radiation physics the Green's function relates

a disturbance to its measurable effect or response. From the viewpoint of engineering $G(x;\xi)$ expresses those inner workings of a linear system which relates its input to its output.

Invariant linear systems constitute one of the most ubiquitous of its kind. They are characterized by invariance under space and/or time translations. Their Green's function have the invariance property

$$G(x+a;\xi+a) = G(x;\xi)$$

under arbitrary translations a. Letting $a = -\xi$, one finds that

$$G(x;\xi) = G(x-\xi;0) \equiv G(x-\xi)$$

Thus Eq.(4.30) becomes

$$u(\xi) = \int_{-\infty}^{\infty} G(\xi - x)\ f(x)\ dx \equiv G \star f\ (\xi)\ . \tag{4.31}$$

In other words the response of an invariant linear system is simply the *convolution* of the input with the system Green's function.

It is virtually impossible to evade the fact that the essence of any linear translation invariant aspect of nature is best grasped by means of the Fourier representation. The input-output relation of a linear invariant system expressed by means of the convolution integral, Eq.(4.31), is no exception. Take the Fourier transform $\mathcal{F}$ of both sides and find

$$\hat{u}(k) = \sqrt{2\pi}\ \hat{G}(k)\hat{f}(k)\ ,$$

where

$$\hat{u}(k) \equiv \mathcal{F}[u](k) \equiv \int_{-\infty}^{\infty} \frac{e^{-ik\xi}}{\sqrt{2\pi}}\ u(\xi)\ d\xi$$

is the Fourier amplitude corresponding to $u(\xi)$. Convolution of functions in the given domain has simplified into multiplication of their Fourier amplitudes in the Fourier domain. For each point in this domain the factor $\hat{f}(k)$ expresses the input of the linear system, $\hat{u}(k)$ expresses its response. In signal processing and in electromagnetic theory the function $\hat{G}(k)$ is called the *filter function*, while in acoustics and optics it is called the *transfer function* [4] of the linear system.

4.5 Pictorial Definition of a Green's Function

One of the most effective ways to use an idea and keep it in the forefront of one's mind is in terms of pictures. A Green's function is no exception. One of the best pictures which illustrates a one-dimensional Green's function is that of a string subjected to a unit point force.

4.5.1 The Simple String and Poisson's Equation

Consider a simple string with a force applied to it. For such a string let

$F_T dx$ = net transverse force acting on the string segment between x and $x + dx$ *due to tension* T only.

= vertical force component

$$= T\left.\frac{du}{dx}\right|_{x+dx} - T\left.\frac{du}{dx}\right|_{x} \qquad \left(\begin{array}{l}\text{assuming } \frac{du}{dx} \ll 1, \text{ so that} \\ \sin = \frac{\frac{du}{dx}}{\sqrt{1+\left(\frac{du}{dx}\right)^2}} \approx \frac{du}{dx} = \tan\end{array}\right)$$

so that

$$\frac{d}{dx}T\frac{du}{dx} = \frac{\text{(Force due to tension)}}{\text{(length)}} \begin{cases} > 0\text{(upward)} & \text{if string curvature} > 0 \\ < 0\text{(downward)} & \text{if string curvature} < 0 \end{cases}.$$

Let $F(x) =$ *applied* force density $\left(\frac{\text{(Force)}}{\text{(length)}}\right)$.
If the string is in equilibrium then there is no acceleration. Consequently, the total force density is zero:

$$\frac{d}{dx}T\frac{du}{dx} + F(x) = 0$$

or

$$\frac{d}{dx}T\frac{du}{dx} = -F(x)$$

For constant tension one obtains

$$\boxed{\frac{d^2u}{dx^2} = -f(x) \quad \text{where } f(x) = \frac{F(x)}{T}} \quad . \tag{4.32}$$

This is the one-dimensional *Poisson equation.*

Example
Consider a cable whose linear mass density is ρ and which is suspended between two horizontal points, $x = 0$ and $x = L$, in a uniform gravitational field whose acceleration is g. The force density on such a cable is ρg. If the

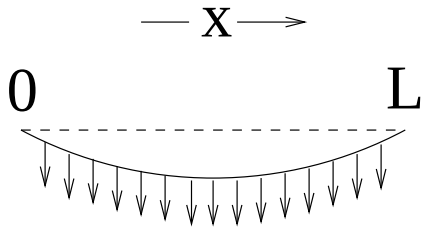

Figure 4.1: A cable of length L suspended between two horizontal points. If its slope is small then its deviation away from the dotted horizontal is governed by Poisson's equation. If the slope is not small then the deviation is described by a *catenary*.

tension in the cable is T, then the equilibrium profile $u(x)$ is governed by

$$\frac{d^2u}{dx^2} = \frac{\rho g}{T} \quad .$$

The solution is evidently

$$u(x) = c_1 + c_2 x + \frac{1}{2}\frac{\rho g}{T}x^2 \quad ,$$

where the integration constants are determined by $u(0) = 0$ and $u(L) = 0$. It follows that the cable's profile away from the straight horizontal is

$$u(x) = x(x - L)\frac{\rho g}{2T} \quad .$$

Exercise 4.5.1 (ADJOINT OF AN OPERATOR)
Find the adjoint differential operator L^* and the space on which it acts if

(a) $Lu = u'' + a(x)u' + b(x)u$ where

$$u(0) = u'(1) \quad \text{and} \quad u(1) = u'(0).$$

(b) $Lu = -\left(p\left(x\right)u'\right)' + q(x)u$ where

$$u(0) = u(1) \quad \text{and} \quad u'(0) = u'(1).$$

Assume that the scalar product is

$$\langle u, v\rangle = \int_0^1 u\ v\ dx.$$

Exercise 4.5.2 (ADJOINT EIGENVALUE PROBLEM)
Let L be a differential operator defined over that domain $\mathcal{S}$ of functions which satisfy the given homogeneous boundary condition $B_1(u) = 0$ and $B_2(u) = 0$. Let L^* be the corresponding adjoint operator defined on the domain $\mathcal{S}^*$ of functions which satisfy the corresponding adjoint boundary conditions, $B_1^*(v) = 0$ and $B_2^*(v) = 0$. Let $u \in \mathcal{S}$ be an eigenfunction of L:

$$Lu = \lambda u$$

Similarly let $v \in \mathcal{S}^*$ be an eigenfunction of L^*:

$$L^*v = \lambda' v.$$

(i) Make a guess as to the relationship between the eigenvalues λ of L and the eigenvalues λ' of L^* and give a reason why.

(ii) Prove: If $\lambda \neq \bar{\lambda'}$ then $\langle v, u\rangle = 0$. i.e. An eigenfunction of L corresponding to the eigenvalue λ is *orthogonal* to every eigenfunction of L^* which does not correspond to $\bar{\lambda}$. Here the overline means complex conjugate, of course.

Exercise 4.5.3 (BESSEL OPERATORS)
Find the Green's function for the Bessel operators

(a) $Lu(x) = \dfrac{d}{dx}\ x\ \dfrac{du(x)}{dx}$

(b) $Lu(x) = \dfrac{d}{dx}\ x\ \dfrac{du(x)}{dx} - \dfrac{n^2}{x}u(x)$ with $y(0)$ finite and $y(1) = 0$,

i.e. solve the equations $Lu = -\delta(x - \xi)$ with the given boundary conditions.

Exercise 4.5.4 (DIFFERENT ENDPOINT CONDITIONS)

1. Find the Green's function for the operator with

$$L = \frac{d^2}{dx^2} + \omega^2 \quad \text{with} \quad \begin{matrix} u(a) = 0 \\ u(b) = 0 \end{matrix} \qquad a < b$$

and ω^2 a fixed constant. i.e. solve $Lu = -\delta(x - \xi)$ with the given boundary conditions.

2. Does this Green's function exist for all values of ω? If NO, what are the exceptional values of ω?

3. Having found the Green's function in part (1), suppose one wishes to find the Green's function for the same differential equation, but with different end point conditions, namely $u(a) = 0$ and $u'(a) = 0$. How would one find this new Green's function with a minimal amount of work? Go ahead, find it.

Exercise 4.5.5 (ADJOINT FOR GENERIC ENDPOINT CONDITIONS) Suppose that $Lu = u''$ where

$$a_1 u(0) + b_1 u'(0) + c_1 u(1) + d_1 u'(1) = 0$$

and

$$a_2 u(0) + b_2 u'(0) + c_2 u(1) + d_2 u'(1) = 0.$$

1. Find L^* and the space on which it acts if one uses the scalar product $\langle u, v \rangle = \int_0^1 u\ v\ dx$.

2. For what values of the constants $a_1, b_1, \ldots, c_2, d_2$ is the operator self adjoint?

Lecture 29

4.5.2 Point Force Applied to the System

A *unit force localized* at a point is a unit force *distributed* over an ε-interval surrounding the given point. The density of this distributed force is inversely proportional to ε. More precisely, one defines

$$\delta_\varepsilon(x-\xi) \equiv \begin{cases} \dfrac{1}{\varepsilon} & |x-\xi| \le \dfrac{\varepsilon}{2} \\ 0 & |x-\xi| > \dfrac{\varepsilon}{2} \end{cases} \left[\frac{\text{(Force)}}{\text{(length)}}\right],$$

the *finite approximation to the Dirac distribution*, whose integral, the total force,

$$\int_a^b \delta_\varepsilon(x-\xi)\ dx = \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \delta_\varepsilon(x-\xi)\ dx = 1 \quad ,$$

is unity. Let us apply such a force density,

$$F(x) = \delta_\varepsilon(x - \xi) \quad ,$$

to a string with constant horizontal tension T. The response of this string is governed by the Poisson equation (4.32), namely

$$T\frac{d^2G}{dx^2} = -\delta_\varepsilon(x - \xi) \quad .$$

Note that the sum of *all* the vertical forces is necessarily zero. This equilib-

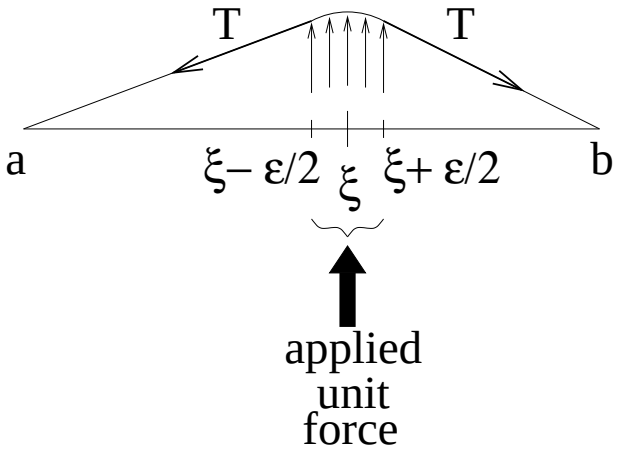

Figure 4.2: A distributed unit force applied to a string with tension T. The force is concentrated in an interval of size ε. As $\varepsilon \to 0$, the response of the string tends towards the Green's function $G(x;\xi)$.

rium condition is expressed by the statement that (see Fig. 4.2)

$$TG'(\xi^+) - TG'(\xi^-) + 1 = 0$$

or by

$$\boxed{G'(\xi^+) - G'(\xi^-) = -\frac{1}{T}} \quad ,$$

which is known as the *jump condition*. Here we are using the notation $\xi^\pm = \xi \pm \varepsilon/2$ with ε neglegibly small. The other condition that the response G must satisfy is that it be continuous at $x = \xi$, i.e.

$$\boxed{G(\xi^+) - G(\xi^-) = 0} \quad .$$

This *continuity condition*, the jump condition, together with the boundary conditions lead to a unique response, the Green's function $G(x;\xi)$ of the string.(Why?)

4.6 Properties and Utility of a Green's Function

More generally, a unit force applied to a general linear system yields a response which is governed by the equation

$$LG_\varepsilon = -\delta_\varepsilon(x-\xi) \quad ,$$

where L is the repeatedly used linear operator

$$L = \frac{d}{dx}p(x)\frac{d}{dx} + \gamma(x) \quad .$$

Integrate both sides and obtain

$$\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \left(\frac{d}{dx}p\frac{d}{dx}G_\varepsilon + \gamma G_\varepsilon\right) dx = -1 \tag{4.33}$$

What happens to G_ε as $\varepsilon \to 0$? The physical properties of the linear system imply that the response G_ε remain a continuous function of x, and its mathematical formulation should reflect this fact. Indeed, this continuity is guaranteed by the fact that the equation

$$\frac{d}{dx}p(x)\frac{d}{dx}G + \gamma(x)G = -\delta(x-\xi)$$

be satisfied. If $G \equiv \lim_{\varepsilon\to 0} G_\varepsilon$ were not continuous, then the first term of the differential equation,

$$p\frac{d^2G}{dx^2} \quad ,$$

would yield the *derivative* of a Dirac delta function, and there is no such expression on the right hand side.

The continuity of $G(x;\xi)$ and the evaluation of the integral Eq.(4.33) lead to the two key conditions which the unit impulse response G must satisfy,

$$\boxed{G(\xi^+) - G(\xi^-) = 0} \qquad \text{“continuity for all } a < x < b\text{”} \ .$$

and

$$\boxed{\left.\frac{dG}{dx}\right|^{\xi^+} - \left.\frac{dG}{dx}\right|^{\xi^-} = -\frac{1}{p(\xi)}} \quad \text{"jump condition at } x = \xi\text{"} \quad .$$

A more careful statement of these properties is provided by the following

Theorem 4.6.1 (Fundamental Theorem for Green's Functions) *Let $G(x;\xi)$ be a function which*

(a) considered as a function of x, satisfies the differential equation

$$\left[\frac{d}{dx}p(x)\frac{d}{dx} + \gamma(x)\right] G(x;\xi) \equiv LG(x;\xi) = 0$$

in (a,b) except *at the point* $x = \xi$,

(b) satisfies the given homogeneous boundary conditions,

(c) for fixed ξ is continuous, *even at* $x = \xi$,

(d) has continuous 1^{st} and 2^{nd} derivatives everywhere in (a,b), except *at* $x = \xi$, *where it has a* jump discontinuity *given by*

$$\left.\frac{d}{dx}G(x;\xi)\right|_{\xi^-}^{\xi^+} = \frac{-1}{p(\xi)} \quad .$$

Conclusion:

$$u(x) = \int_a^b G(x;\xi)\ f(\xi)\ d\xi \iff \begin{array}{l} u(x) \text{ satisfies} \\ (i) \text{ the given boundary conditions} \\ (ii) \quad Lu = -f(x), \\ \qquad \text{where } f \text{ is piecewise} \\ \qquad \text{continuous in } (a,b) \end{array} \tag{4.34}$$

Comment. A function which satisfies properties (a)-(d) is, of course, the Green's function for the boundary value problem stated in the conclusion, equivalently given by Eq.(4.28). Even though there is more than one way of constructing such a function (if it exists), the result is always the same. In other words, one has the following

Theorem 4.6.2 (Uniqueness of a Green's function)
The Green's function of a given linear system is unique.

It is easy to verify the validity of this theorem. If there were two such functions:

$$LG_1(x;\xi) = -\delta(x-\xi)$$
$$LG_2(x;\xi) = -\delta(x-\xi) \ ,$$

then their difference satisfies the homogeneous equation

$$L(G_1 - G_2) = 0 \ .$$

Consider the Green's function H adjoint to either G_1 or G_2. It satisfies Eq.(4.29),

$$L^*H(x;\xi') = -\delta(x-\xi') \ .$$

Consequently,

$$\begin{aligned} 0 &= \langle H, L(G_1 - G_2)\rangle \\ &= \langle L^*H, (G_1 - G_2)\rangle \\ &= \int_a^b (-)\delta(x-\xi')\left(G_1(x;\xi) - G_2(x;\xi)\right) dx \\ &= G_2(\xi';\xi) - G_1(\xi';\xi) \ . \end{aligned}$$

Thus the Greens function is unique:

$$G_1(\xi';\xi) = G_2(\xi';\xi) \ .$$

It is informative to restate this calculation algebraically: Starting with the fact that the difference satisfies $G_1 - G_2$ satisfies the homogeneous problem, one recalls that such a problem furnishes us with only two alternatives:

(i) the trivial solution,which is the zero solution. In this case the difference between the two Green's functions vanishes identically. This means the Green's function is unique.

(ii) a nontrivial solution, which implies that the nullspace of the homogeneous adjoint problem is nonzero. In this case the inner product of this solution with the inhomogeneity, the Dirac delta function, does not vanish. Hence the existence of a solution to the inhomogeneous problem is impossible. In other words, the Green's function does not exist.

The two possibilities (i) and (ii) are *mutually exclusive* and *jointly exhaustive*. They illustrate the so-called *Fredholm alternatives* of a linear operator.

Proof of the Fundamental Theorem: The implication "$\Longleftarrow$" has already been demonstrated with Eq.(4.30). To show "$\Longrightarrow$" compute the various derivatives and then form the linear combination Lu. The fact that the slope of G makes a jump at $x = \xi$ demands that the integral for u be split at that point,

$$\begin{aligned} u(x) &= \int_a^{x-0} G(x;\xi)\ f(\xi)\ d\xi \ + \ \int_{x+0}^{b} G(x;\xi)\ f(\xi)\ d\xi \\ u'(x) &= \int_a^{x-0} \frac{dG(x;\xi)}{dx}\ f(\xi)\ d\xi \ + \ \int_{x+0}^{b} \frac{dG(x;\xi)}{dx}\ f(\xi)\ d\xi \\ &+ \quad G(x;x-0)\ f(x-0) \quad - \quad G(x;x+0)\ f(x+0) \end{aligned}$$

By hypothesis (c) the last two terms cancel for all x where $f(x)$ has no jump discontinuity. (If f *does* have a jump discontinuity at, say, x_0 then consider $u(x)$ for the case $x < x_0$ separately from the case $x > x_0$.) Finally, take the second derivative,

$$\begin{aligned} u''(x) &= \int_a^{x-0} \frac{d^2G(x;\xi)}{dx^2}\ f(\xi)\ d\xi \ + \ \int_{x+0}^{b} \frac{d^2G(x;\xi)}{dx^2}\ f(\xi)\ d\xi \\ &+ \quad \left.\frac{dG(x;\xi)}{dx}\right|^{\xi=x-0} f(x-0) \quad - \quad \left.\frac{dG(x;\xi)}{dx}\right|^{\xi=x+0} f(x+0) \end{aligned}$$

Combine these derivatives to form

$$\begin{aligned} Lu(x) &= \int_a^{x-0} \left[p\frac{d^2G}{dx^2} + p'\frac{dg}{dx} + \gamma G\right] f(\xi)\ d\xi \\ &+ \int_{x+0}^{b} \left[p\frac{d^2G}{dx^2} + p'\frac{dg}{dx} + \gamma G\right] f(\xi)\ d\xi \\ &+ \quad p(x)f(x)\ \left[\frac{dG(x;x-0)}{dx} \ - \ \frac{dG(x;x+0)}{dx}\right] \end{aligned}$$

The first two integrals are zero because of hypothesis (a). Compare the last term with the jump discontinuity stipulated by (d),

$$\frac{dG(\xi^+;\xi)}{dx} \ - \ \frac{dG(\xi^-;\xi)}{dx} = \frac{-1}{p(\xi)} \quad .$$

Next compare the first term in this difference with the first term in the square bracket on the right hand side of $Lu(x)$. Note that the first argument ("point of observation") is to the *right* of the second argument ("source point") in both of these first terms.

Comparing the second terms, one finds the same thing, except that the "point of observation" is to the *left* of the "source point". This agreement implies that

$$\frac{dG(x;x-0)}{dx} - \frac{dG(x;x+0)}{dx} = \frac{-1}{p(x)} \quad .$$

Insert this expression into the right hand side of $Lu(x)$ and obtain

$$Lu(x) = -f(x) \quad .$$

This verifies that $u(x)$ as given in the conclusion satisfies the inhomogeneous differential equation indeed.

Lecture 30

4.7 Construction of the Green's Function

The explicit construction of the Green's function is a very intuitive and mechanical process if one has available two independent solutions to the homogeneous (i.e. with zero on the right hand side) differential equation governing the physical system. Indeed, on the interval (a, b) consider the two linearly independent solutions $u_1(x)$ and $u_2(x)$ which satisfy

$$Lu_1(x) = 0$$

and

$$Lu_2(x) = 0 \quad .$$

Let consider first the case where these two functions satisfy boundary conditions at each end point, a and b separately. We shall let these boundary conditions be the mixed Dirichlet-Neumann conditions at a and b respectively,

$$\begin{aligned} 0 &= B_1(u_1) \equiv \alpha u_1(a) + \alpha' u_1'(a) \\ 0 &= B_2(u_2) \equiv \beta u_2(b) + \beta' u_2'(b) \quad . \end{aligned}$$

It is important to note that these boundary conditions do not determine these two functions uniquely. In fact, each one may be multiplied by its own multiplicative factor. Thus, one obtains two families of solutions,

$$c_1 u_1(x) \ : \qquad B_1(c_1 u_1) = 0$$

and

$$c_2 u_2(x) \ : \qquad B_2(c_2 u_2) = 0 \ .$$

Because of the Fundamental Theorem, we must say (i) that the Green's function has the form

$$G(x;\xi) = \begin{cases} c_1 u_1(x) & x < \xi \\ c_2 u_2(x) & \xi < x \end{cases} \tag{4.35}$$

and that (ii) the constants c_1 and c_2 must be adjusted so that at $x = \xi$ the Green's function is *continuous*:

$$c_2 u_2(\xi) - c_1 u_1(\xi) = 0 \tag{4.36}$$

and has the prescribed *jump in its slope*:

$$c_2 u_2'(\xi) - c_1 u_1'(\xi) = \frac{-1}{p(\xi)} \ . \tag{4.37}$$

These are two equations in the two unknowns c_2 and c_1. Thus two unique members of each family of solutions have been determined. Figure 4.3 depicts how the graphs of the two solutions meet so as to fulfill the continuity requirement. Observe that, by itself, continuity at $x = \xi$ does not determine the amplitude at that point. Furthermore, at that point the graph has a kink, an abrupt change in its slope which depends entirely on the as-yet-indeterminate amplitude at that point.

However, from Figure 4.3 one sees that by adjusting the amplitude $G(\xi;\xi)$ to an appropriate value, the magnitude of the change in the slope at $x = \xi$ can be made to equal the required amount, which is $-1/p(\xi)$. This determines $G(x;\xi)$ uniquely.

Note, however, that there is *one* circumstance under which $G(x;\xi)$ does *not exist*, namely, when u_1 and u_2 form a linearly dependent set, i.e. when they are related by

$$u_1(x) = k u_2(x) \qquad a \le x \le b \ .$$

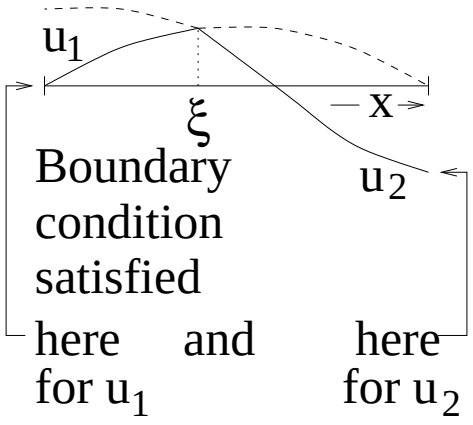

Figure 4.3: Pictorial construction of the Green's function $G(x;\xi)$. At $x = \xi$, where the graphs of the two solutions meet, G must have exactly that amplitude which guarantees that the jump in the slope equals precisely the requisite amount.

It is clear that in this circumstance the continuity condition at $x = \xi$ prevents the existence of any kink at $x = \xi$: regardless how large an amplitude one chooses, the change in the slope will always be zero,

$$\left.\begin{array}{l} u_1(\xi) = k u_2(\xi) \\ c_2 u_2(\xi) - c_1 u_1(\xi) = 0 \end{array}\right\} \Longrightarrow c_2 u_2'(\xi) - c_1 u_1'(\xi) = 0$$

Equation (4.37) will always be violated, and the Green's function does not exist.

If the Green's function does exist (i.e. when u_1 and u_2 form a linearly independent set) then it is given by Eq.(4.35), where c_1 and c_2 are determined uniquely by Eqs.(4.36) and (4.37). This circumstance is summarized by the following

Theorem 4.7.1 (Construction of $G(x;\xi)$)
Given: The functions $u_1(x)$ and $u_2(x)$ which satisfy

$$\begin{array}{rcl} Lu_1(x) & = & 0 \\ \alpha u_1(a) + \alpha' u_1'(a) & = & 0 \end{array} \quad \textit{and} \quad \begin{array}{rcl} Lu_2(x) & = & 0 \\ \beta u_2(b) + \beta' u_2'(b) & = & 0 \end{array} .$$

Conclusion: The Green's function for L is

$$\begin{aligned} G(x;\xi) &= \frac{-1}{c}\begin{cases} u_1(x)u_2(\xi) & \text{for} \quad x<\xi \\ u_1(\xi)u_2(x) & \text{for} \quad \xi<x \end{cases} \\ &\equiv \frac{-1}{c}\ u_1(x_<)u_2(x_>) \end{aligned} \tag{4.38}$$

where

$$c = p(\xi)\ [u_1(\xi)u_2'(\xi) - u_1'(\xi)u_2(\xi)]\ .$$

Remark. (i) The normalization constant c is according to Abel's theorem (Section 1.3.3) always a constant.
(ii) It is evident that the notation introduced in Eq.(4.38),

$$G(x;\xi) = \frac{-1}{c}u_1(x_<)u_2(x_>)$$

is very suggestive. We shall use it repeatedly.
Proof. To verify that the formula given by Eq.(4.38) is the Green's function, simply check that properties (a)-(d) of the Fundamental Theorem are satisfied. Thus

(a) $G(x;\xi)$ obviously satisfies the homogeneous differential equation $LG(x;\xi) = 0$ whenever $x \neq \xi$.

(b) $G(x;\xi)$ does satisfy the given boundary conditions at each endpoint a and b.

(c) $G(x;\xi)$ is obviously continuous.

(d) The derivative $\dfrac{dG}{dx}$ satisfies the correct jump discontinuity at $x = \xi$. Indeed,

$$\begin{aligned} \left.\frac{dG}{dx}\right|^{\xi^+} - \left.\frac{dG}{dx}\right|^{\xi^-} &= \frac{-1}{c}[u_1(\xi)u_2'(\xi^+) - u_1'(\xi^-)u_2(\xi)] \\ &= \frac{-1}{c}\frac{c}{p(\xi)} \\ &= \frac{-1}{p(\xi)} \end{aligned}$$

Thus G as given by formula (4.38) has all the identifying properties of the Green's function indeed.

Example *(Response of a static string)*
Consider the following boundary value problem:

$$\begin{aligned} \frac{d^2u}{dx^2} &= -f(x) \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

Find its Green's function and its solution.
Solution: There are three steps that lead up to the Green's function:

$$\begin{aligned} u''=0 \Rightarrow\ a)\quad u_1 &= x && \text{so that } u_1(0)=0 \\ b)\quad u_2 &= 1-x && \text{so that } u_2(1)=0 \\ c)\quad c &= p(x)\,[u_1u_2' - u_1'u_2] \\ &= 1\,[x(-1)-1(1-x)] \\ &= -1 \end{aligned}$$

Consequently,

$$\begin{aligned} d)\quad G(x;\xi) &= \begin{cases} x(1-\xi) & x<\xi \\ \xi(1-x) & \xi<x \end{cases} \\ &\equiv x_<(1-x_>) \end{aligned}$$

The solution is

$$\begin{aligned} e)\quad u(x) &= \int_0^1 G(x;\xi)f(\xi)\,d\xi \\ &= \int_0^x \xi(1-x)f(\xi)\,d\xi + \int_x^1 x(1-\xi)f(\xi)\,d\xi \end{aligned}$$

Lecture 31

4.8 Unit Impulse Response: General Homogeneous Boundary Conditions

From the viewpoint of technique, the Green's function most easily constructed is the one satisfying the separated boundary conditions. This Green's

function is

$$G(x;\xi) = \frac{-1}{c} u_1(x_<) u_2(x_>) \quad .$$

It satisfies

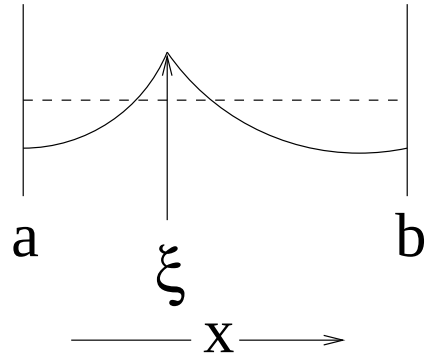

Figure 4.4: Response to a unit impulse applied at $x = \xi$.

$$\begin{aligned} \alpha G(a;\xi) + \alpha' G'(a;\xi) &= 0 \\ \beta G(b;\xi) + \beta' G'(b;\xi) &= 0 \end{aligned}$$

and

$$LG(x;\xi) = -\delta(x-\xi) \quad , \quad \text{where} \quad L = \frac{d}{dx} p(x) \frac{d}{dx} + \gamma(x) \quad .$$

The graph of such a unit impulse response is depicted in Figure 4.4. Such a Green's function is obviously the simplest to construct: find any solution to the homogeneous problem for the left hand interval, then find any solution for the right hand interval, and for all intent and purposes one is done. The only remaining question is: What is the Green's function if the homogeneous boundary conditions are different?

The answer is illustrated by the following problem:

Given: The above Green's function $G(x;\xi)$.

Find: (a) The Green's function $G^R(x;\xi)$ which satisfies the "initial conditions"

$$G^R(a;\xi) = 0$$

and

$$\left. \frac{dG^R(x;\xi)}{dx} \right|^{x=a} = 0$$

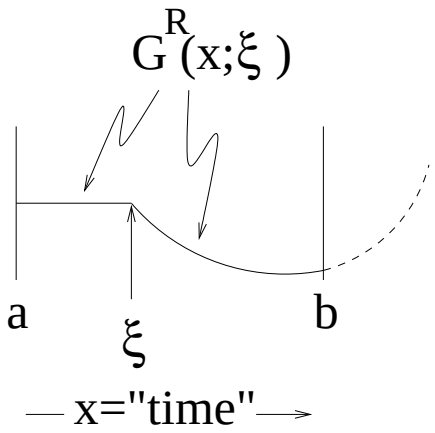

Figure 4.5: Unit impulse response of a system satisfying Dirichlet *and* Neumann conditions at $x = a$. Boundary conditions imposed at the same point are called *initial* conditions. If these two conditions are imposed at the starting point (and x is "time"), then the response is called *causal* or *retarded*. If the two conditions were imposed at a later point, then the response would be called *acausal* or *advanced*.

(b) The Green's function, say G^A, which is *adjoint* to G^R.

(c) The *adjoint* boundary conditions.

(d) A qualitative graph of G^A.

Solution: Use $L(G^R - G) = 0$ and the theorem of Section 4.2.

Remark: If x is the time, then G^R would be the so-called *causal* or *retarded* Green's funtion depicted in Figure 4.5, while G^A would be the so-called *acausal* or *advanced* Green's function.

The general philosophy is this: If it is too difficult to find the Green's function for a desired set of boundary conditions, consider alternative boundary conditions for which the Green's function can readily be found. The desired Green's function is obtained by adding that solution to the homogeneous differential equation which guarantees that the desired boundary conditions are fullfilled.

4.9 The Totally Inhomogeneous Boundary Value Problem

The utility of the Green's function extends to the inhomogeneous boundary value problem where the Dirichlet-Neumann boundary conditions are inhomogeneous.

$$\begin{aligned} Lu &= -f(x) \qquad a < x < b \\ B_1(u) &= d \\ B_2(u) &= e \quad . \end{aligned}$$

The solution is expressed in terms of the Green's function in the previous section, and it is given by

$$u(x) = \int_a^b G(x;\xi)\ f(\xi)\ d\xi + c_1 u_1(x) + c_2 u_2(x) \quad ,$$

where, as before, u_1 and u_2 are two independent solutions to the homogeneous differential equation, and c_1 and c_2 are adjusted so as to satisfy the given boundary condition,

$$\begin{array}{ccccccccc} d & = & B_1(u) & = & 0 & + & 0 & + & c_2 B_1(u_2) \\ e & = & B_2(u) & = & 0 & + & c_1 B_2(u_1) & + & 0 \end{array}$$

Consequently, the solution to the problem is

$$u(x) = \int_a^b G(x;\xi)\ f(\xi)\ d\xi + \frac{e}{B_2(u_1)} u_1(x) + \frac{d}{B_1(u_2)} u_2(x) \quad ,$$

The mathematically most perspicuous aspect of this expression is the fact that it can be written as

$$u(x) = u_p(x) \ + \ u_h(x) \quad .$$

Here $u_p(x)$ is a *particular* solution to the inhomogeneous differential equation,

$$Lu_p(x) = -f(x) \quad ,$$

while $u_h(x)$ is a *complementary* function which satisfies the homogeneous differential equation,

$$Lu_h(x) = 0 \quad .$$

The motivation for adding the appropriate solution of this equation to a particular solution is precisely the one already stated, namely to satisfy the given boundary conditions at the endpoints.

Exercise 4.9.1 (NON-SELFADJOINT BOUNDARY CONDITIONS)
Let $L = -\frac{d^2}{dx^2}$ with boundary conditions $u(0) = 0$, $u'(0) = u(1)$, so that the domain of L is $\mathcal{S} = \{u : Lu \text{ is square integrable; } u(0) = 0,\ u'(0) = u(1)\}$.

(a) For the above differential operator FIND $\mathcal{S}^*$ for the adjoint with respect to

$$\langle v, u \rangle = \int_0^1 \bar{v}\, u\, dx\,.$$

and compare $\mathcal{S}$ with $\mathcal{S}^*$.

(b) COMPARE the eigenvalues λ_n of

$$Lu_n = \lambda_n u_n \qquad n = 0, 1, 2, \ldots$$

with the eigenvalues λ_n^* of

$$L^* v_n = \lambda_n^* v_n \qquad n = 0, 1, 2, \ldots$$

If the two sequences of eigenvalues are different, point out the distinction; if you find they are the same, justify that result.

(c) EXHIBIT the corresponding eigenfunctions.

(d) Is $\lambda = 0$ an eigenvalue? Why or why not?

(e) VERIFY that $\int_0^1 \bar{v}_n u_m\, dx = 0$ for $n \neq m$.

Exercise 4.9.2 (TWO-COMPONENT EIGENVALUE PROBLEM)
Attack the eigenvalue problem

$$\begin{aligned} -u''(x) &= \lambda u(x) \qquad 0 < x < 1 \\ u'(1) &= \lambda u(1) \\ u(0) &= 0 \end{aligned}$$

as follows:

Let $U = \begin{pmatrix} u(x) \\ u_1 \end{pmatrix}$ be a two-component vector whose first component is a twice differentiable function $u(x)$, and whose second component is a real number u_1. Consider the corresponding vector space $\mathcal{H}$ with inner product

$$\langle U, V \rangle \equiv \int_0^1 u(x)v(x)dx + u_1 v_1$$

Let $\mathcal{S} \subset \mathcal{H}$ be the subspace

$$\mathcal{S} = \{U : U = \begin{pmatrix} u(x) \\ u(1) \end{pmatrix}; \quad u(0) = 0\}$$

and let

$$LU = \begin{pmatrix} -u''(x) \\ u'(1) \end{pmatrix} .$$

The above eigenvalue problem can now be rewritten in standard form

$$LU = \lambda U \quad \text{with} \quad U \in \mathcal{S} .$$

(a) PROVE or DISPROVE that L is self adjoint, i.e. that $\langle V, LU \rangle = \langle LV, U \rangle$.

(b) PROVE or DISPROVE that L is positive-definite, i.e. that $\langle U, LU \rangle > 0$ for $U \neq \vec{0}$. (Reminder: "positive-definiteness" applies to *all* vectors, not only to eigenvectors of L.

(c) FIND the (transcendental) equation for the eigenvalues of L.

(d) Denoting these eigenvalues by $\lambda_1, \lambda_2, \lambda_3, \cdots$, EXHIBIT the orthonormalized eigenvectors U_n, $n = 1, 2, 3, \cdots$, associated with these eigenvalues.

Exercise 4.9.3 (ASYMPTOTIC EIGENVALUE SPECTRUM)
The eigenvalue equation for the Exercise on the previous page ("Non-selfadjoint Boundary Value Conditions") is

$$\sin \lambda^{1/2} = \lambda^{1/2}$$

Prove or *disprove* that an asymptotic formula for the roots is

$$\lambda^{1/2} \sim (2m + \frac{1}{2})\pi - \frac{2\log(4m+1)\pi}{(4m+1)\pi} \pm i \log(4m+1)\pi$$

(You might put $\lambda^{1/2} = \alpha + i\beta$ so that

$$\begin{aligned} \sin\alpha \cosh\beta &= \alpha \qquad (1) \\ \cos\alpha \sinh\beta &= \beta \qquad (2) \end{aligned}$$

For large α, Eq. (1) implies β is large. If β is large then $\beta/\sinh\beta$ approaches zero so that $\alpha = (n + \frac{1}{2})\pi + \epsilon_n$ where $\epsilon_n \to 0$. From Eq. (1), since $\cos\epsilon_n \sim 1$, one has $\cosh\beta = (2m + \frac{1}{2})\pi$, where $n = 2m$)

Lecture 32

4.10 Spectral Representation

Once the Green's function for a system has been obtained one knows everything about the system, in particular its set of orthonormal basis. An example from linear algebra illustrates the point.

4.10.1 Spectral Resolution of the Resolvent of a Matrix

Consider the problem

$$(A - \lambda I)\vec{u} = \vec{b}$$

of solving N equations for N unknowns; in other words, given $\vec{b}$ and the matrix A, find $\vec{u}$. Here λ is a fixed parameter.

This problem is solved by solving the alternate problem

$$(A - \lambda I)G_\lambda = I \tag{4.39}$$

for

$$G_\lambda = (A - \lambda I)^{-1} \quad .$$

The matrix G_λ is called the *resolvent* of the operator A.

The solution $\vec{u}$ is given by

$$\vec{u} = G_\lambda \vec{b} = (A - \lambda I)^{-1}\vec{b} \quad .$$

This corresponds to expressing the solution in terms of the Green's function.

Continuing with the illustrative example from linear algebra, let us assume that A is Hermitian. Consequently, it has a complete set of eigen

vectors $\{\vec{\xi}_i : \ i = 1, \cdots, N\}$, and

$$\begin{aligned} A &= \sum_{i=1}^{N} \lambda_i \vec{\xi}_i \vec{\xi}_i^H \\ I &= \sum_{i=1}^{N} \vec{\xi}_i \vec{\xi}_i^H \qquad (4.40) \\ G_\lambda &= \sum_{j=1}^{N} \frac{\vec{\xi}_j \vec{\xi}_j^H}{\lambda_j - \lambda} \qquad (4.41) \end{aligned}$$

are the spectral representations of A, I, and of the resolvent $G_\lambda = (A - \lambda I)^{-1}$. The last one follows from the spectral representation

$$A - \lambda I = \sum_{i=1}^{N} (\lambda_i - \lambda) \vec{\xi}_i \vec{\xi}_i^H$$

and the orthonormality of the eigenvectors:

$$\vec{\xi}_i^H \vec{\xi}_j = \delta_{ij} \quad .$$

That G_λ is the resolvent can be readily checked by verifying Eq.(4.39):

$$\begin{aligned} (A - \lambda I) G_\lambda &= \sum_{i=1}^{N} \sum_{j=1}^{N} (\lambda_i - \lambda) \vec{\xi}_i \vec{\xi}_i^H \frac{\vec{\xi}_j \vec{\xi}_j^H}{\lambda_j - \lambda} \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{\xi}_i \delta_{ij} \vec{\xi}_j^H \frac{\lambda_i - \lambda}{\lambda_j - \lambda} \\ &= \sum_{i=1}^{N} \vec{\xi}_i \vec{\xi}_i^H \\ &= I \end{aligned}$$

It is clear that the resolvent G_λ, Eq.(4.41), viewed as a function of the complex variable λ has singularities which are located at $\lambda_1, \cdots, \lambda_N$, the eigenvalues of A. It follows that a contour integral in the complex λ plane around these eigenvalues will recover the spectral representation of the identity, Eq.(4.40). The complex integration leading to this conclusion is straight

forward. Consider the integral around the contour C,

$$\frac{1}{2\pi i}\oint_C G_\lambda \, d\lambda = \frac{1}{2\pi i}\oint_C \sum_{j=1}^{N} \frac{\vec{\xi}_j \vec{\xi}_j^H}{\lambda_j - \lambda} \, d\lambda \tag{4.42}$$

Let the closed integration contour C be large enough to enclose *all* the

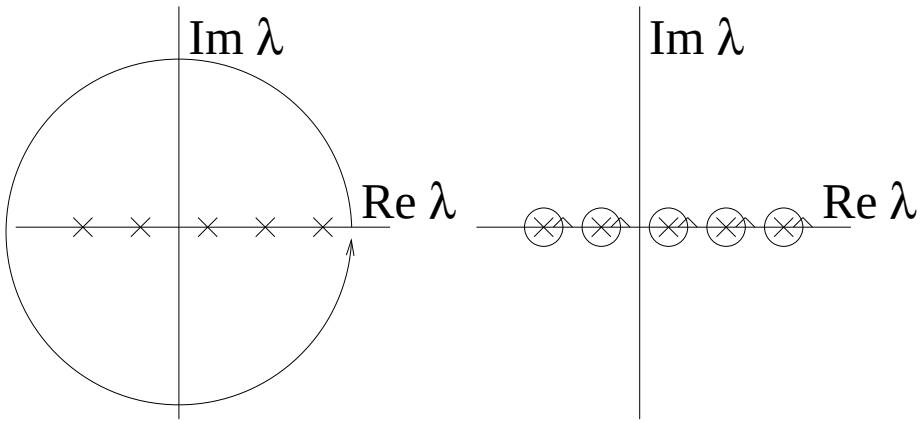

Figure 4.6: The integration path in the complex λ-plane is large enough so as to enclose all the poles of the resolvent. That integration path is deformed into a union of small circles, each one enclosing a single pole.

singularities of G_λ. Use the Cauchy-Goursat theorem to deform the closed integration path without changing the value of the integral. Have the deformed integration path consist of the union of cirles, each one enclosing its respective singularity of G_λ. This is done in Figure 4.6. As a consequence, the integral over the large contour C becomes a sum of integrals, each one over a small circle surrounding a pole of G_λ. Apply Cauchy's integral theorem to each integral. Its value equals the $2\pi i$ times the residue of G_λ at that pole, an eigenvalue of A. The residue at the ith eigenvalue is $-\vec{\xi}_j \vec{\xi}_j^H$. Inserting this result into the right hand side of Eq.(4.42), cancelling out the factor $2\pi i$, one is left with

$$\frac{1}{2\pi i}\oint_C G_\lambda \, d\lambda = -\sum_{j=1}^{N} \vec{\xi}_j \vec{\xi}_j^H$$

This formula is an expression of the following

Proposition: *If the contour encloses* all *singularities of the resolvent* G_λ *in the complex* λ*-plane, then the contour integral yields the* complete *set of* orthonormalized *eigenvectors of* A *or (in view of the completeness of such a set of vectors)*
the resolvent of A *yields, via complex integration, the spectral representation of the identity,*

$$\boxed{\frac{1}{2\pi i}\oint_C G_\lambda \, d\lambda = -Identity} \quad ,$$

where

$$Identity = \sum_{j=1}^{N} \vec{\xi}_j \vec{\xi}_j^H \quad .$$

The uniqueness of the resolvent of A guarantees this result no matter how one obtained G_λ in the first place. Thus, *if one somehow can determine* G_λ*, then by an integration in the complex* λ*-plane one can readily obtain all normalized eigenvectors of the operator* A*.*

Furthermore, it is worth while to emphasize that the nonexistence of G_λ for certain values of λ, far from being a source of trouble or dispair, is, in fact, an inordinate physical and mathematical boon. As we shall see, from the physical point of view, the nonexistence furnishes us with the resonances of the system, while from the mathematical viewpoint it furnishes us with the orthonormalized eigenvectors or eigenfunctions of the system.

Lecture 33

4.10.2 Spectral Resolution of the Green's Function

Instead of representing the Green's function $G_\lambda(x;\xi)$ of a system in terms of two solutions to the homogeneous differential equation, as by Eq.(4.38), we shall now represent it as a generalized Fourier series. The Green's function is a solution to the inhomogeneous boundary value problem

$$\left(\frac{1}{\rho(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx}+q(x)\right]-\lambda\right)u=\frac{f(x)}{\rho(x)}$$

or

$$(\mathcal{L}-\lambda)u=\frac{f}{\rho} \tag{4.43}$$

with

$$\begin{aligned} B_1(u) &= 0 \\ B_2(u) &= 0 \end{aligned}$$

Here $\rho(x)$ is the weight function of the Sturm-Liouville differential equation.

We are led to the spectral representation of G_λ by the following three-step line of reasoning:

(1) The non-trivial solutions to the homogeneous boundary value problem are the eigenfunctions, $u_n(x) \quad n = 1, 2, \cdots$, each of which satisfies

$$\left.\begin{aligned} \mathcal{L}u_n &= \lambda u_n \\ B_1(u_n) &= 0 \\ B_2(u_n) &= 0 \end{aligned}\right\} \quad n = 1, 2, \cdots$$

These eigenfunction are used to represent the solution $u(x)$ as a generalized Fourier series,

$$u = \sum_{m=1}^{\infty} c_m u_m \quad ,$$

whose coefficients c_n are to be determined. To obtain them, take the inner product of Eq.(4.43) with these eigenfunctions u_n,

$$\langle u_n, \mathcal{L}u\rangle - \lambda\langle u_n, u\rangle = \langle u_n, f/\rho\rangle \tag{4.44}$$

(2) For illustrative purposes consider the case where $\mathcal{L}$ is *self-adjoint* with respect to the inner product

$$\langle u_n, u\rangle \equiv \int_a^b \overline{u_n}(x)\, u(x)\, \rho(x)\, dx$$

Consequently, the first term of Eq.(4.44) is

$$\begin{aligned} \langle u_n, \mathcal{L}u\rangle &= \langle \mathcal{L}u_n, u\rangle \\ &= \lambda_n\langle u_n, u\rangle \\ &= \lambda_n c_n \quad . \end{aligned}$$

The equation itself becomes

$$(\lambda_n - \lambda)c_n = \langle u_n, f/\rho\rangle$$

and thus determines the coefficient c_n. Thus the Fourier series representation of the solution is

$$\boxed{u(x) = \sum_{n=1}^{\infty} u_n \frac{\langle u_n, f/\rho \rangle}{\lambda_n - \lambda}}$$

(3) This representation can be applied to the Green's function. Letting $f(x) = \delta(x-\xi)/\rho(x)$, one obtains

$$\langle u_n, \frac{\delta(x-\xi)}{\rho(x)} \rangle = \overline{u}_n(\xi) \rangle \quad .$$

Thus the Fourier series becomes

$$\boxed{G_\lambda(x;\xi) = \sum_{n=1}^{\infty} \frac{u_n(x)\overline{u}_n(\xi)}{\lambda_n - \lambda}} \tag{4.45}$$

This is the *spectral representation* of the Green's function.

For the purpose of comparison consider the spectral representation of the identity. It is obtained from the Fourier representation of a generic function satisfying the given boundary conditions,

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} u_n(x) c_n \\ &= \sum_{n=1}^{\infty} u_n(x) \int_a^b \overline{u}_n(\xi) \rho u(\xi) \, d\xi \\ &= \int_a^b \sum_{n=1}^{\infty} u_n(x) \rho \overline{u}_n(\xi) u(\xi) \, d\xi \quad . \end{aligned}$$

This holds for any function u. Consequently, the expression in front of $u(\xi)$ is the Dirac delta,

$$\sum_{n=1}^{\infty} u_n(x) \rho \overline{u}_n(\xi) = \delta(x-\xi)$$

or

$$\boxed{\frac{\delta(x-\xi)}{\rho(\xi)} = \sum_{n=1}^{\infty} u_n(x)\overline{u}_n(\xi)}$$

This is the *spectral representation* of the identity operator in the Hilbert space, the same as Eq.(2.20) on page 87, which was obtained by essentially the same line of reasoning. Note the perspicuous similarity with the Green's function $G_\lambda(x;\xi)$, the resolvent of the Sturm-Liouville operator in this Hilbert space.

Exercise 4.10.1 (REPRESENTATION VIA EIGENFUNCTIONS) Consider the eigenvalue problem

$$Lu = \lambda u \quad L = \alpha\frac{d^2}{dx^2} + \beta\frac{d}{dx} + \gamma$$

$$B_1(u) = 0$$

$$B_2(u) = 0$$

and its adjoint

$$L^*v = \bar{\lambda}v$$

$$B_1^*(v) = 0$$

$$B_2^*(v) = 0$$

with respect to the inner product $\langle v, u\rangle = \int_a^b \bar{v}(x)u(x)dx$. One can show, and you may safely assume, that the eigenvalue spectra of these two problems are complex conjugates of each other.

(a) Prove that the solution $u(x;\lambda)$ for the problem

$$Lu - \lambda u = -f(x)$$

$$B_1(u) = 0\,; \quad B_2(u) = 0$$

is given by

$$u(x;\lambda) = \sum_n \frac{\langle v_n, f\rangle}{\lambda - \lambda_n} u_n(x)$$

Here u_n and v_n are the eigenfunctions of L and L^* and they have been normalized by the condition

$$\langle v_n,\, u_m\rangle = \delta_{nm}\,.$$

(b) Show that the Green's function is

$$G_\lambda(x|\xi) = \sum_n \frac{u_n(x)\bar{v}_n(\xi)}{\lambda_n - \lambda}$$

Exercise 4.10.2 (NORMAL MODE PROFILES VIA COMPLEX INTEGRATION) Obtain the *o.n.* set of eigenfunctions for the Sturm-Liouville problem

$$Lu \equiv -\frac{d^2u}{dx^2} = \omega^2 u$$

$$u(a) = u(b) = 0$$

by applying the complex integration technique to the Green's function $G_\omega(x;\xi)$

$$(L-\omega^2)G \equiv -\frac{d^2G_\omega}{dx^2} - \omega^2 G_\omega = \delta(x-\xi) \qquad a < x\,, \quad \xi < b$$

$$\begin{matrix} G_\omega(a;\xi) = 0 \\ G_\omega(b;\xi) = 0 \end{matrix} \qquad a < \xi < b$$

4.10.3 Green's Function as the Fountainhead of the Eigenvalues and Eigenvectors of a System

The spectral representation of G is a second way of writing the Green's function. The first way was Eq.(4.38) on page 253 in terms of the two independent solutions ("elements of the null space") of the homogeneous Eq.(4.43) on page 263 with $f = 0$.

This raises an important question: Are the null space representation and the spectral representation really one and the same function? The answer is, of course, "yes" because of the uniqueness of the Green's function for a given problem. This fact was the subject of Green's function uniqueness theorem on page 248.

We shall now take advantage of this uniqueness in order to obtain from the null space representation of the Green's function the complete set of orthonormal eigenfunctions of the linear system. The beauty of this method is that one directly obtains the eigenvalues and these eigenfunctions without having to evaluate any normalization integrals. Thus the Green's function of a system lives up to its reputation of being able to give everything one wants

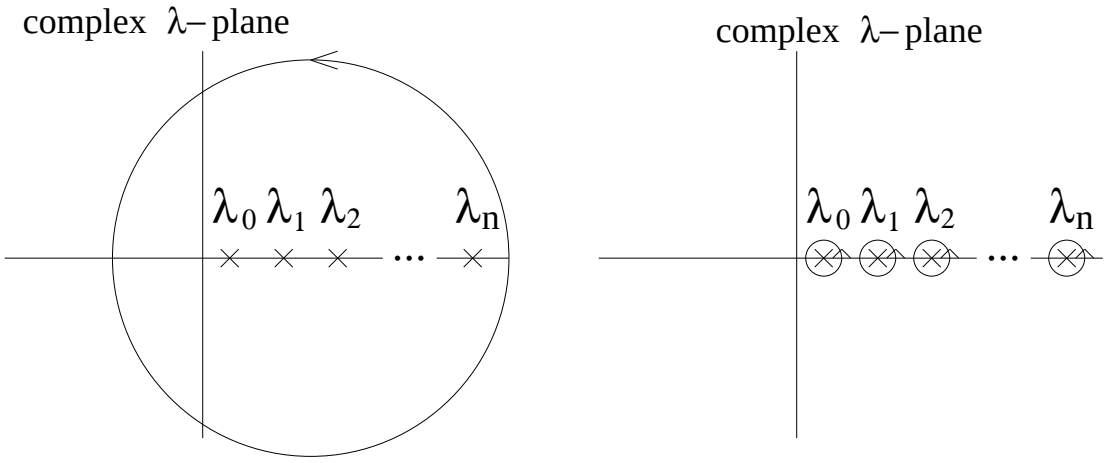

Figure 4.7: The integration path in the complex λ-plane is to be large enough to enclose all the eigenvalues of the Sturm-Liouville problem, the poles of the Green's function. That integration path is deformed into a union of small circles, each one enclosing a single eigenvalue.

to know about the internal workings of a linear system but never dared to ask.

However, the story does not end there. The Green's function gives one a quick method for checking the completeness of a set of eigenfunctions, i.e. whether they span the whole vector space of functions satisfying the same homogeneous boundary conditions. This completeness was already validated in Section 1.7 (page 54) using Rayleigh's quotient as the starting point. But with the Green's function at hand, the completeness property can be easily validated by merely evaluating an integral.

Spectral Representation

Consider the spectral representation of the Green's function $G_\lambda(x;\xi)$. Viewed as a function of the variable λ, this function has poles in the complex λ-plane. These poles are the eigenvalues of the Sturm-Liouville problem. If the problem is self-adjoint, these poles lie along the real axis (Theorem 2 on page 24). However, in general they may lie anywhere in the complex plane.

Suppose we consider the contour integral

$$\frac{1}{2\pi i}\oint G_\lambda(x;\xi)\ d\lambda$$

around a closed integration path which is large enough to enclose all the poles of the Green's function. According to the Cauchy-Goursat theorem, if one deforms the integration contour of this integral from the large circle in Figure 4.7 into the union of the little circles around the poles $\lambda_0, \lambda_1, \lambda_2, \cdots$ of the integrand, then the value of the integral will not change; in other words,

$$\begin{aligned} \frac{1}{2\pi i}\oint_C G_\lambda(x;\xi)\, d\lambda &= \frac{1}{2\pi i}\oint_{\bigcup_n C_n} G_\lambda(x;\xi)\, d\lambda \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\oint_{C_n} G_\lambda(x;\xi)\, d\lambda \end{aligned} \tag{4.46}$$

Each term on the right hand side equals the residue of G_λ at its respective pole λ_n. According to Eq.(4.45) this residue is

$$\begin{aligned} \operatorname*{Res}_{\lambda=\lambda_n} G_\lambda(x;\xi) &= \lim_{\lambda\to\lambda_n} (\lambda-\lambda_n) G_\lambda(x;\xi) \\ &= -u_n(x)\overline{u}_n(\xi) \end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i}\oint_{C_n} G_\lambda(x;\xi)\, d\lambda = -\sum_{n=0}^{\infty} u_n(x)\overline{u}_n(\xi) \tag{4.47}$$

Consequently, the contour integral, Eq.(4.46), is

$$\boxed{\frac{1}{2\pi i}\oint_C G_\lambda(x;\xi)\, d\lambda = -\sum_{n=0}^{\infty} u_n(x)\overline{u}_n(\xi)} \quad , \tag{4.48}$$

This is a remarkable relation. It says that if one somehow can determine the λ-parametrized Green's function for the problem

$$\begin{aligned} \left(\frac{d}{dx}p(x)\frac{d}{dx} - q(x) + \lambda\rho(x)\right) G_\lambda(x;\xi) &= -\delta(x-\xi) \\ B_1(G_\lambda) &= 0 \\ B_2(G_\lambda) &= 0 \quad , \end{aligned}$$

then this function yields the corresponding complete set of *orthonormalized* eigenfunctions of the Sturm-Liouville operator. This can be done whenever one can find a closed expression for $G_\lambda(x;\xi)$. The example below illustrates this.

Completeness via Green's function

What can one say about the value of $\frac{1}{2\pi i}\oint_C G_\lambda(x;\xi)\ d\lambda$ on the left hand side? If one knows that the set of eigenfunctions $\{u_n(x)\}$ forms a complete set, then Eq.(2.20) on page 87 tells us that

$$\frac{\delta(x-\xi)}{\rho(x)} = \sum_{k=1}^{\infty} u_k(x)\overline{u}_k(\xi)\ . \tag{4.49}$$

This is a necessary and sufficient condition for completeness. Combining it with Eq.(4.48), one finds that

$$\boxed{\oint_C G_\lambda(x;\xi)\ d\lambda = \frac{\delta(x-\xi)}{\rho(x)}}\ . \tag{4.50}$$

This is the new criterion for completeness:
Equation (4.50) holds if and only if the spectral representation of G_λ is based on a complete set of eigenfunctions.
This means that, if upon evaluating the left hand side of Eq.(4.46) along an asymptotically infinite circular contour C, one finds that Eq.(4.50) holds, then one in guaranteed that the set of eigenfunctions obtained from Eq.(4.47) forms a complete set. The example of a free string in the next section illustrates this computational criterion.

4.10.4 String with Free Ends: Green's Function, Spectrum, and Completeness

Consider the following boundary value problem:

$$\begin{aligned} \frac{d^2u}{dx^2} + \lambda u &= 0 \\ u'(0) &= 0 \\ u'(\ell) &= 0\ . \end{aligned}$$

Green's Function

To construct the Green's function for this system, find any two solutions, say $w(x,\lambda)$ and $z(x,\lambda)$, which satisfy the two respective boundary conditions:

$$\begin{aligned} w(x,\lambda) &= \cos\sqrt{\lambda}x\ , \\ z(x,\lambda) &= \cos\sqrt{\lambda}(x-\ell)\ . \end{aligned}$$

The Green's function is

$$\begin{aligned} G_\lambda(x;\xi) &= \frac{-1}{c} w(x_<,\lambda) z(x_>,\lambda) \\ &= \frac{-1}{c(\lambda)} \begin{cases} \cos\sqrt{\lambda}x \ \cos\sqrt{\lambda}(\xi-\ell) & x \le \xi \\ \cos\sqrt{\lambda}\xi \ \cos\sqrt{\lambda}(x-\ell) & \xi \le x \end{cases} . \end{aligned}$$

Here

$$\begin{aligned} c(\lambda) &= p(x) \ (w(x)z'(x) \ - \ w'(x)z(x)) \\ &= 1 \ \left(\cos\sqrt{\lambda}x \ (-)\sqrt{\lambda}\sin\sqrt{\lambda}(x-\ell) \ - \ \sqrt{\lambda}\sin\sqrt{\lambda}x\cos\sqrt{\lambda}(x-\ell)\right) \\ &= -\sqrt{\lambda} \ \sin\sqrt{\lambda}(x-\ell-x) \ = \ \sqrt{\lambda}\sin\sqrt{\lambda}\ell \end{aligned}$$

Thus

$$G_\lambda(x;\xi) = \begin{cases} -\dfrac{\cos\sqrt{\lambda}x \ \cos\sqrt{\lambda}(\xi-\ell)}{\sqrt{\lambda}\sin\sqrt{\lambda}\ell} & x \le \xi \\ -\dfrac{\cos\sqrt{\lambda}\xi \ \cos\sqrt{\lambda}(x-\ell)}{\sqrt{\lambda}\sin\sqrt{\lambda}\ell} & \xi \le x \end{cases} .$$

Spectrum via Green's Function

In order to evaluate the contour integral of $G_\lambda(x;\xi)$ one must know its singular points in the complex λ-plane. It is clear that on this domain the Green's function has the form

$$G_\lambda(x;\xi) = \frac{g(\lambda)}{c(\lambda)} \ ,$$

where both $g(\lambda)$ and $c(\lambda)$ are analytic for all λ, even though each one depends manifestly on the nonanalytic function $\sqrt{\lambda}$. Thus the singular points of G_λ are located at the zeroes of $c(\lambda)$, the eigenvalues of the Sturm-Liouville system:

$$c(\lambda) \equiv \sqrt{\lambda}\sin\sqrt{\lambda}\ell = 0 \quad \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 0,1,2,\cdots \ .$$

At these points $c'(\lambda_n) \neq 0$. Consequently, $\lambda = \lambda_n$ is a simple pole in whose neighborhood the ratio g/c has the expansion

$$G_\lambda = \frac{g(\lambda)}{c(\lambda)} = \frac{\alpha_0}{\lambda - \lambda_n} + \alpha_1 + \alpha_2(\lambda - \lambda_n) + \cdots \ .$$

Here α_0 is the residue of G_λ, and one must find it. To do this, consider

$$\frac{g(\lambda)}{c(\lambda)}(\lambda - \lambda_n) = \alpha_0 + \alpha_1(\lambda - \lambda_n) + \alpha_2(\lambda - \lambda_n)^2 + \cdots$$

and take the limit. Thus

$$\begin{aligned} \alpha_0 &= \lim_{\lambda \to \lambda_n} \frac{g(\lambda)(\lambda - \lambda_n)}{c(\lambda)} \\ &= \lim_{\lambda \to \lambda_n} \frac{g(\lambda)}{c'(\lambda)} \\ &= \frac{g(\lambda_n)}{c'(\lambda_n)} \end{aligned}$$

where the second step used L'Hospital's rule. The residue of G_λ is therefore

$$\operatorname*{Res}_{\lambda = \lambda_n} \quad G_\lambda(x;\xi) = \frac{g(\lambda_n)}{c'(\lambda_n)}$$

Its evaluation is based on the following expressions

$$\begin{aligned} g(\lambda) &= -\cos\sqrt{\lambda}x_< \ \cos\sqrt{\lambda}(x_> - \ell) \\ c(\lambda) &= \sqrt{\lambda}\sin\sqrt{\lambda}\ell \\ c(\lambda_n) &= 0 \quad \rightarrow \quad \sqrt{\lambda_n} = \frac{n\pi}{\ell}, \quad n = 0, 1, 2, \cdots \\ c'(\lambda_n) &= \frac{1}{2\sqrt{\lambda}}\sin\sqrt{\lambda}\ell \ + \ \sqrt{\lambda}(\cos\sqrt{\lambda}\ell)\frac{\ell}{2\sqrt{\lambda}}\bigg|_{\lambda=\lambda_n} \\ &= \frac{\ell}{2}\cos n\pi \quad \text{when} \quad \lambda_n \neq 0 \\ &= \frac{\ell}{2} + \frac{\ell}{2} \quad \text{when} \quad \lambda_n = 0 \end{aligned}$$

It follows that G_λ has a closed contour integral given by

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C G_\lambda(x;\xi)\,d\lambda &= -\frac{1}{\ell} - \sum_{n=1}^{\infty} \frac{\cos\sqrt{\lambda}x_<\ \cos\sqrt{\lambda}(x_>-\ell)}{c'(\lambda)}\bigg|_{\sqrt{\lambda}=\frac{n\pi}{\ell}} \\
&= -\frac{1}{\ell} - \sum_{n=1}^{\infty} \frac{\cos\frac{n\pi}{\ell}x_<\ \cos\frac{n\pi}{\ell}(x_>-\ell)}{\frac{\ell}{2}\cos n\pi} \\
&= -\frac{1}{\ell} - \frac{2}{\ell}\sum_{n=1}^{\infty} \frac{\cos\frac{n\pi}{\ell}x_<\ \cos\frac{n\pi}{\ell}x_>\cos n\pi}{\cos n\pi} \\
&= -\frac{1}{\ell} - \frac{2}{\ell}\sum_{n=1}^{\infty} \cos\frac{n\pi}{\ell}x\ \cos\frac{n\pi}{\ell}\xi \qquad (4.51)
\end{aligned}$$

Compare this bilinear expression with the fundamental formula, Eq.(4.48) on page 269, and read out the complete set of orthonormalized eigenfunctions

$$\{u_m(x)\} = \left\{\sqrt{\frac{1}{\ell}}, \sqrt{\frac{2}{\ell}}\cos\frac{m\pi}{\ell}x \quad : \quad m = 1, 2, \cdots\right\} \quad .$$

Completeness

To validate the completeness of these eigenfunctions one must evaluate

$$\frac{1}{2\pi i}\oint_C G_\lambda(x;\xi)\,d\lambda = \frac{1}{2\pi i}\oint_C \frac{-\cos\sqrt{\lambda}x\ \cos\sqrt{\lambda}(\xi-\ell)}{\sqrt{\lambda}\sin\sqrt{\lambda}\ell}d\lambda \qquad (4.52)$$

along the very large (in the limit infinite) circular contour

$$C = \{\lambda = Re^{i\theta} : \quad 0 < \theta < 2\pi\}$$

before we deformed it into $\bigcup_{n=0}^{\infty} C_n$. This evaluation is facilitated by introducing

$$\sqrt{\lambda} = k \ \text{ and hence } \ \frac{d\lambda}{\sqrt{\lambda}} = 2dk \ .$$

This transforms the integration contour into a very large semicircle

$$\begin{aligned}
k &= \sqrt{R}e^{i\phi} \\
&= \sqrt{R}\cos\phi + i\sqrt{R}\sin\phi, \quad 0 < \phi < \pi \qquad (4.53)
\end{aligned}$$

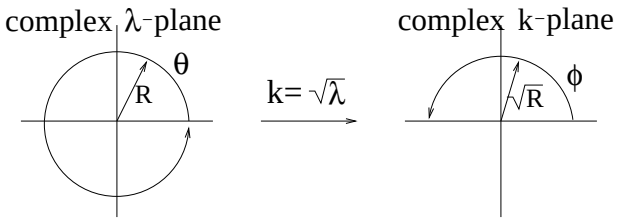

Figure 4.8: Integration contour C in the λ-plane and its semicircular image the k-plane

The integral to be evaluated is therefore

$$\frac{1}{2\pi i}\oint_C G_\lambda(x;\xi)\,d\lambda = \frac{-1}{2\pi i}\int_{\sqrt{R}}^{\sqrt{R}e^{i\pi}} \underbrace{\frac{\cos kx\ \cos k(\xi-\ell)}{\sin k\ell}}_{\frac{N}{D}} 2dk \tag{4.54}$$

In light of Eq.(4.53) one finds that

$$\cos kx = \frac{e^{ikx}+e^{-ikx}}{2} \quad \to \quad \frac{e^{-ikx}}{2} \quad \text{as} \quad \sqrt{R}\to\infty \tag{4.55}$$

$$\cos k(\xi-\ell) = \frac{e^{ik(\xi-\ell)}+e^{-ik(\xi-\ell)}}{2} \quad \to \quad \frac{e^{ik(\xi-\ell)}}{2} \quad \text{as} \quad \sqrt{R}\to\infty \tag{4.56}$$

$$\sin k\ell = \frac{e^{ik\ell}-e^{-ik\ell}}{2i} \quad \to \quad -\frac{e^{-ik\ell}}{2i} \quad \text{as} \quad \sqrt{R}\to\infty \tag{4.57}$$

so that

$$\tag{4.58}$$

$$\frac{N}{D} \quad \to \quad \frac{(e^{-ikx}/2)(e^{ik(\xi-\ell)}/2)}{-(e^{-ik\ell})/2} = \frac{-i}{2}e^{ik(\xi-x)} \tag{4.59}$$

Consequently,

$$\frac{1}{2\pi i}\oint_C G_\lambda(x;\xi)\,d\lambda = \frac{-1}{2\pi}\int_{-\sqrt{R}}^{\sqrt{R}} e^{ik(\xi-x)}dk \tag{4.60}$$

The integrand is analytic in the semidisk bounded by the semicircle and the the real interval $[-\sqrt{R},\sqrt{R}]$ as in the righthand picture of Figure 4.8. Thus one can use the Cauchy-Goursat theorem to deform the semicircular contour

into a straight line just barely above the real k-axis. This changes Eq.(4.60) into an integral along the real axis,

$$\frac{1}{2\pi i}\oint_C G_\lambda(x;\xi)\, d\lambda = \frac{-1}{2\pi}\int_{-\sqrt{R}}^{\sqrt{R}} e^{ik(\xi-x)}dk \;\rightarrow\; -\delta(\xi-x) \text{ as } R\to\infty\ . \tag{4.61}$$

This shows that the set of orthonormal eigenfunctions forms a complete set. Indeed, comparing this expression with Eq.(4.51) one has

$$\delta(x-\xi) = \frac{1}{\ell} + \frac{2}{\ell}\sum_{m=1}^{\infty}\cos\frac{m\pi x}{\ell}\ \cos\frac{m\pi\xi}{\ell} \quad \text{whenever} \quad 0<x,\xi<\ell\ .$$

This is the requisite completeness relation.

From a different perspective this relation is also the spectral representation of the Dirac delta function, which one may compare with that of the Green's function,

$$G_\lambda(x;\xi) = -\frac{1}{\ell(\lambda-0)} - \frac{2}{\ell}\sum_{m=1}^{\infty}\frac{\cos\dfrac{m\pi x}{\ell}\ \cos\dfrac{m\pi\xi}{\ell}}{(\lambda-\dfrac{m^2\pi^2}{\ell^2})}\ .$$

Lecture 34

4.11 Boundary Value Problem via Green's Function: Integral Equation

It is difficult to overstate the importance of Green's functions. This is true in particular in regard to their role in boundary value problems, be they scattering problems, where the domain is infinite (*exterior* boundary value problems), or cavity problems, where the domain is finite (*interior* boundary value or bound state problems).

4.11.1 One-dimensional Scattering Problem: Exterior Boundary Value Problem

Scattering of radiation by bodies is ubiquitous. The mathematical formulation of this process can be reduced, more often than not, to an *exterior* boundary value problem, namely finding the solution to the following homogeneous b.v. problem:

$$-\frac{d}{dx}p(x)\frac{d\psi}{dx} - v(x)\psi = \lambda\rho(x)\psi \quad a < x < \infty \tag{4.62}$$

$$\psi(a) = 0 \tag{4.63}$$

$$\lim_{x\to\infty}\psi(x) = finite \tag{4.64}$$

The scattering is due to $v(x)$, the potential of the body. The body is finite. Consequently, its potential vanishes for large x:

$$\lim_{x\to\infty} v(x) = 0 \ .$$

If the scattering body is absent, there is no scattering potential at all. In that case the boundary value problem is

$$-\frac{d}{dx}p(x)\frac{d\psi_0}{dx} = \lambda\rho(x)\psi_0 \quad a < x < \infty \tag{4.65}$$

$$\psi_0(a) = 0 \tag{4.66}$$

$$\lim_{x\to\infty}\psi_0(x) = finite \ . \tag{4.67}$$

The solution to this problem, ψ_0, is called the unscattered solution or the *incident wave*. It is characterized by the physical parameter λ, which usually expresses a squared frequency, a squared wave number, an energy or something else, depending on the nature of the wave.

If the scattering body is present, $v(x) \neq 0$. To find the scattered wave, i.e. the solution to the homogeneous Eqs.(4.62)-(4.64), one writes this system with the help of Eqs.(4.65)-(4.67) in the form

$$\frac{d}{dx}p(x)\frac{d\{\psi(x)-\psi_0(x)\}}{dx} + \lambda\rho(x)\{\psi(x)-\psi_0(x)\} = -v(x)\psi \quad a < x < \infty \tag{4.68}$$

$$\{\psi(a)-\psi_0(a)\} = 0 \tag{4.69}$$

$$\lim_{x\to\infty}\{\psi(x)-\psi_0(x)\} = 0 \ , \tag{4.70}$$

and views the r.h.s., $v(x)\psi(x)$, as an inhomogeneity for the corresponding Green's function problem

$$\left[\frac{d}{dx}p(x)\frac{d\psi}{dx}+\lambda\rho(x)\right]G_\lambda(x;\xi)=-\delta(x-\xi)\quad a<x<\infty$$
$$G_\lambda(a;\xi)=0$$
$$\lim_{x\to\infty}G_\lambda(x;\xi)=0\ .$$

Solutions to problems like this one are discussed in the next Section 4.12.3 starting on page 291. It follows from Eq.(4.34) on page 247 that

$$\psi(x)=\psi_0(x)+\int_a^\infty G_\lambda(x;x')v(x')\psi(x')\,dx'\ . \tag{4.71}$$

However, unlike Eq.(4.34), Eq.(4.71) is not an explicit solution because the unknown function ψ appears inside the integral. The physical reason is that the source of the scattered wave on the r.h.s. of Eq.(4.68) is nonzero if and only if an incident wave ψ_0 is present, i.e.

$$\psi_0(x)\neq 0\Longleftrightarrow\psi(x)\neq 0\quad a<x<\infty\ .$$

Equation (4.71) is an *integral equation* for the to-be-determined solution $\psi(x)$. This equation not only implies the differential Eq.(4.62), but also the associated boundary conditions. In fact, the integral equation is mathematically equivalent to the homogeneous boundary value problem, Eqs.(4.62)-(4.64).

The reformulation in terms of an integral equation constitutes a step forward. By condensing three concepts into one, one has implemented the principle of unit-economy[5], and thereby identified the essence – the most consequential aspect – of the external boundary value problem. That this is indeed the case is borne out by the fact that Eq.(4.71) lends itself to being solved by a process of iteration without having to worry about boundary conditions.

The first iterative term, with $\psi(x')$ inside the integral replaced by $\psi_0(x')$, yields what in scattering theory is called the *first Born approximation*:

$$\begin{aligned}\psi_1(x)&=\psi_0(x)+\int_a^\infty G_\lambda(x;x')v(x')\psi_0(x')\,dx'\\&=\psi_0(x)+\Delta^{(1)}\psi(x)\ .\end{aligned}$$

[5]As identified in the footnote on page 192

Successive terms in this iteration yield

$$\begin{aligned}
\psi_2(x) &= \psi_0(x) + \int_a^\infty G_\lambda(x;x')v(x')\psi_1(x')\,dx' \\
&= \overbrace{\psi_0(x) + \int_a^\infty G_\lambda(x;x')v(x')\psi_0(x')\,dx'}^{\psi_1(x)} \\
&\qquad + \int_a^\infty \int_a^\infty G_\lambda(x;x')v(x')G_\lambda(x';x'')v(x'')\psi_0(x'')\,dx''dx' \\
&\equiv \psi_0(x) + \Delta^{(1)}\psi(x) + \Delta^{(2)}\psi(x) \\
&\vdots
\end{aligned}$$

The n^{th} iteration involves a multiple integral of the form

$$\begin{aligned}
\Delta^{(n)}\psi(x) = \int_a^\infty \int_a^\infty \cdots \int_a^\infty & G_\lambda(x;x')v(x')G_\lambda(x';x'')v(x'') \\
& \cdots G_\lambda(x^{(n-1)};x^{(n)})v(x^{(n)})\psi_0(x^{(n)})\,dx^{(n)}\cdots dx''dx'
\end{aligned}$$

For $n = 1$ such a term corresponds to a *scattering process* in which the incident wave is scattered by the potential at x' before it arrives at x. The integration over x' expresses the fact that the total total wave amplitude at x is a linear superposition of the waves due to the scattering process taking place at x'.

By induction one concludes that for any n such a term refers to a *multiple scattering process*: the incident wave is scattered by the potential at $x^{(n)}, x^{(n-1)}, \cdots, x'', x'$ before it arrives at x, where it is observed.

Thus the solution to the external boundary value problem, Eq.(4.71), has the form

$$\psi(x) = \psi_0(x) + \Delta^{(1)}\psi(x) + \Delta^{(2)}\psi(x) + \cdots + \Delta^{(n)}\psi(x) + \cdots\ .$$

The scattered wave is represented by a *Born series*, a sum of the unscattered wave $\psi_0(x)$, a wave $\Delta^{(1)}\psi(x)$ that was scattered once, a wave $\Delta^{(2)}\psi(x)$ that was scattered twice, and so on. The Born series converges if the scattering potential is small enough. By truncating this series one obtains an approximate solution to the given exterior boundary value problem.

4.11.2 One-dimensional Cavity Problem: Interior Boundary Value Problem

The process of solving the inhomogeneous boundary value problem

$$\begin{aligned} \frac{d}{dx}p\frac{du}{dx} + [q(x) + \lambda\rho(x)]u &= -f(x) \qquad a < x < b \\ B_1(u) &= d \\ B_2(u) &= e \end{aligned}$$

is somewhat awkward from a numerical and even a conceptual point of view. Solving the differential equation is a local process: one determines the function and its properties at $x + dx$ from those at x. One repeats this step-like process until one has found $u(x)$ for $a \leq x \leq b$. Upon completion one checks whether the boundary conditions B_1 and B_2 have been satisfied. If not, one alters the function u at the point where one started solving the differential equation and then starts all over again. Thus one might have to solve the differential equation many times before one finally obtains the solution to the desired degree of accuracy.

It is evident that this undersirable drudgery is due to the fact that the key property, boundary conditions, which determine the qualitatively important features of the solution u, are stated separately and are not an intrinsic part of the differential equation.

This deficiency can be removed by recasting the boundary value problem in the form of an integral equation. The one-dimensional Sturm-Liouville system with, say, inhomogeneous Dirichlet boundary conditions,

$$\begin{aligned} \frac{d}{dx}p\frac{du}{dx} + [q(x) + \lambda\rho(x)]u &= 0 \\ u(a) &= d \\ u(b) &= e\,, \end{aligned}$$

illustrates the general principle. To convert this sytem into a single integral equation, one considers the corresponding Green's function problem

$$\begin{aligned} \left[\frac{d}{dx}p\frac{d}{dx} + q(x)\right] G(x;\xi) &= -\delta(x-\xi) \\ G(a;\xi) &= 0 \\ G(b;\xi) &= 0\,. \end{aligned}$$

One transposes the term $\lambda\rho(x)u(x)$ to the right hand side of the S-L equation and considers it as an inhomogeneous equation. Multiply this equation by $G(x;\xi)$, multiply the Green's function equation by $u(x)$. One finds

$$G(x;\xi)\left[\frac{d}{dx}p\frac{du}{dx}+q(x)u\right]=-\lambda G(x;\xi)\rho(x)u$$

and

$$u(x)\left[\frac{d}{dx}p\frac{d}{dx}+q(x)\right]G(x;\xi)=-\delta(x-\xi)u(x)\ .$$

Upon subtracting one finds that the $q(x)$-terms cancel and that the left hand side becomes a total derivative (Lagrange's identity!):

$$\begin{aligned}\text{l.h.s.} &= G(x;\xi)\frac{d}{dx}p\frac{du}{dx}-u(x)\frac{d}{dx}p\frac{dG(x;\xi)}{dx}\\ &= \frac{d}{dx}\left\{G(x;\xi)p(x)\frac{du(x)}{dx}-u(x)p(x)\frac{dG(x;\xi)}{dx}\right\}\ .\end{aligned}$$

The r.h.s. becomes

$$\text{r.h.s.}=-\lambda G(x;\xi)\rho(x)u(x)+\delta(x-\xi)u(x)\ .$$

Integration of l.h.s.=r.h.s. yields

$$p(x)\left[G(x;\xi)\frac{du(x)}{dx}-\frac{dG(x;\xi)}{dx}u(x)\right]_{x=a}^{x=b}=-\lambda\int_a^b G(x;\xi)\rho(x)u(x)dx+u(\xi)\,.$$

Switching variables $x\leftrightarrow\xi$, one finds

$$\begin{aligned}u(x) &= \lambda\int_a^b G(\xi;x)\rho(\xi)u(\xi)d\xi \\ &\quad +p(a)u(a)\left.\frac{dG(\xi;x)}{d\xi}\right|_{\xi=a}-p(b)u(b)\left.\frac{dG(\xi;x)}{d\xi}\right|_{\xi=b}\ .\end{aligned}\tag{4.72}$$

This is an integral equation for $u(x)$. Note that the boundary conditions for $u(x)$ are an intrinsic part of the equation: the boundary conditions do not have to be stated separately. Also note that if $u(x)$ satisfies the homogeneous Dirichlet conditions $u(a)=0$, $u(b)=0$, then the integral equation becomes

$$u(x) = \lambda\int_a^b G(x;\xi)\rho(\xi)u(\xi)d\xi\,,\tag{4.73}$$

which is an eigenvalue equation for the function u.

4.11.3 Eigenfunctions via Integral Equations

To illustrate this integral equation, consider the boundary value problem for several of the familiar orthogonal functions.

1. Trigonometric functions:

$$\begin{aligned} \frac{d^2u}{dx^2} + \lambda u &= 0 \qquad u(0) = u(\ell) = 0 \\ u(x) &= \lambda \int_0^\ell G(x;\xi) u(\xi) d\xi \\ G(x;\xi) &= \frac{1}{\ell} \begin{cases} x(\ell - \xi) & \text{when } x < \xi \\ \xi(\ell - x) & \text{when } \xi < x \end{cases} \end{aligned}$$

   Eigenfunction: $u_n(x) = \sin \frac{n\pi x}{\ell}$; $\lambda = \left(\frac{n\pi}{\ell}\right)^2$; $n =$ integer.

2. Bessel functions:

$$\begin{aligned} \frac{1}{x}\frac{d}{dx} x \frac{du}{dx} + \left(\lambda - \frac{n^2}{x^2}\right) u &= 0 \qquad u \text{ finite at } x = 0\,,\ \infty \\ u(x) &= \lambda \int_0^\infty G(x;\xi) u(\xi) \xi d\xi \\ G(x;\xi) &= \frac{1}{2n} \begin{cases} \left(\frac{x}{\xi}\right)^n & \text{when } x < \xi \\ \left(\frac{\xi}{x}\right)^n & \text{when } \xi < x \end{cases} \end{aligned}$$

   Eigenfunction: $u_n(x) = J_n(\sqrt{\lambda}x)$; $0 < \lambda < \infty$

3. Legendre polynomials:

$$\begin{aligned} \frac{d}{dx}(1 - x^2)\frac{du}{dx} + \lambda u &= 0 \qquad u \text{ is finite at } x = \pm 1 \\ u(x) &= \lambda \int_{-1}^1 G(x;\xi) u(\xi) d\xi - \frac{1}{2}\int_{-1}^1 u(\xi) d\xi \\ G(x;\xi) &= \frac{1}{2} \begin{cases} \ln\left(\frac{1+x}{1-\xi}\right) & \text{when } x < \xi \\ \ln\left(\frac{1+\xi}{1-x}\right) & \text{when } \xi < x \end{cases} \\ u_n(x) &= P_n(x)\,; \quad \lambda = n(n+1)\,; \quad n = \text{integer}\,. \end{aligned}$$

4.11.4 Types of Integral Equations

It is evident that different types of boundary value problems give rise to different types of integral equations.

A. Fredholm Equations

The inhomogeneous boundary value problem gave rise to Eq.(4.72), whose form is

$$u(x) \;=\; \lambda \int_a^b K(x;\xi)u(\xi)d\xi + \varphi(x)\,. \tag{4.74}$$

In this case, $K(x;\xi) = G(x;\rho)\rho(\xi)$ and φ are known functions, and u is the unknown function.

The integration limits a and b are fixed. An integral equation for $u(x)$ of the form Eq. (4.74) is called *inhomogeneous Fredholm equation* of the *second kind*. The expression $K(x;\xi)$ is called the "kernel" of the integral equation.

A *homogeneous Fredholm equation of the second kind* is obtained by dropping the function $\varphi(x)$,

$$u(x) = \lambda \int_a^b K(x;\xi)u(\xi)d\xi\,.$$

Equation (4.73) and the subsequent eigenvalue equations are examples of such equations.

A *Fredholm equation of the first kind* has the form

$$\varphi(x) = \int_a^b K(x;\xi)u(\xi)d\xi$$

whenever $\varphi(x)$ is a known function and $u(\xi)$ is the unknown function.

B. Volterra Equations

Fredholm equations are based on definite integrals. If the integration limits are variable, then the corresponding integral equations are *Volterra equations*. An *inhomogeneous Volterra equation of the second kind*, corresponding to Eq. (4.74), has the form

$$u(x) \;=\; \int_a^x K(x;\xi)u(\xi)d\xi + \varphi(\xi)\,. \tag{4.75}$$

If $\varphi = 0$, then one has a *homogeneous* Volterra equation of the second kind. By contrast, a *Volterra equation of the first kind* has the form

$$\varphi(x) = \int_a^x K(x,\xi)u(\xi)d\xi\,,$$

where φ is known and u is the unknown function. A Volterra integral equation may be viewed as a Fredholm equation whose kernel vanishes for $x < \xi$. Thus letting

$$\begin{aligned} M(x;\xi) &= 0 & x < \xi \\ M(x;\xi) &= K(x;\xi) & \xi < x\,, \end{aligned}$$

one finds that the Volterra Eq. (4.75) becomes

$$u(x) = \int_a^b M(x;\xi)u(\xi)d\xi + \varphi(\xi)\,,$$

whose form is that of a Fredholm equation.

One of the prominent examples giving rise to Volterra's integral equations are initial value problems. To illustrate this point, consider the motion of a simple harmonic oscillator governed by the equation

$$\frac{d^2u}{d\tau^2} + \omega^2 u = 0 \tag{4.76}$$

and the initial conditions

$$\begin{aligned} u(0) &= d \\ \dot{u}(0) &= e\,. \end{aligned}$$

The Green's function for this problem is depicted in Figure 4.5 on page 256. It is the response to the impulse $\delta(t-\tau)$, and it satisfies

$$\frac{d^2G^R(t;\tau)}{dt^2} = -\delta(t-\tau)\,;\quad G^R(0;\tau) = 0\,,\quad \left.\frac{dG^R(t;\tau)}{dt}\right|_{t=0} = 0$$

or

$$\frac{d^2G^R(t;\tau)}{d\tau^2} = -\delta(t-\tau)\,;\quad G^R(t,\tau) = 0\,,\quad \text{for all}\ \ t < \tau\,, \tag{4.77}$$

in spite of the fact that $G^R(t;\tau) \neq G^R(\tau;t)$ (Why? Hint: what is the second derivative of a function that depends only on $t-\tau$?). To obtain the integral equation multiply Eq. (4.76) by $G^R(t;\tau)$ and Eq. (4.77) by $u(\tau)$. One finds

$$\begin{aligned} G^R(t;\tau)\left[\frac{d^2u}{d\tau^2} + \omega^2\right]u &= 0 \\ u(\tau)\frac{d^2G^R(t;\tau)}{d\tau^2} &= -\delta(t-\tau)u(\tau) \end{aligned}$$

Subtraction yields a l.h.s. whose second derivative terms consolidate into a total derivative (Lagrange's identity!):

$$\frac{d}{d\tau}\left\{G^R(t;\tau)\frac{du}{d\tau} - u(\tau)\frac{dG^R(t;\tau)}{d\tau}\right\} + \omega^2 G^R(t;\tau)u(\tau) = -\delta(t-\tau)u(\tau)$$

Next perform the integration $\int_0^{t^+}\cdots d\tau$, where t^+ signifies taking the limit of the integral as $\tau \to t$ from the side for which $\tau > t$. One obtains

$$\int_0^{t^+} d\tau\left\{G^R(t;\tau)\frac{d^2u(\tau)}{d\tau^2} - u(\tau)\frac{d^2G^R(t,\tau)}{d\tau^2}\right\} + \omega^2\int_0^{t^+} G^R(t,\tau)u(\tau)d\tau = u(t)$$

or with the help of the property $G^R(t;\tau) = 0$ whenever $t < \tau$,

$$u(t) = \omega^2\int_0^{t^+} G^R(t;\tau)u(\tau)d\tau + u(0)\left.\frac{dG^R(t;\tau)}{d\tau}\right|_{\tau=0} - \dot{u}(0)G^R(t;0)\,.$$

Here $u(0)$ and $\dot{u}(0)$ are the initial amplitude and velocity of the simple harmonic oscillator, and they are now intrinsically incorporated in an inhomogeneous Volterra equation of the second kind. In this integral equation $u(\tau)$ is the unknown function to be determined. However, the utility of this integral equation, which is based on the Green's function $G^R(t;\tau)$, is eclipsed by an integral equation which is based a another Green's function, say $g^R(t;\tau)$, satisfying

$$\frac{d^2g^R(t;\tau)}{d\tau^2} + \omega^2 g^R(t;\tau) = -\delta(t-\tau)\,;\quad g^R(t,\tau) = 0\,,\quad \text{for all}\ \ t < \tau$$

similar to Eq.(4.76). Following the same derivation steps, one finds that the ω^2-term gets cancelled.

$$u(t) = u(0)\left.\frac{dg^R(t;\tau)}{d\tau}\right|_{\tau=0} - \dot{u}(0)g^R(t;0)\ .$$

The integral has diappeared. One is left with the solution to the problem one is actually trying to solve. The overall conclusion is this: *Picking the right Green's function for the problem speeds up the process of reaching one's goal.*

Exercise 4.11.1 (TRANSLATION INVARIANT INTEGRATION KERNEL)
Consider the inhomogeneous Fredholm equation of the second kind,

$$u(x) = \lambda \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x;\xi)\ u(\xi)\ d\xi \quad + \quad \phi(x).$$

Here λ is a parameter and ϕ is a known and given function. So is the integration kernel K, which in this problem is given to be translation invariant, i.e. you should assume that $K(x;\xi) = v(x-\xi)$, where v is a given function whose Fourier transform

$$\hat{v}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} v(x)\ dx$$

exists. SOLVE the integral equation by finding the function $u(x)$ in terms of what is given.

Exercise 4.11.2
Look up an integral equation of the *2nd kind,* either of the Volterra or of the Fredholm type. Submit it and its solution.

Lecture 35

4.12 Singular Boundary Value Problem: Infinite Domain

All our observations of nature are specific and hence finite. In order to extend our grasp from the finite to those aspects nature termed "infinite", one starts with a one parameter family of (finite) systems. By letting this parameter become asymptotically large one can ask: are there any properties of the system that don't change as we let that parameter become arbitrarily large? An affirmative answer to this question yields a new perspective on the system. The constellation of properties subsumed under the existence of this mathematical limit is a new concept, the "infinite system" corresponding to the finite sytems giving rise to it.

The purpose of the method of the "infinite system" is to extend our grasp of nature from the direct perceptual level of awareness to the conceptual level which has no spatial and temporal bounds.

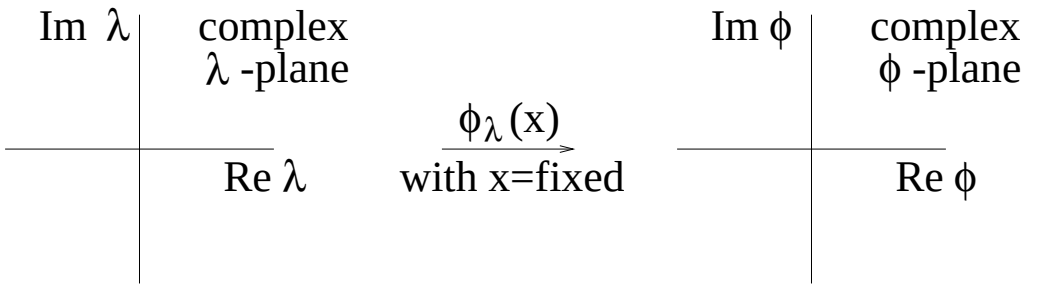

Figure 4.9: The λ-parametrized family of functions $\phi_\lambda(x)$ as a map from the complex λ-plane onto the complex ϕ-plane

The archetypical system we shall study is the uniform string of length ℓ. We shall determine its response to a unit force, and after that let $\ell \to \infty$. This is done in Section 4.12.4 on Page 294. We shall find the remarkable (but not necessarily unexpected) result that as $\ell \to \infty$, the string's response becomes *independent* of the particular mixed Dirichlet-Neumann boundary conditions one may have imposed at ℓ.

It will turn out that when one lets ℓ become infinitely large, it is necessary to impose some other homogeneous boundary condition. The mixed D-N conditions are simply inappropriate for an (in the limit) infinite string. They get replaced by the so-called "ingoing" or "outgoing" wave conditions. They comply with what is observed.

In order to become familiar with the key attributes of an infinite string consider again the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \ , \quad -\infty < x < \infty$$

but without specifying any boundary conditions as yet. The general solution to this differential equation has the form

$$\begin{array}{rcll} \phi_\lambda(x) & = & A \ + \ Bx & \text{for} \ \ \lambda \ = 0 \\ \phi_\lambda(x) & = & Ce^{i\lambda^{1/2}x} \ + \ De^{-i\lambda^{1/2}x} & \text{for} \ \ \lambda \ \neq 0 \end{array}$$

The expression for $\phi_\lambda(x)$ is a λ-parametrized family of solutions to a λ-parametrized family of differential equations. The parameter may be, and in general is, complex. Consequently, $\phi_\lambda(x)$ should be viewed as a function of the complex variable λ. For *fixed* x this function maps the complex λ-plane

into the complex ϕ-plane

$$\begin{array}{ccc} \text{Complex} & \phi(x) & \text{Complex} \\ \lambda-\text{plane} & \longrightarrow & \phi-\text{plane} \end{array}$$

$$\lambda \qquad \rightsquigarrow \qquad \phi_\lambda(x) \;=\; Ce^{i\sqrt{\lambda}x} \;+\; De^{-i\sqrt{\lambda}x}$$

The analytic properties of this function depend on the properties of the square root function $\lambda^{1/2}$, which has two analytic branches

$$\lambda^{1/2} = \alpha + i\beta = \begin{cases} \sqrt{\lambda} & \beta > 0 \\ -\sqrt{\lambda} & \beta < 0 \end{cases}$$

and a branch cut which separates them. They all play a key role in determining the behaviour of the function $\phi_\lambda(x)$. Thus a quick review is appropriate.

4.12.1 Review: Branches, Branch Cuts, and Riemann Sheets

The square root function and its branches are defined as follows:

(i) The square root function $\lambda^{1/2}$

Let $\lambda = |\lambda|e^{i\theta}$. A function is defined by giving its *formula* and specifying its *domain*. The square root function $\lambda^{1/2}$ is defined by

$$\lambda^{1/2} = |\lambda|^{1/2}e^{i\theta/2} \qquad \text{(Formula)}$$

where

$$\theta \text{ is any real number} \qquad \text{(Domain)}$$

(ii) The first branch of $\lambda^{1/2}$.

This function, denoted by $\sqrt{\lambda}$, is defined by

$$\sqrt{\lambda} = |\lambda|^{1/2}e^{i\theta/2} \qquad \text{(Formula)}$$

where

$$0 \le \theta < 2\pi \qquad \text{(Domain)}$$

More succinctly, we have

$$\begin{aligned} 1^{st} \text{ branch of } \lambda^{1/2} &\equiv \sqrt{\lambda} \\ &= |\lambda|^{1/2}e^{i\theta/2} \quad 0 \le \theta < 2\pi \end{aligned}$$

See Figure 4.10

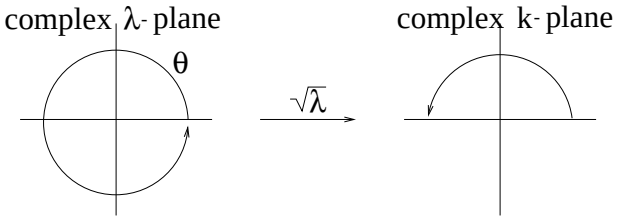

Figure 4.10: The first branch of the map $\lambda^{1/2}$ maps the complex λ-plane onto the upper half of the complex k-plane

(iii) The second branch of $\lambda^{1/2}$.

This function, denoted by $-\sqrt{\lambda}$, is obtained by *restricting* the domain of $\lambda^{1/2}$ to $2\pi \leq \theta < 4\pi$. In other words, $-\sqrt{\lambda}$ is defined by

$$-\sqrt{\lambda} = |\lambda|^{1/2} e^{i\theta/2} \qquad \text{(Formula)}$$

where

$$2\pi \leq \theta < 4\pi \qquad \text{(Domain)}$$

Equivalently, the second branch is defined by

$$-\sqrt{\lambda} = -|\lambda|^{1/2} e^{i\theta'/2} \qquad \text{(Formula)}$$

where

$$0 \leq \theta' < 2\pi \qquad \text{(Domain)}$$

See Figure 4.11

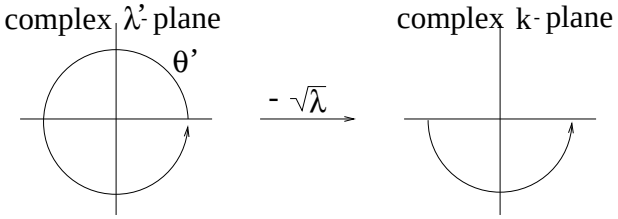

Figure 4.11: The second branch of the map $\lambda^{1/2}$ maps the complex λ'-plane onto the lower half of the complex k-plane

(iv) The Riemann sheets of $\lambda^{1/2}$.

The two branches $\sqrt{\lambda}$ and $-\sqrt{\lambda}$ are the two components of the single function $\lambda^{1/2}$ whose domain consists of *two* copies of the complex λ-plane. The points along the real λ-axis are glued together (i.e. identified) in the manner depicted in Figure 4.12.

We are forced to accept these two copies if one writes the image of λ as

$$[\lambda]^{1/2} = \alpha + i\beta$$

in the k-plane of Figure 4.12. Then the first Riemann sheet consists of the set of λ's for which $\beta > 0$, while the second Riemann sheet consists of those λ's for which $\beta < 0$. These sheets are joined continuously along their positive x-axes by the requirement that the function

$$[\lambda]^{1/2} = |\lambda|^{1/2} e^{i\theta/2} \tag{4.78}$$

be continuous for all values of θ. As a result of this requirement the two sheets are joined in the manner depicted in Figure 4.12.

It is quite true that, by itself, the complex number λ does not tell whether λ is on the first or the second Riemann sheet. This information is found neither in the real nor in the imaginary part of λ. Instead, it is inferred from the square root function, Eq.(4.78) Indeed, one has

$$\begin{aligned} 0 < \theta < 2\pi \quad &\Longrightarrow \quad \lambda \in \text{1st Riemann sheet} \\ 2\pi < \theta < 4\pi \quad &\Longrightarrow \quad \lambda \in \text{2nd Riemann sheet} \end{aligned} \tag{4.79}$$

The set of points $\theta = 2\pi$ lies on both Riemann sheets; so does the set of points $\theta = 0$. However, these two sets are distinct: they lie on opposite sides of the branch cut of the upper or the lower λ-plane.

An alternative way of characterizing the square root function is to collapse its domain, the two Riemann sheets, into a single λ-plane. Such a simplification comes, however, at a price: the square root function is now two-valued, it has two formulas, the two branches $\sqrt{\lambda}$ and $-\sqrt{\lambda}$. The domain for both of them is the λ-plane with a branch cut along the positive real axis across which each branch is discontinuous.

4.12.2 Square Integrability

Let us determine how the location of the point λ controls the square-integrability of the exponential solution $e^{i\lambda^{1/2}x}$ on the inteval $[0, \infty)$.

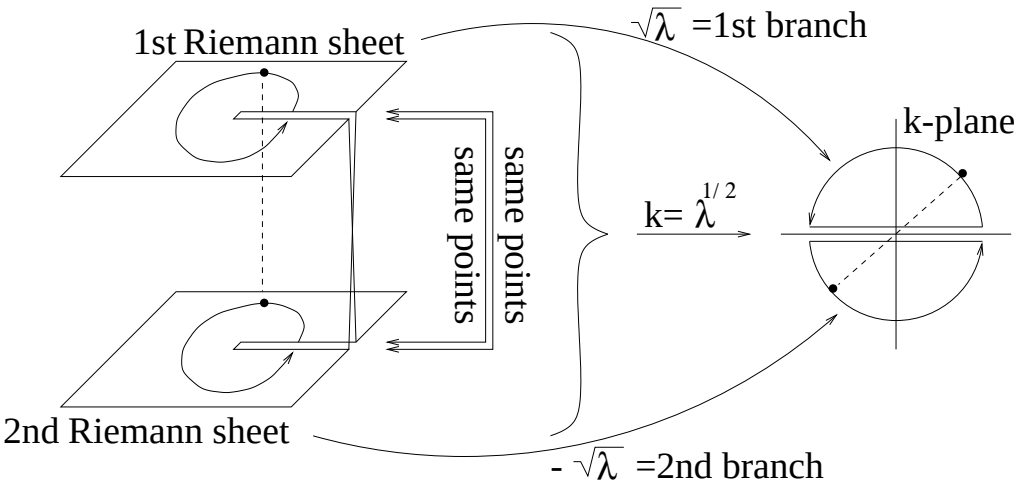

Figure 4.12: The map $\lambda^{1/2}$ whose domain consists of the upper and the lower Riemann sheets and whose range is the the complex k-plane. The circle in the first Riemann sheet gets mapped into a semi-circle in the upper k-plane, and the cirle in the second Riemann sheet gets mapped into a semi-circle in the lower k-plane. A pair of points one above the other in the two Riemann sheets gets mapped into a pair of diametrically opposite points in the k-plane. The points common to the two Riemann sheets get mapped into the real axis in the k-plane.

With

$$\lambda^{1/2} \equiv \alpha + i\beta$$

the λ-parametrized function

$$\phi_\lambda(x) = e^{i\lambda^{1/2}x} = \begin{cases} e^{i\sqrt{\lambda}x} & = e^{i\alpha x}e^{-\beta x} \quad 0 < \beta \\ e^{-i\sqrt{\lambda}x} & = e^{i\alpha x}e^{-\beta x} \quad \beta \leq 0 \end{cases}$$

has entirely different integrability properties depending on whether λ lies in the first Riemann sheet ($\lambda^{1/2} = \sqrt{\lambda}$, i.e. $\beta > 0$) or in the second Riemann sheet ($\lambda^{1/2} = -\sqrt{\lambda}$, i.e.$\beta < 0$), that is to say, in the domain of which branch of $\lambda^{1/2}$ the point λ happens to lie. In fact, from

$$\begin{aligned} \int_0^\infty |\exp\{i\lambda^{1/2}x\}|^2 \, dx &= \int_0^\infty |\exp\{i(\alpha + i\beta)x\}|^2 \, dx \\ &= \int_0^\infty e^{-2\beta x} \, dx = \frac{1}{2\beta} \quad \text{for all } \alpha, \text{ but } \beta > 0 \end{aligned} \tag{4.80}$$

we see that $\exp\{i\lambda^{1/2}x\}$ is square-integrable ($\in L^2[0,\infty)$) only when $\beta > 0$, but the integral diverges whenever $\beta \leq 0$. In other words,
$\exp\{i\lambda^{1/2}x\}$ *is square-integrable on* $[0,\infty)$ *whenever* λ *lies on the 1st Riemann sheet,* and not on the real λ-axis, nor on the 2nd Riemann sheet.

An analogous statement holds for $\exp\{i\lambda^{1/2}x\}$ being square-integrable on $(-\infty, 0]$. In that circumstance the requirement is that λ lies on the 2nd Riemann sheet:

$$\int_{-\infty}^{0} |\exp\{i\lambda^{1/2}x\}|^2 \, dx = \int_{0}^{\infty} |\exp\{-i(\alpha + i\beta)x\}|^2 \, dx$$
$$= \int_{0}^{\infty} e^{2\beta x} \, dx = \frac{-1}{2\beta} \quad \text{for all } \alpha, \text{ but } \beta < 0 \quad (4.81)$$

Thus the requirement of square integrability relates the Riemann sheets of $\lambda^{1/2}$ to the two semi-infinite integration domains of $\exp(i\lambda^{1/2}x)$:

$$\lambda \in \text{1st Riemann sheet} \Longleftrightarrow \exp(i\lambda^{1/2}x) \in L^2[0,\infty)$$

and

$$\lambda \in \text{2st Riemann sheet} \Longleftrightarrow \exp(i\lambda^{1/2}x) \in L^2(-\infty, 0]$$

whenever $\lambda \neq$ real. Thus,

$$\exp(i\lambda^{1/2}x) \text{ is square integrable on } [0,\infty) \Rightarrow \exp(i\lambda^{1/2}x) = \exp(i\sqrt{\lambda}x)$$

and

$$\exp(i\lambda^{1/2}x) \text{ is square integrable on } (-\infty, 0] \Rightarrow \exp(i\lambda^{1/2}x) = \exp(-i\sqrt{\lambda}x) \; .$$

Lecture 36

4.12.3 Infinite String

Even though in nature one never observes an infinite string, such a string is a concept with properties which are directly observable and which lend themselves to easy mathematical analysis. This means that the infinite string

is a natural way by which to grasp the properties and behavior of any system which exhibits these attributes. All the essential properties of a string are contained in the solution to the following

Problem: Construct the Green's function for the system

$$\frac{d^2G_\lambda}{dx^2} + \lambda G_\lambda = -\delta(x-\xi) \qquad 0 < x < \infty$$

subject to

1. $G_\lambda(0;\xi) = 0$
2. $G_\lambda(x;\xi)$ expresses an "outgoing" wave for very large x, i.e., $G_\lambda \sim e^{i\lambda^{1/2}x}$.
3. $G_\lambda(x;\xi)$ is square-integrable.

Comment

Such a boundary value problem arises in the solution to a vibrating semi-infinite string which is imbedded in an elastic medium, and which responds to a harmonically varying force:

$$\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial t^2} - k^2\psi = -f(x)e^{-i\omega t}\,.$$

The steady state solution to this system is

$$\psi(x,t) = u(x)e^{-i\omega t}$$

where

$$\frac{d^2u}{dx^2} + \lambda u(x) = -f(x)$$

with $\lambda = \omega^2 - k^2$. If the harmonic driving force is localized to a point, then the solution is

$$\psi(x,t) = G_\lambda(x;\xi)e^{-i\omega t}\,.$$

Being square integrable, for large $(\xi < x)$ the solution is

$$\psi \sim e^{i\sqrt{\lambda}x}e^{-i\omega t} = e^{i(\alpha+i\beta)x}e^{-i\omega t},,$$

where

$$\sqrt{\lambda} = \pm|\alpha| + i\beta \quad \text{with} \quad 0 < \beta \lll 1\,.$$

It is evident that the upper sign expresses an *outgoing* wave and the lower sign an *incoming* wave on $0 < x < \infty$. This is because the locus of the constant amplitude

$$\pm|\alpha|x - \omega t = \text{const}$$

is a point with phase velocity

$$\frac{dx}{dt} = \pm\frac{\omega}{|\alpha|}\,.$$

The upper sign refers to a wave moving towards larger x. It is an outgoing wave. The difference between an *outgoing* and an *incoming* wave is the difference between the driving force emitting and absorbing wave energy. In both cases $0 < \beta$[6] .

The Green's function is constructed in the usual way:

$$G_\lambda(x;\xi) = \frac{-1}{c}u_1(x_<)u_2(x_>)$$

where

$$\begin{aligned}
u_1(x) &= \sin\lambda^{1/2}x \qquad (u_1(0) = 0)\\
u_2(x) &= \exp i\lambda^{1/2}x \qquad (\text{outgoing b.c. : } \lambda^{1/2} = |\alpha| + i\epsilon)\\
c &= u_1u_2' - u_1'u_2\\
&= \lambda^{1/2}(i\sin\lambda^{1/2}x - \cos\lambda^{1/2}x)e^{i\lambda^{1/2}x}\\
&= -\lambda^{1/2}\,.
\end{aligned}$$

Thus the Green function is

$$G_\lambda(x;\xi) = \frac{\sin\lambda^{1/2}x_<}{\lambda^{1/2}}e^{i\lambda^{1/2}x_>}\ .$$

Here

$$\lambda^{1/2} = \begin{cases} |\lambda|^{1/2} + i\epsilon \text{ for "outgoing" wave at large } x\\ -|\lambda|^{1/2} + i\epsilon \text{ for "incoming" wave at large } x\end{cases}$$

[6]On the other hand, if the wave propagation domain were $-\infty < x < 0$, then an outgoing (resp. incoming) wave

$$e^{-i\sqrt{\lambda}x}e^{-i\omega t} = e^{i(\alpha+i\beta)x}e^{-i\omega t}$$

would be characterized by $\alpha = -|\alpha|$ (resp. $\alpha = |\alpha|$), while square integrability demands that $\beta < 0$, i.e. that $\lambda^{1/2}$ be evaluated on its 2nd branch.

From the perspective of physics, a non-zero but neglegible ϵ expresses the presence of damping. The exponential damping factor $e^{-\epsilon x}$ gurantees that G_λ is square integrable on $[0,\infty)$, but it has no effect on the shape of the Green's function because $\epsilon \to 0$. It can thus be written as

$$G_\lambda^{\text{outgoing}}(x;\xi) = \frac{\sin \lambda^{1/2} x_<}{\lambda^{1/2}} e^{(i|\lambda|^{1/2} - 0^+)x_>} \ ,$$

while

$$G_\lambda^{\text{incoming}}(x;\xi) = \frac{\sin \lambda^{1/2} x_<}{\lambda^{1/2}} e^{(-i|\lambda|^{1/2} - 0^+)x_>} \ .$$

The branch cut of $\lambda^{1/2}$ has a significant effect on the exponential. However, the sinc function remains uneffected because it is an analytic function of λ.

4.12.4 Infinite String as the Limit of a Finite String

It seems that the properties and behavior of an infinite string are irreconcilably different from those of a finite string. However, it is possible to consider the former as a limiting form of the latter. This fact, together with the the associated role of the square root function, is brought out by comparing the solutions of two simple strings, both of length $|\ell|$. One extends into the positive, the other into the negative x-direction:

Problem: Consider the Green's functions g_+ and g_- of the two linear systems with symmetrically located domains.

(i) The first one is governed by the differential

$$\frac{d^2 g_+}{dx^2} + \lambda g_+ = -\delta(x-\xi) \quad 0 < x, \xi < \ell \tag{4.82}$$

$$\tag{4.83}$$

with Dirichelet boundary conditions at $x = 0$:

$$g_+(x=0) = 0 \ ,$$

and with mixed Dirichelet-Neumann boundary conditions at $x = \ell$:

$$b_1 g_+(\ell) + b_2 g'_+(\ell) = 0 \ . \tag{4.84}$$

(ii) The second one is governed by the same differential equation, but on a domain which extends symmetrically to the left:

$$\frac{d^2 g_-}{dx^2} + \lambda g_- = -\delta(x-\xi) \quad \ell < x, \xi < 0 \tag{4.85}$$

$$\tag{4.86}$$

with Dirichelet boundary conditions also at $x = 0$:

$$g_-(x=0) = 0 \ ,$$

and with the mixed Dirichelet-Neumann boundary conditions, but located at $x = \ell = -|\ell|$:

$$b_1 g_-(\ell) + b_2 g'_-(\ell) = 0 \ . \tag{4.87}$$

Thus the domain of g_+ is $[0, \ell]$, while that of $g_-$is $[-|\ell|, 0]$.

Compare the respective Green's functions g_+ and g_- in the limit as $|\ell| \to \infty$.

Remark: We shall find three noteworthy results: First of all, each Green's function has *two* asymptotic limits: one is square integrable, the other is not. Second, *these limits are entirely independent of the mixed Dirichlet-Neumann conditions*, Eq.(4.84), (4.87). Finally, each of these limits accomodates two radiation conditions: *outgoing* and *incoming*.

Solution: (i) The first step consists of constructing the Green's function in the usual way. This task is based on

$$u_1 = \sin \lambda^{1/2} x \quad ,$$

which satisfies $u_1(0) = 0$, and on

$$u_2 = A \cos \lambda^{1/2} x + B \sin \lambda^{1/2} x$$

whose coefficients A and B are related so as to satisfy the given boundary conditions, Eqs.(4.84) and (4.87) at $x = \ell$. These boundary conditions demand that

$$A(\underbrace{b_1 \cos \lambda^{1/2}\ell - b_2 \lambda^{1/2} \sin \lambda^{1/2}\ell}_{D}) + B(\underbrace{b_1 \sin \lambda^{1/2}\ell + b_2 \lambda^{1/2} \cos \lambda^{1/2}\ell}_{N}) = 0$$

or

$$A = -B\frac{N}{D} \quad . \tag{4.88}$$

To construct the Green's functions

$$g_{\pm}(x;\xi) = \begin{cases} \frac{-1}{c}\, u_1(x)\, u_2(\xi) & \text{when } |x| < |\xi| \\ \frac{-1}{c}\, u_1(\xi)\, u_2(x) & \text{when } |\xi| < |x| \end{cases}$$

we need the Wronskian determinant

$$\begin{aligned} c &= \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \\ &= \begin{vmatrix} \sin\lambda^{1/2}x & A\cos\lambda^{1/2}x + B\sin\lambda^{1/2}x \\ \lambda^{1/2}\cos\lambda^{1/2}x & -\lambda^{1/2}A\sin\lambda^{1/2}x + \lambda^{1/2}B\cos\lambda^{1/2}x \end{vmatrix} \\ &= -\lambda^{1/2}\, A \\ &= \lambda^{1/2}\, B\, \frac{N}{D} \end{aligned}$$

Using Eq.(4.88), write down the Green's functions. For $|\xi| < |x|$ one has

$$\begin{aligned} g_{\pm}(x;\xi) &= \frac{-1}{c}\, u_1(\xi)\, u_2(x) \\ &= \frac{-1}{\lambda^{1/2}\, B\, \frac{N}{D}}\, \sin\lambda^{1/2}\xi \left[(-)B\frac{N}{D}\cos\lambda^{1/2}x + B\sin\lambda^{1/2}x\right] \\ &= \underbrace{\frac{\sin\lambda^{1/2}\xi}{\lambda^{1/2}}\cos\lambda^{1/2}x}_{\text{“particular solution”}} - \underbrace{\frac{\sin\lambda^{1/2}\xi}{\lambda^{1/2}}\frac{\sin\lambda^{1/2}x}{\frac{N}{D}}}_{\text{“solution to the homogeneous problem”}} \end{aligned} \tag{4.89}$$

(ii) The second step consists of taking the limit of this Green's function as the boundary $|\ell| \to \infty$. Note that there is no ℓ-dependence whatsoever in the "particular solution" part of $g_{\pm}(x;\xi)$. In fact, it is totally independent of the specific boundary condition that has been imposed at $x = \ell$.

This is different for the second part, the "solution to the homogeneous equation". It depends on the boundary condition by virtue of the ratio

$$\frac{N}{D} = \frac{b_1\sin\lambda^{1/2}\ell + b_2\lambda^{1/2}\cos\lambda^{1/2}\ell}{b_1\cos\lambda^{1/2}\ell - b_2\lambda^{1/2}\sin\lambda^{1/2}\ell}$$

and hence also on the length ℓ. However, in the limit as $\ell \to \infty$ something remarkable happens: the ratio N/D, and hence the Green's function 4.89, becomes *independent of the mixed Dirichlet-Neumann boundary condition at* $x = \ell$. In order to determine the value of the limiting ratio

$$\lim_{\ell\to\infty} \frac{N}{D} \quad ,$$

set

$$\lambda^{1/2} = \alpha + i\beta \quad ,$$

so that

$$\begin{aligned} \sin\lambda^{1/2}\ell &= \frac{1}{2i}(e^{i\alpha\ell}e^{-\beta\ell} - e^{-i\alpha\ell}e^{\beta\ell}) \\ \cos\lambda^{1/2}\ell &= \frac{1}{2}(e^{i\alpha\ell}e^{-\beta\ell} + e^{-i\alpha\ell}e^{\beta\ell}) \quad . \end{aligned}$$

Introduce these expressions into the ratio N/D. It is evident that this ratio has no limit when $\beta = 0$. However, for $\beta \neq 0$ one finds that

$$\begin{aligned} \lim_{\beta\ell\to\infty} \frac{N}{D} &= \lim_{\beta\ell\to\infty} \frac{\frac{b_1}{2i}(-)e^{-i\alpha\ell}e^{\beta\ell} + \frac{b_2\lambda^{1/2}}{2}e^{-i\alpha\ell}e^{\beta\ell}}{\frac{b_1}{2}e^{-i\alpha\ell}e^{\beta\ell} + \frac{b_2\lambda^{1/2}}{2i}e^{-i\alpha\ell}e^{\beta\ell}} \\ &= i \end{aligned}$$

and

$$\lim_{\beta\ell\to-\infty} \frac{N}{D} = -i \ .$$

Applying these two limits to the Green's function, Eq.(4.89), one obtains

$$\lim_{\beta\ell\to\infty} g_\pm(x;\xi) = \frac{\sin\lambda^{1/2}\xi}{\lambda^{1/2}} e^{i\lambda^{1/2}x} \quad \begin{cases} \text{upper sign for } \ell \to \infty \\ \text{lower sign for } \ell \to -\infty \end{cases} \tag{4.90}$$

and

$$\lim_{\beta\ell\to-\infty} g_\pm(x;\xi) = \frac{\sin\lambda^{1/2}\xi}{\lambda^{1/2}} e^{-i\lambda^{1/2}x} \quad \begin{cases} \text{upper sign for } \ell \to \infty \\ \text{lower sign for } \ell \to -\infty \end{cases} \tag{4.91}$$

The allowed values of $\lambda^{1/2} = \alpha + i\beta$ are no longer determined by the Dirichelet-Neumann conditions. Instead, for an (in the limit) infinite system the two parts of $\lambda^{1/2}$ are determined by the two boundary conditions

1. square integrability of $g_\pm$, and
2. outgoing (or incoming) signal propagation condition.

These are *two* boundary conditions which determine the *two* parts of the complex number $\lambda^{1/2} = \alpha + i\beta$, and hence of the eigenvalue $\lambda^{1/2}$. The first condition is met by fulfilling the requirement that

$$\begin{aligned} &\text{for } g_+ : \ \beta > 0 \qquad \text{i.e. } \lambda \in \text{1st Riemann sheet of } \lambda^{1/2} \\ &\text{for } g_- : \ \beta < 0 \qquad \text{i.e. } \lambda \in \text{2nd Riemann sheet of } \lambda^{1/2} \end{aligned}$$

The second condition for outgoing (resp. incoming) signal propagation is met by fulfilling the reqirement that

$$\begin{aligned} \text{for } g_+ : \qquad \lambda^{1/2} &= \begin{cases} \ \ |a| + i\epsilon & \text{outgoing (to the "right") wave} \\ -|a| + i\epsilon & \text{incoming (from the "right") wave} \end{cases} \\ \text{for } g_- : \qquad \lambda^{1/2} &= \begin{cases} -|a| - i\epsilon & \text{outgoing (to the "left") wave} \\ \ \ |a| - i\epsilon & \text{incoming (from the "left") wave} \end{cases} \end{aligned}$$

These are the mathematical conditions on $\lambda^{1/2} = \alpha + i\beta$ for an asymptotically infinite string. Inserting them into Eqs.(4.90) and (4.91), one finds that the Green's functions satisfying these conditions are

$$\left.\begin{aligned} g_\pm^{outgoing} &= \frac{\sin \lambda^{1/2}\xi}{\lambda^{1/2}} e^{\pm i(|a|+i\epsilon)x} \\ g_\pm^{incoming} &= \frac{\sin \lambda^{1/2}\xi}{\lambda^{1/2}} e^{\pm i(-|a|+i\epsilon)x} \end{aligned}\right\} \times e^{-i\omega\tau} \quad \text{for } |\xi| < |x| \ .$$

Figure 4.13 depicts the real part of the graph of these functions.

4.13 Spectral Representation of the Dirac Delta Function

The physical difference between a finite and an infinite system has a profound impact on the mathematical structure of the corresponding Green's functions.

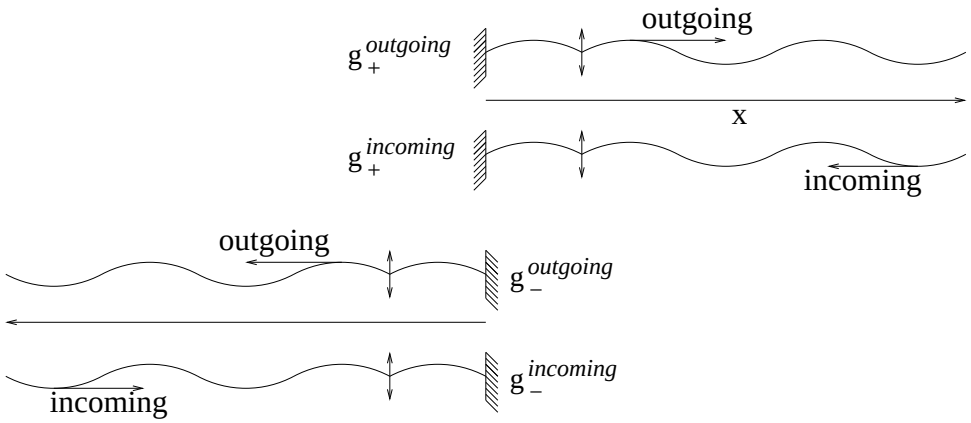

Figure 4.13: Instantaneous amplitude profiles of waves with outgoing and incoming phase velocities. In the first case the source *emits* energy; in the second case the source *absorbs* energy.

For a finite system, exemplified by a finite string, the Green's function is singular only at its poles, the eigenvalues of the system. By contrast, for an infinite system, for example, an infinite string, the Green's function is singular along each point of a line segment, a branch cut in the complex λ-plane.

4.13.1 Coalescence of Poles into a Branch Cut

To appreciate the non-analyticity of the Green's function, consider how the the isolated poles of a finite string Green's function merge so that they form the branch cut when the string becomes infinitely long. To illustrate the point, start with a string of length ℓ which satisfies the Dirichlet boundary conditions at both ends.

$$\begin{aligned} \frac{d^2 g_\lambda}{dx^2} + \lambda g_\lambda &= -\delta(x-\xi) \quad 0 < x, \xi < \ell \\ g_\lambda(0;\xi) &= 0 \\ g_\lambda(\ell;\xi) &= 0 \ . \end{aligned}$$

The Green's function is

$$g_\lambda(x;\xi) = \frac{1}{\lambda^{1/2} \sin \lambda^{1/2}\ell} \sin\{\lambda^{1/2} x_<\} \ \sin\{\lambda^{1/2}(\ell - x_>)\}$$

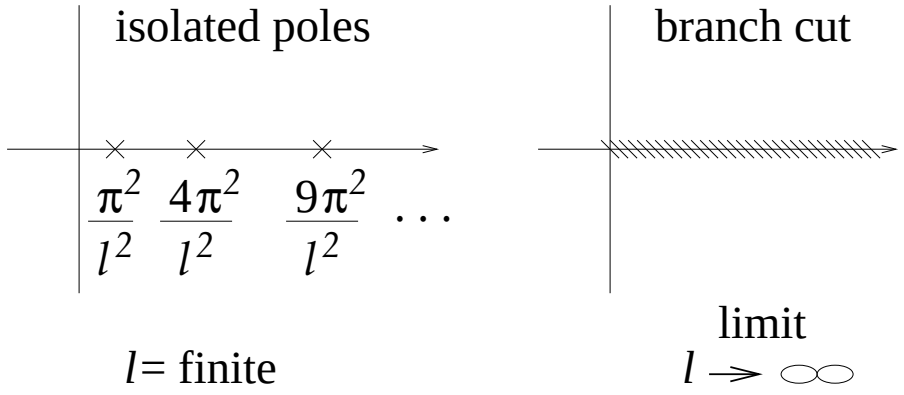

Figure 4.14: Eigenvalue spectra in the complex λ-plane

Observe that in the complex λ-plane its poles are isolated and located at

$$\lambda_n = \frac{n^2\pi^2}{\ell^2} \qquad n = 1, 2, 3, \cdots$$

along the real λ-axis. Their separation

$$\begin{aligned} \triangle\lambda_n &= \lambda_{n+1} - \lambda_n \\ &= \frac{\pi^2}{\ell^2}(2n+1) \end{aligned}$$

tends to zero as $\ell \to \infty$. Thus, as depicted in Figure 4.14, as $\ell \to \infty$ the *isolated poles* of $g_\lambda(x;\xi)$ *coalesce into a continuous barrier, the branch cut*, which separates the "outgoing" from the "incoming" wave numbers $\lambda^{1/2}$ on the same Riemann sheet of $\lambda^{1/2}$.

Remark. How would a change in boundary conditions, from Dirichlet to, say, mixed Dirichlet-Neumann conditions, have altered the coalescence of the poles of the Green's function? It is evident that the positions of these poles depend continuously on the parameters that specify the Dirichlet-Neumann boundary conditions. A change in these boundary conditions would merely have shifted these poles along the real λ axis in a continuous way. However, as $\ell \to \infty$, they still would have coalesced and formed the branch cut across which the limiting Green's function is discontinuous in the λ plane.

4.13.2 Contour Integration Around the Branch Cut

Lecture 37

If the poles of a Green's function coalesce into a branch cut, can one expect that the sum over the discrete eigenfunctions, Eq.(4.46), mutates into a corresponding integral? The answer is 'yes', and this means that instead of representing a function as a discrete sum of eigenfunctions, one now represents a function as an integral transform. The Green's function for a semi-infinite string furnishes us with the archetypical recipe for obtaining this integral transform. It is a two step process:

1. Evaluate the contour integral of $G_\lambda(x;\xi)$ over a circle with unlimited large radius:

$$\oint G_\lambda(x;\xi)\, d\lambda = \oint \frac{\sin\sqrt{\lambda}\, x}{\sqrt{\lambda}} e^{i\sqrt{\lambda}\,\xi}\, d\lambda \quad 0 < x < \xi$$

When $0 < \xi < x$ one interchanges x and ξ on the right hand side. The contour path of integration is

$$\lambda(\theta) = Re^{i\theta} \quad 0 < \theta < 2\pi \ .$$

In terms of the complex variable

$$k = \sqrt{\lambda}$$

this contour integral extends over a semicircle from $k = |k|$ to $k = |k|e^{i\pi}$

$$\oint G_\lambda\, d\lambda = \lim_{|k|\to\infty} \int_{|k|}^{|k|e^{i\pi}} \frac{e^{ik(x+\xi)} - e^{-ik(x-\xi)}}{2i} 2\, dk \ .$$

The integrand is analytic in k. Consequently, the semicircle can be straightened out into a line segment along the real axis. The integral becomes therefore

$$\begin{aligned} \oint G_\lambda\, d\lambda &= i\int_{-\infty}^{\infty} \left[e^{ik(x+\xi)} - e^{-ik(x-\xi)}\right]\, dk \\ &= 2\pi i\left[\delta(x+\xi) - \delta(x-\xi)\right] \end{aligned}$$

For a semi-infinite string the domain variables are only positive, $0 < x < \xi$. Therefore the first Dirac delta function vanishes. We are left with

$$\frac{1}{2\pi i}\oint G_\lambda\, d\lambda = -\delta(x-\xi) \ . \tag{4.92}$$

2. The second step also starts with the closed contour integral

$$\oint G_\lambda(x;\xi)\ d\lambda\ , \tag{4.93}$$

but this time the circular contour gets deformed into two linear paths on either side of the positive real axis $[0,\infty)$, the branch cut of

$$G_\lambda(x;\xi) = \frac{\sin\sqrt{\lambda}x_<}{\sqrt{\lambda}} e^{i\sqrt{\lambda}\,x_>}\ .$$

Designate the two values of $G_\lambda(x;\xi)$ on opposite sides of the branch cut by G_+ and G_-. The integral is therefore

$$\begin{aligned}\oint G_\lambda(x;\xi)\ d\lambda &= \int_\infty^0 G_+\ d\lambda + \int_0^\infty G_-\ d\lambda \\ &= \int_0^\infty [G_- - G_+]\ d\lambda\ . \end{aligned}\tag{4.94}$$

To evaluate the difference $G_- - G_+$ note that the value of $\sqrt{\lambda}$ is

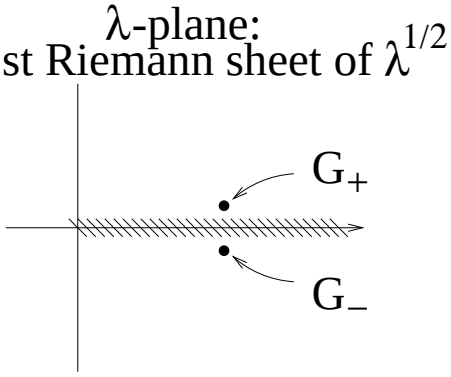

Figure 4.15: Evaluation of the Green's function just above and just below the branch cut of $\lambda^{1/2}$ on its first Riemann sheet.

$$\sqrt{\lambda} = \begin{cases} |\lambda|^{1/2} \text{ just above the branch cut} \\ -|\lambda|^{1/2} \text{ just below the branch cut} \end{cases}$$

Consequently, the value of the Green's function at these locations is

$$\begin{aligned} G_+ &= \frac{\sin|\lambda|^{1/2}x_<}{|\lambda|^{1/2}} \exp\{i|\lambda|^{1/2}x_>\} \\ G_- &= \frac{\sin|\lambda|^{1/2}x_<}{|\lambda|^{1/2}} \exp\{-i|\lambda|^{1/2}x_>\} \end{aligned}$$

The discontinuity across the branch cut is therefore

$$\begin{aligned} G_+ - G_- \equiv [G_\lambda] &= \frac{2i}{|\lambda|^{1/2}} \sin|\lambda|^{1/2} x_< \ \sin|\lambda|^{1/2} x_> \\ &= \frac{2i}{|\lambda|^{1/2}} \sin|\lambda|^{1/2} x \ \sin|\lambda|^{1/2} \xi \end{aligned} \tag{4.95}$$

Insert this result into Eq.(4.94), change the integration variable to $|\lambda|^{1/2} = k$ and obtain the result that

$$\begin{aligned} \frac{1}{2\pi i} \oint G_\lambda(x;\xi)\ d\lambda &= \frac{-1}{\pi} \int_0^\infty \frac{\sin|\lambda|^{1/2} x \ \sin|\lambda|^{1/2} \xi}{|\lambda|^{1/2}}\ d\lambda \\ &= \frac{-2}{\pi} \int_0^\infty \sin kx \ \sin k\xi \ dk \end{aligned} \tag{4.96}$$

This two step procedure yields two alternative expressions, Eqs.(4.92) and (4.96) for the contour integral of the Green's function. Their equality yields the spectral representation of the Dirac delta function for a semi-infinite string,

$$\delta(x-\xi) = \frac{2}{\pi} \int_0^\infty \sin kx \ \sin k\xi \ dk \tag{4.97}$$

4.13.3 Fourier Sine Theorem

Spectral representations like Eq.(4.97) yield pairs of functions which are transforms of each other. Let $f(x)$ be an integrable function $f(x)$ defined on the real interval $0 \le x < \infty$. Multiply Eq.(4.97) by $f(\xi)$ and integrate over the half line $0 \le \xi < \infty$. The result is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(k) \sin kx \ dk \quad ,$$

where

$$F(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi) \sin k\xi \ d\xi$$

These two function are the *Fourier sine transforms* of each other.

Exercise 4.13.1 (VERY LONG STRING: STEADY STATE RESPONSE) Over the interval $-\infty < x < \infty$ consider

$$\begin{aligned} \frac{d^2G}{dx^2} + \lambda G &= -\delta(x-\xi) \\ \frac{d^2u}{dx^2} + \lambda u &= -f(x) \ \text{ and } \ \frac{d^2\varphi}{dx^2} + \lambda\phi = 0. \end{aligned}$$

We are looking for solutions in $\mathcal{L}^2(-\infty,\infty)$ and assume that f is in $\mathcal{L}^2(-\infty,\infty)$.

(a) Show that there are two candidates for G, namely

$$\begin{aligned} G = G^{out}(x|\xi;\lambda) &= \frac{i}{2\sqrt{\lambda}} \exp(-i\sqrt{\lambda}\xi)\exp(i\sqrt{\lambda}x) \quad \xi < x \\ &= \frac{i}{2\sqrt{\lambda}} \exp(-i\sqrt{\lambda}x)\exp(i\sqrt{\lambda}\xi) \quad x < \xi \\ &= \frac{i}{2\sqrt{\lambda}} \exp(i\sqrt{\lambda}|x-\xi|). \end{aligned}$$

and

$$G = G^{in}(x|\xi;\lambda) = \frac{-i}{2\sqrt{\lambda}} \exp(-i\sqrt{\lambda}|x-\xi|).$$

(b) Given the fact that $\sqrt{\lambda} = \alpha + i\beta$ with $\beta > 0$, point out why only one of them is square-integrable.

(c) Consider the contour integral $\oint G(x|\xi;\lambda)\, d\lambda$ over a large circle of radius R. Demonstrate that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \oint G(x|\xi;\lambda)\, d\lambda = -\delta(x-\xi)\ .$$

(d) Next deform the contour until it fits snugly around the branch cut of $\sqrt{\lambda}$, and show that

$$\delta(x-\xi) = \int_0^\infty \cdots \ d\lambda \qquad\qquad (*)$$

and then show that $(*)$ can be rewritten as

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(x-\xi)} d\omega \qquad \text{for } x < \xi \ \text{ and } \ \xi < x\ .$$

(d) Express $u(x)$ as a Fourier integral in terms of f.

(e) Express $G(x|\xi;\lambda)$ in the same way, i.e. obtain the bilinear expansion for G.

Exercise 4.13.2 (STEADY STATE RESPONSE VIA FOURIER)
Again consider

$$\begin{aligned} \frac{d^2G}{dx^2} + \lambda G &= -\delta(x-\xi) \\ \frac{d^2u}{dx^2} + \lambda u &= -f(x) \quad \text{and} \quad \frac{d^2\varphi}{dx^2} + \lambda\phi = 0. \end{aligned}$$

over the interval $-\infty < x < \infty$, but leave the boundary conditions as-yet-unspecfied.

(a) Express $u(x)$ as a Fourier integral in terms of f.

(b) Express $G(x|\xi;\lambda)$ in the same way, i.e. obtain the bilinear expansion for G.

(c) How, do you think, should one incorporate boundary conditions into these expressions?

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Chapter 5

Special Function Theory

We shall now reconsider the eigenvalue problem

$$Lu = \lambda u \quad ,$$

but we take λ to be a *degenerate* eigenvalue. This means that we take λ to have more than one eigenvector. These eigenvectors span a subspace, the eigenspace of L corresponding to λ. This subspace has a basis of eigenvectors, but its choice is not unique.

In spite of this we ask: Is there a way of constructing a basis which is dictated by objective criteria (for our purposes, by geometry and/or physics) and not by subjective preferences?

The answer to this question is "yes" whenever one can identify a linear transformation, call it T, with the following three properties:

(i) The domain of T coincides with that of L,

(ii) the transformation T commutes with L, i.e.

$$TL = LT \quad ,$$

and

(iii) the eigenvalues of T are non-degenerate.

A transformation with these properties determines a unique eigenbasis for each eigenspace of the original eigenvalue problem. Indeed, let u be an eigenvector of T:

$$Tu = \tau u \quad .$$

Then

$$T(Lu) = L(Tu) = \tau\ Lu \quad ,$$

i.e., Lu is again an eigenvector of T corresponding to the same eigenvalue τ. The non-degeneracy of τ implies that Lu is a multiple of u; in other words,

$$Lu = \lambda u \quad .$$

Thus u is also an eigenvector of L. Conversely, if u belongs to the λ-eigenspace of L, then Tu also belongs to this subspace. The set of all the eigenvectors of T which lie in this λ-subspace form a basis for this subspace. This basis is orthonormal if T and L are hermitian. The elements of this T-determined basis are uniquely labelled by the real eigenvalues τ and, of course, by the subspace label λ. A set of commuting linear transformations, such as L and T, whose eigenvalues uniquely label their common eigenvectors, is called a *complete set of commuting operators*.

The operator T is not unique. Suppose there is another hermitian operator, say S, which together with L forms another complete set of commuting operators. This means that one now has two orthonormal bases for the λ-eigenspace of L, one consisting of the eigenvectors of T, the second consisting of the eigenvectors of S. Furthermore, these two bases are related by a unitary transformation, i.e. by a rotation in the complex eigenspace of L.

One of the most far reaching applications of this geometrical framework consists of identifying

- the operator L with the *Laplacian* on E^2, the Euclidean two-dimensional plane,
- the operator T with the *generator of translations* in E^2,
- the operator S with the *generator of rotations* in E^2,
- the eigenvectors of T with the *plane-wave solutions* to the Helmholtz equation,
- the eigenvectors of S with the *cylinder (Bessel) solutions* to the Helmholtz equation, and
- the unitary transformation with the *Fourier series representation* of a plane wave in terms of the Bessel solutions.

5.1 The Helmholtz Equation

Lecture 38

If the Sturm Liouville equation is the most important equation in one dimension, then the Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0$$

is the most important, and simplest, eigenvalue equation in two dimensions. The two-dimensional domains we consider are first the Euclidean plane and later the surface of a sphere.

The Helmholtz equation can be written down and then solved relative to any one of many coordinate systems. In three dimensional Euclidean space there are at least eleven such coordinate systems.

5.1.1 Cartesian versus Polar Coordinates

Relative to the standard rectilinear *Cartesian coordinates* Helmholtz's equation has the form

$$-\nabla^2\psi \equiv -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = k^2\psi\,.$$

If one uses

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

to make a transition to polar coordinates, the Helmholtz equation assumes the form

$$-\nabla^2\psi \equiv -\left(\frac{1}{r}\,\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta}\right)\psi = k^2\psi\,.$$

(*Nota bene*: To show that the Laplacian relative to polars has the form indicated, it is easiest to actually start with that polar expression and then use the above coordinate transformation to recover the Cartesian expression for ∇^2. Going the other way is, of course, equivalent but takes a little extra work.)

Given these two representations of the Laplacian ∇^2, how do their eigenfunctions compare and how are they related?

This is a very fruitful question to ask, because in answering it, we shall not only obtain a deep and thorough understanding of waves on the flat Euclidean plane, but also develop the framework for dealing with waves on a sphere as well as with waves in three dimensional Euclidean space.

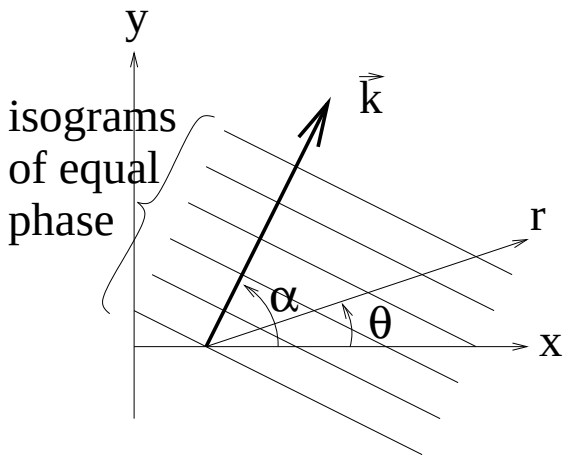

Figure 5.1: An instantaneous plane wave consists of a set of parallel phase fronts, the isograms of the phase function. Its gradient, which is perpendicular to these isograms, is the wave propagation vector $\vec{k}$.

Plane wave solutions play a key role in the development. Thus we must have a natural and precise way of identifying them relative to Cartesian as well as polar coordinates.

The solutions to

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k^2\psi = 0$$

are the *"plane wave"* solutions

$$e^{i\vec{k}\cdot\vec{x}} = e^{i(k_x x + k_y y)} \qquad \text{(relative to Cartesians)}.$$

Such a solution is characterized by its *wave propogation vector*

$$\vec{k} = (k_x, k_y)\,.$$

The polar representation of this vector,

$$\vec{k} = (k\cos\alpha, k\sin\alpha)$$

where

$$k^2 = k_x^2 + k_y^2 \quad ,$$

is appropriate relative to polar coordinates. The wave propagation vector $\vec{k}$ is the gradient of the *phase* for a plane wave solution. This phase has the form

$$\begin{aligned} \text{phase} &\equiv k_x x + k_y y \qquad\qquad \text{(relative to Cartesians)} \\ &= kr(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ &= kr\cos(\alpha - \theta) \qquad\qquad \text{(relative to polars)}\,. \end{aligned}$$

Consequently,

$$e^{i\vec{k}\cdot x} = e^{ikr\cos(\alpha-\theta)} \qquad \text{(relative to polars)}\,.$$

Thus relative to polar coordinates, a plane wave is represented by the magnitude k, and the direction angle α of its propagation vector $\vec{k}$.

5.1.2 Degenerate Eigenvalues

Every eigenvalue of the eigenvalue equation

$$-\nabla^2\psi = k^2\psi$$

is highly degenerate. In fact, each eigenvalue k^2 is infinitely degenerate. This means that for one and the same eigenvalue k^2, there is an infinite set of eigenfunctions, namely,

$$\{e^{i(k_x x + k_y y)} \colon k_x^2 + k_y^2 = k^2\}$$

or

$$\{e^{ikr\cos(\alpha-\theta)} \colon \alpha \text{ is a constant}\}\,.$$

These solutions form a basis for the subspace of solutions to the Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0 \quad .$$

Any solution to this equation is a unique superposition of the basis elements. We shall refer to this subspace as the *eigenspace* of the (degenerate) eigenvalue k^2.

A matrix, and more generally an operator, is diagonal relative to its eigenvector basis. The Helmholtz operator $-\nabla^2$ can, therefore, be viewed as an infinite diagonal matrix

$$-\nabla^2 = \begin{bmatrix} \ddots & & & \\ & k^2 & & 0 \\ & & \ddots & \\ & 0 & & k^2 \\ & & & \ddots \end{bmatrix}$$

with degenerate eigenvalues k^2 along the diagonal.

The question now is, how does one tell the difference between the eigenfunctions having the same eigenvalue k^2? Physically one says that these eigenfunctions are plane waves propagating into different directions. However, one also would like to express the difference algebraically.

5.1.3 Complete Set of Commuting Operators

There is only one way of doing this. It is very direct, and it consists of exhibiting another "matrix", i.e., operator, which

1. has the same domain as ∇^2,
2. has the same eigenvectors that ∇^2 has, but
3. has eigenvalues which are different for different eigenvectors.

Examples of such "matrices" are

$$\frac{1}{i}\,\frac{\partial}{\partial x} \equiv P_x \qquad \text{and} \qquad \frac{1}{i}\,\frac{\partial}{\partial y} \equiv P_y\,.$$

Their eigenvectors are the plane wave solutions,

$$P_x e^{i\vec{k}\cdot\vec{x}} = k_x e^{i\vec{k}\cdot\vec{x}} \qquad \text{and} \qquad P_y e^{i\vec{k}\cdot\vec{x}} = k_y e^{i\vec{k}\cdot\vec{x}}\,,$$

a fact which is also the case for the Helmholtz operator,

$$-\nabla^2 e^{i\vec{k}\cdot\vec{x}} = k^2 \cdot e^{i\vec{k}\cdot\vec{x}}\,.$$

However, note that the eigenvalues, k_x and k_y, are *different* for different plane wave solutions. Thus one has available a very succinct way of characterizing the elements of each degenerate subspace for each eigenvalue k^2 of $-\nabla^2$. This way consists of the statement that the eigenbasis spanning this subspace be labelled by the eigenvalue triplet

$$(k_x, k_y, k^2)$$

of the corresponding three operators

$$\{P_x, P_y, -\nabla^2\}\,.$$

This labelling is unique, i.e., the correspondence

$$\{(k_x, k_y, k^2)\} \leftrightarrow \{e^{i(k_x x + k_y y)} = \psi_{k_x,k_y,k^2}(x, y)\}$$

is unique.The operators $\{P_x, P_y, \nabla^2\}$ form what is called a *complete set of commuting operators* because their eigenvalues (k_x, k_y, k^2) serve as sufficient labels to uniquely identify each of their (common) eigenbasis elements for the vector space of solutions to the Hermholtz equation. No additional labels are necessary.

In addition, notice the following: that the three operators $(P_x, P_y, -\nabla^2 \equiv P_x^2 + P_y^2)$ have the same eigenvectors implies that they *commute*

$$[P_x, P_y] = [P_x, \nabla^2] = [P_y, \nabla^2] = 0\,.$$

In fact, one can show that two operators, each having an eigenbasis for the vector space, commute if and only if they have an eigenbasis in common. This commutativity is a representation independent way of stating that the invariant subspaces of one operator coincide with the invariant subspaces of the other operator, even though their eigenvalues do not. An alternate way of saying this is that

$$\text{span}\{e^{i(k_x x + k_y y)} \colon k_x^2 + k_y^2 = k^2\}$$

is a subspace invariant under P_x, P_y, and ∇^2. To illustrate the commonality of these subspaces, consider the one-dimensional subspace spanned by the eigenvector of the nondegenerate eigenvalue k_x of P_x,

$$P_x\psi = k_x\psi.$$

What can one say about $\nabla^2\psi$? To find out consider $P_x\nabla^2\psi$. Using the fact that $P_x\nabla^2 = \nabla^2 P_x$ one has

$$P_x\nabla^2\psi = \nabla^2 P_x\psi = k_x\nabla^2\psi \ .$$

The fact that the eigenvalue k_x is nondegenerate implies that $\nabla^2\psi$ is a multiple of ψ:

$$\nabla^2\psi = \lambda\psi \ .$$

Thus ψ is also an eigenvector of ∇^2. Thus we have proved an important
Theorem
Suppose that

$$[P_x, \nabla^2] = 0$$

and ψ is an eigenvector belonging to the nondegenerate eigenvalue k_x:

$$P_x\psi = k_x\psi;$$

then ψ is also an eigenvector of ∇^2.

5.1.4 Translations and Rotations in the Euclidean Plane

Lecture 39

What is the significance of the operators

$$P_x = \frac{1}{i}\,\frac{\partial}{\partial x} \qquad\qquad P_y = \frac{1}{i}\,\frac{\partial}{\partial y}\,,$$

and what are they good for? The answer is that they express the translation invariance of the Euclidean plane and that they generate the rectilinear translations of the wave system governed by the Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0\,.$$

Let us see what this means and why this is so.

Point Transformations

The Euclidean plane is characterized by various *symmetry transformations* which leave invariant the distance

$$ds^2 \equiv dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

as well as the Laplacian

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\,\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2}\,. \tag{5.1}$$

There are two obvious types of such transformations:

(i) Translations in the Euclidean plane E^2

(a) along the x-axis by a fixed amount a:

$$\begin{aligned} X_a : \quad & E^2 \to E^2 \\ & (x,y) \rightsquigarrow (x',y') = (x+a, y) \end{aligned} \tag{5.2a}$$

(b) along the y-axis by a fixed amount b:

$$\begin{aligned} Y_b : \quad & E^2 \to E^2 \\ & (x,y) \rightsquigarrow (x',y') = (x, y+b) \end{aligned} \tag{5.2b}$$

(c) and more generally along some generic direction by the fixed vectorial amount $\vec{a}$:

$$\begin{aligned} T_{\vec{a}} : \quad & E^2 \to E^2 \\ & (x,y) \rightsquigarrow (x',y') = (x+a, y+b) \equiv T_{\vec{a}}(x,y); \end{aligned} \tag{5.2}$$

(ii) Rotations in the Euclidean plane around a chosen origin by an angle γ:

$$\begin{aligned} R_\gamma : \quad & E^2 \to E^2 \\ & (r,\theta) \rightsquigarrow (r',\theta') = (r, \theta + \gamma\,). \end{aligned} \tag{5.3}$$

These are point transformations. Even though a transformation takes each point of the Euclidean plane into another, the distance between a pair of points before the transformation is the same as the distance after this pair has been transformed to a new location. This is expressed by the equality

$$\begin{aligned} dx'^2 + dy'^2 &= dx^2 + dy^2 \qquad \text{(invariant)} \\ dr'^2 + r^2 d\theta'^2 &= dr^2 + r^2 d\theta^2 \quad \text{(invariant)} \end{aligned}$$

or, in brief,

$$ds'^2 = ds^2 \quad \text{(invariant)}$$

i.e., the *distance* ds^2 in the Euclidean plane is invariant under translations and rotations. It is also obvious that

$$\nabla'^2 = \nabla^2 \quad . \quad \text{(invariant)}$$

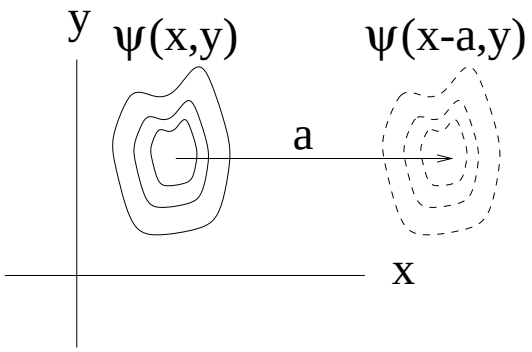

Figure 5.2: The point transformation $x \to x' = x + a$ induces a transformation which acts on functions according to the rule: $\psi(x, y) \to \psi(x - a, y)$. Because of the minus sign, the transformed function is called the "pull-back" of $\psi(x, y)$. What is the line of reasoning giving rise to this pullback? The key is the observation that *the value of the new function, say* ψ' *at the new point, say* x'*, must equal the old function* ψ *at the old point* x. In other words, $\psi(x, y) = \psi'(x', y)$. In light of the fact that $x = x' - a$, one has $\psi(x' - a, y) = \psi'(x', y)$ for *all* x'. Dropping the "prime", one arrives at $\psi(x, y) \to \psi'(x, y) = \psi(x - a, y)$.

Transformations on the Space of Functions

Point transformations such as Eqs.(5.2)-(5.3) induce corresponding transformations $T_{\vec{a}*}$ on the vector space $\mathcal{H}$ of functions defined on E^2,

$$\begin{aligned} &\mathcal{H} \xrightarrow{T_{\vec{a}*}} \mathcal{H} \\ &\psi \rightsquigarrow T_{\vec{a}*}[\psi] = \psi'. \end{aligned} \tag{5.4}$$

The logical leap from the given function ψ to its transform ψ' consists of the requirement that *the new function at the new point be qual to the old function at the corresponding old point*, and that this equality hold for all transformed points. As shown in Figure 5.2, this means that the isograms of ψ ("loci of points where the function has the same value") get pushed forward by $T_{\vec{a}*}$,

$$\psi'(x', y') = \psi(x, y).$$

In light of Eq.(5.2) one has

$$\psi'(x', y') = \psi(x' - a, y' - b) \quad \text{for all } (x', y'). \tag{5.5}$$

Dropping the prime, one finds that the transformed function is

$$\psi'(x, y) = \psi(x - a, y - b). \tag{5.6}$$

This is the explicit form of the function induced by $T_{\vec{a}}$, Eq.(5.2):

$$T_{\vec{a}*}[\psi](x, y) = \psi(x - a, y - b). \tag{5.7}$$

Similarly, corresponding to the point transformations Eqs.(5.2a)-(5.3), one has

$$X_{a*}[\psi](x, y) = \psi(x - a, y), \tag{5.7 a}$$
$$Y_{b*}[\psi](x, y) = \psi(x, y - b), \tag{5.7 b}$$

and

$$R_{\gamma*}[\psi](r, \theta) = \psi(r, \theta - \gamma). \tag{5.8}$$

Each of the functions on the r.h.s. of Eqs.(5.7)-(5.8) is evaluated at a point shifted by a fixed amount. This suggests a Taylor series around the unshifted point. For example,

$$\psi(x-a,y) = \sum_0^\infty \frac{(-a)^n}{n!} \frac{\partial^n \psi(x,y)}{\partial x^n} \tag{5.9}$$

$$\equiv e^{-a\frac{\partial}{\partial x}} \psi(x,y) \tag{5.10}$$

$$= e^{-iaP_x} \psi(x,y)\,. \tag{5.11}$$

See Figure 5.2. Thus, by exponentiating the operator $P_x = \frac{1}{i}\frac{\partial}{\partial x}$ in a way which is identical to exponentiating a matrix, one obtains a linear operator which expresses a translation along the x-axis. This operator

$$e^{-iaP_x} = 1 - iaP_x + \frac{(-ia)^2}{2!}P_x^2 + \cdots \equiv X_{a*}$$

is, therefore, called a *translation operator*. It translates a wave pattern, a solution to the Helmholtz equation from one location to another, i.e.,

$$X_{a*}\psi(x,y) = \psi(x-a,y)\,.$$

This translation transformation is evidently generated by the *translation generator*

$$P_x = \frac{1}{i}\frac{\partial}{\partial x}\,.$$

The effect of the translation operator X_{a*} is particularly simple when that operator is applied to an "eigenvector" of P_x,

$$P_x e^{i(k_x x + k_y y)} = k_x e^{i(k_x x + k_y y)}\,.$$

In that case, one obtains a power series in the eigenvalue k_x,

$$\begin{aligned} X_{a*}e^{i(k_x x+k_y y)} &= e^{-iaP_x}e^{i(k_x x+k_y y)} \\ &= \left[1 - iak_x + \frac{(-ia)^2}{2!}k_x^2 + \cdots\right] e^{i(k_x x+k_y y)} \\ &= e^{-ik_x a}e^{i(k_x x+k_y y)}\,. \end{aligned}$$

Thus, except for the phase factor $e^{-ik_x a}$, the plane wave remaines unchanged. It is a *translation eigenfunction*. In other words, a plane wave is *invariant*

(i.e. gets changed only by a constant phase factor) under translation along the x-axis. This result is the physical significance of the mathematical fact that a plane wave solution is an "eigenvector" of $P_x = \frac{1}{i}\,\frac{\partial}{\partial x}$. *It expresses the physical fact that a plane wave is a translation invariant solution of the Helmholtz equation.*

Analogous considerations lead to the definition of translations along the y-axis and rotations around the origin. Thus, corresponding to the three point transformations (i), (ii), and (iii) earlier in this section, one has the three generators

1. $P_x = \frac{1}{i}\,\frac{\partial}{\partial x}$ "x-translation generator"
2. $P_y = \frac{1}{i}\,\frac{\partial}{\partial y}$ "y-translation generator"
3. $\vec{P} = P_x + P_y$ "generic translation generator"
4. $L_\theta = \frac{1}{i}\,\frac{\partial}{\partial \theta}$ "rotation generator"

which generate the finite transformations

1. $X_{a*} = e^{-i\,a\,P_x}$ "x-translation by a"
2. $Y_{b*} = e^{-i\,b\,P_y}$ "y-translation by b"
3. $T_{\vec{a}*} = e^{-i\,\vec{a}\cdot\vec{P}}$ "generic translation by $\vec{a}$"
4. $R_{\gamma*} = e^{-i\,\gamma\,L_\theta}$ "θ-rotation by γ"

when they are applied to functions defined on the Euclidean plane. For example, the application of the rotation operator $R_{\gamma*}$ to $\psi(r,\theta)$ yields

$$R_{\gamma*}\psi(r,\theta) = \psi(r,\theta-\gamma)\,.$$

5.1.5 Symmetries of the Helmholtz Equations

It is easy to see that if

$$(\nabla^2 + k^2)\psi = 0$$

then

$$X_{a*}\psi\,,\quad Y_{b*}\psi \quad\text{and}\quad R_{\gamma*}\psi$$

are also solutions to the Hermholtz equation. In other words,

$$(\nabla^2 + k^2)(X_a\psi) = 0\,, \quad \text{etc.}$$

This is because the partial derivative can be interchanged and the coefficient of ∇^2 are independent of x, y, and θ. One refers to this independence by saying that x, y and θ are *cyclic* coordinates, or equivalently, that X_{a*}, Y_{b*}, and $R_{\gamma*}$ are *symmetries* of ∇^2.

This independence implies that the eigenspace of ∇^2 is invariant under X_{a*}, Y_{b*}, and also $R_{\gamma*}$. This is a very powerful result. It says that if ψ is a solution, then one obtains the additional solutions

$$X_{a*}\psi, \quad Y_{b*}\psi, \quad \text{and} \quad R_{\gamma*}\psi \quad ,$$

which are parametrized by the translation parameter a and b, and by the angle γ respectively.

5.1.6 Wanted: Rotation Invariant Solutions to the Helmholtz Equation

A plane wave solution $e^{i\vec{k}\cdot\vec{x}}$ is also an *eigenfunction* of the translation operator:

$$X_{a*}e^{i\vec{k}\cdot\vec{x}} = e^{-ik_x a}e^{i\vec{k}\cdot\vec{x}}$$
$$Y_{b*}e^{i\vec{k}\cdot\vec{x}} = e^{-ik_y b}e^{i\vec{k}\cdot\vec{x}}$$

but

$$R_{\gamma*}e^{ikr\cos(\alpha-\theta)} \equiv e^{ikr\cos(\alpha-\theta-\gamma)} \neq \lambda e^{ikr\cos(\alpha-\theta)}$$

for any λ!. In other words, a plane wave solution is *not* an eigenfunction of the rotation operator! Nevertheless, we know that R_γ is a transformation which takes eigenfunctions of ∇^2 into eigenfucntions belonging to the same eigenvalue. This leads to the following question:
Which linear combination of plane waves (having the same k^2) is an eigenfunction of R_γ?

We need a solution to the Helmholtz equation of the form

$$\psi = Z(kr)e^{i\nu\theta} \qquad (\nu = \text{ complex constant})$$

so that

$$R_{\gamma *}\psi = e^{-i\nu\gamma}\psi \qquad (\text{“Rotation eigenfunction } \psi\text{”})\,.$$

If we can find $Z(kr)$ such that

$$(\nabla^2 + k^2)Z(kr)e^{i\nu\theta} = 0\,,$$

then we shall have what we are looking for, namely a solution which is also an *eigenfunction of the rotation operator*.

Using the polar representation of ∇^2, and cancelling out the factor $e^{i\nu\theta}$, we have

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\,\frac{d}{dr} + \left(k^2 - \frac{\nu^2}{r^2}\right)\right\} Z(kr) = 0 \quad ,$$

or with $\rho = kr$,

$$\left\{\frac{d^2}{d\rho^2} + \frac{1}{\rho}\,\frac{d}{d\rho} + \left(1 - \frac{\nu^2}{\rho^2}\right)\right\} Z(\rho) = 0\,.$$

In other words, $Z(\rho)$ must satisfy Bessel's equation.

The first impulse is to solve this equation using infinite series. However, we shall take note of STOKE'S observation: *“series solutions have the advantage of being generally applicable, but are wholly devoid of elegance”*. In our case “elegance” means ability to capture the geometric and physical properties of the Euclidean plane.

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Instead of a series solution, we shall take the question on the previous page seriously and construct an appropriate *superposition* of *plane wave solutions* with a direction-dependent phase shift that varies linearly ($\propto \alpha$) from one plane wave to the next. Such a phase shift is expressed by the phase factor

$$e^{i\nu\alpha}\,,$$

where ν is a constant. The superposition is therefore given by

$$\psi = \int_{\alpha_1}^{\alpha_2} e^{ikr\cos(\alpha-\theta)} e^{i\nu\alpha} d\alpha \quad . \tag{5.12}$$

Different choices of ν will yield different linear combinations. Is each such solutions invariant (i.e. gets changed only by a constant phase factor) under a rotation? To find out, let $\alpha = \overline{\alpha} + \theta$ so that

$$\psi = \underbrace{\int_{\overline{\alpha}_1=\alpha_1-\theta}^{\overline{\alpha}_2=\alpha_2-\theta} e^{ikr\cos\overline{\alpha}} e^{i\nu\overline{\alpha}} d\overline{\alpha}}_{Z(kr)} e^{i\nu\theta} .$$

This superposition has the desired form

$$Z(kr)e^{i\nu\theta}$$

provided the effect of the θ-dependence in the integration limits can be removed. In other words, expression (5.12), which is a solution of

$$0 = \nabla^2 + k^2 = \left(\frac{1}{r}\,\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2} + k^2\right)\psi \quad ,$$

is an eigenfunction of $L_\theta = \frac{1}{i}\,\frac{\partial}{\partial\theta}$ if

$$Z \equiv \int_{\alpha_1-\theta}^{\alpha_2-\theta} e^{ikr\cos\overline{\alpha}} e^{i\nu\overline{\alpha}} d\overline{\alpha}$$

can be shown to be independent of θ. In that case $Z = Z(kr)$, and it necessarily satisfies

$$\left[\frac{1}{r}\,\frac{d}{dr} r \frac{d}{dr} + k^2 - \frac{\nu^2}{r^2}\right] Z(kr) = 0 ,$$

which is Bessel's equation, with ν equal to any complex constant.

Let us, therefore, consider more closely the complex line integral

$$Z_\nu(\rho) = \int_C e^{i\rho\cos\alpha + i\nu\alpha} d\alpha$$

Here we assume, for the time being, that $\rho = |\rho|$ because

$$\rho = kr ,$$

a product of two positive numbers. The integration contour C is a curve in the complex α-plane, whose points are

$$\alpha = p + iq \qquad\qquad p, q \text{ real} .$$

We shall find that the chosen integration contour will start far away from the origin at a point with large positive or negative imaginary part, $q = \pm\infty$, and terminate at another such point, again with $q = \infty$ or $q = -\infty$. This choice has a dual purpose. (i) It guarantees, as we shall see, that the integral converges, and (ii) it guarantees, as we shall see, that the contour integral will be independent of the real angle θ, which is the amount by which the two end points get shifted horizontally in the complex α-plane. The value of the integral itself is independent of the integration path because the integrand is analytic in the whole complex α-plane.

Where shall the starting and termination points of the contour integral be located? This question is answered by the asymptotic behaviour of the dominant terms in the exponent of the integrand,

$$\begin{aligned} i\cos\alpha &= i\cos(p+iq) \\ &= i\cos p\cosh q + \sin p\sinh q\,. \end{aligned}$$

When the real part of this expression becomes large and negative ($\sin p\sinh q \to -\infty$), then the convergence of the integral will be guaranteed because in that case the term $\sin p\sinh q$ dominates over all other contributions to the exponent of the integrand. This is true for all complex numbers ν. The integration

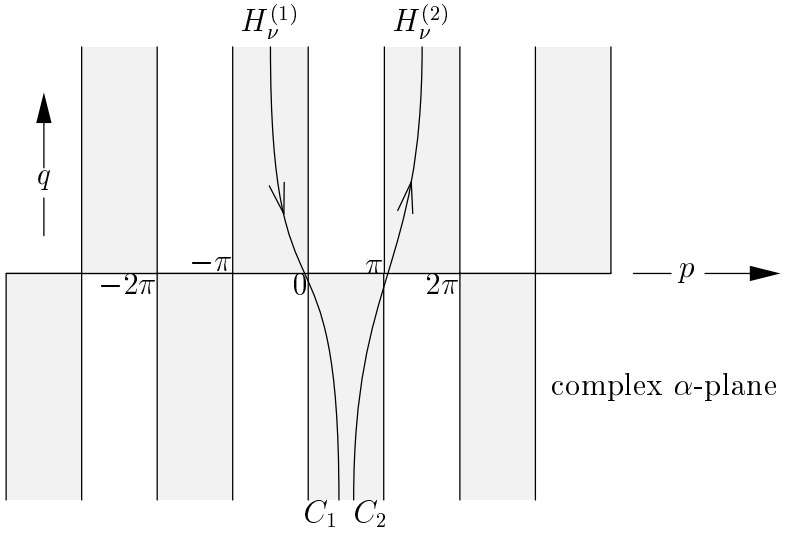


Figure 5.3: Contour integration paths C_1 and C_2 for the two Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$. The shaded regions are the regions of convergence as $q \to \pm\infty$.

contour we choose has endpoints which lie far in the upper α-plane or in the lower plane ($q \to \infty$ or $q \to -\infty$).

To obtain an integral which converges, one must have $\sin p \sinh q \to -\infty$ at both endpoints. This implies that if $q \to \infty$, then the value of p must satisfy

$$\sin p < 0\,, \qquad \text{i.e.,} \quad -\pi < p < 0 \qquad \mod 2\pi\,.$$

On the other hand, if $q \to -\infty$, then the value of p must satisfy

$$0 < \sin p\,, \qquad \text{i.e.,} \quad 0 < p < \pi \qquad \mod 2\pi\,.$$

Thus the integration contour can start and terminate only in one of the shaded regions in the complex α-plane of Figure 5.3.

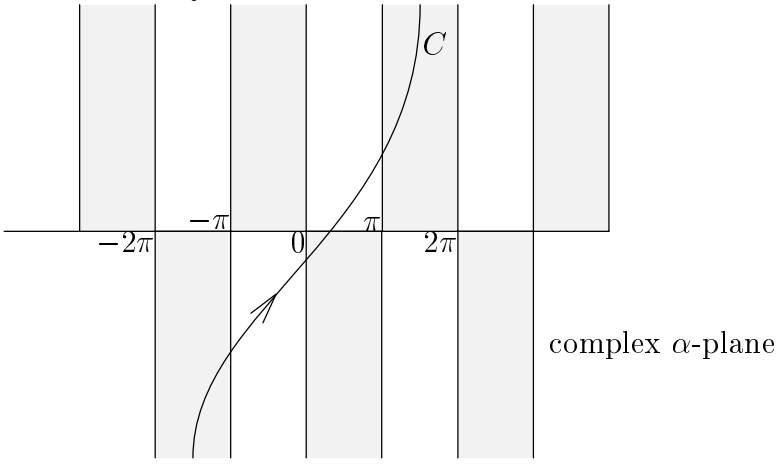
There are only two basic contour integrals that one needs to consider, and they give rise to the two kinds of fundamental functions. They are $H_\nu^{(1)}(\rho)$, the *Hankel function of the first kind* , and $H_\nu^{(2)}(\rho)$, the *Hankel function of the second kind*. All other integration contours give rise to contour integrals which merely are linear combinations of these two fundamental functions.

Moving forward, we shall use in the next subsection these two functions to deduce 23 of their mathematical wave mechanical properties and applications.

Exercise 5.1.1 (DIFFERENT INTEGRATION CONTOUR)
Evaluate the integral

$$\int_C e^{i\rho\cos\alpha + i\nu\alpha} d\alpha$$

along the curve C (in the complex α-plane below) in terms of the two kinds of Hankel functions $H_\nu^{(1)}(\rho)$ and $H_\nu^{(2)}(\rho)$



Exercise 5.1.2 (STRIPS OF CONVERGENCE)
In the complex β-plane determine those semi-infinite strip regions where the line integral

$$\int_C e^{i\rho\cosh\beta - i\omega\beta} d\beta$$

converges if the integration limits of the integration path C are extended to infinity in each of a pair of such strips.

Exercise 5.1.3 (HANKEL FUNCTION AS A DEFINITE INTEGRAL)
By slightly deforming the integration path prove or disprove that the integral

$$\int_{-\infty}^{\infty} e^{i\rho\cosh\beta - i\omega\beta} d\beta$$

can be expressed in terms of a Hankel function. Which kind and which order?

Exercise 5.1.4 (WAVE EQUATION IN PSEUDOPOLAR COORDINATES)
Instead of applying

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

to the Helmholtz equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k^2\psi = 0$$

to obtain

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + k^2\psi = 0 \quad ,$$

apply

$$\begin{aligned} t &= \xi \cosh\tau \qquad 0 < \xi < \infty \\ z &= \xi \sinh\tau \qquad -\infty < \tau < \infty \end{aligned}$$

to the wave equation

$$-\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial z^2} - k^2\psi = 0 \tag{5.13}$$

in order to obtain the wave equation relative to the coordinates ξ and τ. To do this, take advantage of the fact that letting

$$\begin{aligned} r &= \xi \\ \theta &= i\tau \end{aligned}$$

and

$$\begin{aligned} x &= t \\ y &= iz \end{aligned}$$

yields the hyperbolic transformation and the wave equation (5.13).

a) *Write down* the wave equation in terms of the ("pseudo") polar coordinates ξ and τ.

b) Consider a solution which is a ("pseudo") rotation eigenfunction ψ_ω:

$$\frac{\partial\psi_\omega}{\partial\tau} = -i\omega\psi_\omega$$

and *determine* the differential equation

$$[\alpha(\xi)\frac{d^2}{d\xi^2} + \beta(\xi)\frac{d}{d\xi} + \gamma(\xi)]\psi_\omega = 0$$

it satisfies.

c) *Verify* that the translation (in the t, z-plane) eigenfunction

$$\psi = e^{-i(k_0 t - k_z z)}$$

is a solution to the wave Eq.(5.13) whenever the two constants k_0 ("frequency") and k_z ("wave number") satisfy the *dispersion* relation

$$k_0^2 - k_z^2 = k^2 \quad .$$

Then, using $k_0 = k\cosh\alpha$, $k_z = k\sinh\alpha$ (with $k > 0$) and $t = \xi\cosh\tau$, $z = \xi\sinh\tau$, and the hyperbolic angle addition formula, *rewrite* the phase and hence the wave function ψ in terms of ξ and τ.

d) *Construct* a superposition (as an integral over α) of waves ψ which is a ("pseudo") rotation eigenfunction, i.e. satisfies

$$\frac{\partial \psi_\omega}{\partial \tau} = -i\omega\psi_\omega,$$

where ψ_ω is that superposition.

e) *Exhibit* two independent solutions ψ_ω to Eq.(5.13) corresponding to two different integration contours. What are they? If your solutions are proportional to Hankel functions, *specify* what kind, and *identify* their order.

5.2 Properties of Hankel and Bessel Functions

Associated with the two kinds of Hankel functions are two solutions to the Helmholtz equation. They are the "cylinder harmonics" or order ν,

$$\begin{aligned}\psi_1(kr,\theta) &= c_1 \int_{-\varepsilon+i\infty}^{\varepsilon-i\infty} e^{i\rho\cos(\alpha-\theta)+i\nu\alpha} d\alpha \\ &= c_1 \int_{-\varepsilon+i\infty-\theta}^{\varepsilon-i\infty-\theta} e^{i\rho\cos\alpha+i\nu\alpha} d\alpha e^{i\nu\theta} \equiv H_\nu^{(1)}\ e^{i\nu\theta} \qquad (5.14)\end{aligned}$$

and

$$\begin{aligned}\psi_2(kr,\theta) &= c_2 \int_{\pi-\varepsilon-i\infty}^{\pi+\varepsilon+i\infty} e^{i\rho\cos(\alpha-\theta)+i\nu\alpha} d\alpha \\ &= c_2 \int_{\pi-\varepsilon-i\infty-\theta}^{\pi+\varepsilon+i\infty-\theta} e^{i\rho\cos\alpha+i\nu\alpha} d\alpha e^{i\nu\theta} \equiv H_\nu^{(2)}\ e^{i\nu\theta} \qquad (5.15)\end{aligned}$$

Here

$$c_1 = c_2 = \frac{e^{-i\nu\pi/2}}{\pi}$$

are normalization constants whose values are derived below (see Property 11 below).

The name "cylinder harmonic" arises from the fact that these two functions emerge from those solutions of the Helmholtz equation whose level surfaces mold themselves naturally to the cylindrical geometry of its domain. These functions have the following properties:

Property 1
They are linear superpositions of plane waves.

Property 2
Their integration contours in the complex α-plane are as indicated in Figure 5.3.

Property 3 (No angular dependence)
The integral representatives

$$\left.\begin{matrix} H_\nu^{(1)} \\ H_\nu^{(2)} \end{matrix}\right\} = \frac{e^{-i\nu\pi/2}}{\pi} \int_{\alpha_1-\theta}^{\alpha_2-\theta} e^{i\rho\cos\alpha} e^{i\nu\alpha} d\alpha$$

of the two Hankel functions do not depend on any real changes in the integration limits.

This means that the θ-dependent shift in the limits of the integral has no effect on the value of the integral itself, whenever the integration limits α_1 and α_2 each lie near infinity in a strip of convergence of the integral.

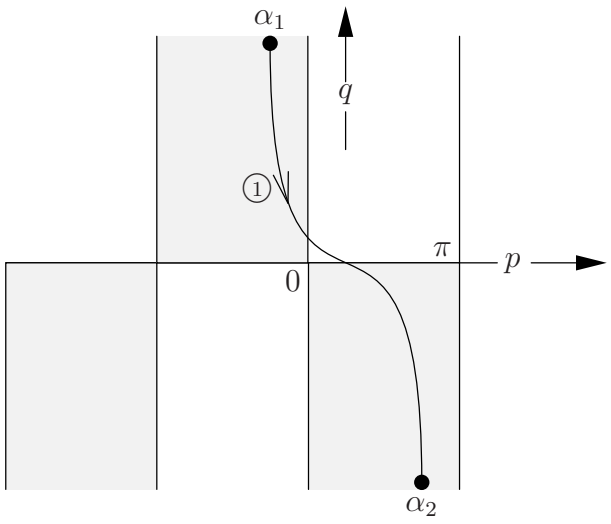


Figure 5.4: Integration contour for a Hankel function. If the integration limits are such that $Im\ \alpha_1 = +\infty$ and $Im\ \alpha_2 = -\infty$ then the integral does not change under horizontal shifts.

Suppose the integration contour is taken to be the curve labelled ①, where α_1 is near the vertical line $p = 0$ and α_2 is near $p = \pi$. Then for $0 \leq \theta < \pi$ we see that

$$\int_{\alpha_1}^{\alpha_2} e^{i\rho\cos\alpha + i\nu\alpha} d\alpha = \int_{\alpha_1-\theta}^{\alpha_2-\theta} e^{i\rho\cos\alpha + i\nu\alpha} d\alpha \,. \tag{5.16}$$

This equality is a result of two facts:

1. The dominant contribution comes from the path between $\alpha_1 - \theta$ and $\alpha_2 - \theta$, and that path can be deformed into the original curve ①and by the Cauchy-Goursat theorem the integral will remain unchanged.

2. The θ-dependent change due to the shift in the end points $\alpha_1 - \theta$ and $\alpha_2 - \theta$ is neglegible, because the integrand is already neglegibly small at these points.

When $-\pi < \theta \leq 0$, then the path of integration can again be deformed into a standard one, but in that case one starts with α_1 and α_2 are near $p = -\pi$ and $p = 0$ instead.

To summarize:

$$\int_{\alpha_1}^{\alpha_2} e^{i\rho\cos(\alpha-\theta) + i\nu\alpha} d\alpha$$

represents a continuous function of θ whenever $-\pi < \theta < \pi$. When this inequality is fulfilled one has

$$\int_{\alpha_1}^{\alpha_2} e^{i\rho\cos(\alpha-\theta) + i\nu\alpha} d\alpha = \int_{\alpha_1}^{\alpha_2} e^{i\rho\cos\alpha + i\nu\alpha} d\alpha \, e^{i\nu\theta} \tag{5.17}$$

If this inequality is not fulfilled, then the right hand side of this equation diverges and is therefore not valid. This is because in this case the contour of the integral on r.h.s. of Eq.(5.16) cannot be deformed into that of the l.h.s.: The integration limits would fall outside the shaded strips of convergence, the integral would diverge, and the r.h.s. of Eq.(5.17) would loose its meaning.

As usual, the circumstance $\theta = -\pi$ or $\theta = \pi$ are defined in terms of limits as $\theta \to \pm\pi$ from the inside of the interval..

Thus we conclude that $H_\nu^{(1)} = H_\nu^{(1)}(kr)$ and $H_\nu^{(2)} = H_\nu^{(2)}(kr)$ are independent of θ indeed. The result is that the two cylinder harmonics have the form

$$\psi(r,\theta) = Z_\nu(\rho) e^{i\nu\theta} \,, \qquad \rho = kr$$

a product of two functions, each one depending separately on its own variable, but independent of the other.

Property 4 (Eigenfunction of rotations in the Euclidean plane)
The cylinder harmonics are eigenfunctions of the rotation generator $L_\theta = \frac{1}{i}\frac{\partial}{\partial\theta}$,

$$L_\theta\psi = \nu\psi\,;$$

that is to say, they are invariant (modulo a constant multiplicative factor) under rotation around the origin

$$\begin{aligned} R_{\gamma *}\psi &\equiv e^{-i\gamma L_\theta}\psi \\ &= e^{-i\gamma\nu}Z_\nu(\rho)e^{i\nu\theta}\,. \end{aligned}$$

Property 5 (Solution to the Helmholtz equation)
They satisfy the Helmholtz's equation, which in polar coordinates becomes Bessel's equation

$$\begin{aligned} 0 &= [\nabla^2 + k^2]\psi \\ &= \left[\frac{1}{r}\,\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + k^2 + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}\right]\psi \\ &= \left[\frac{1}{r}\,\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + k^2 - \frac{\nu^2}{r^2}\right]Z_\nu(kr)e^{i\nu\theta} \\ &= k^2\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\,\frac{d}{d\rho} + 1 - \frac{\nu^2}{\rho^2}\right]\left\{\begin{array}{c} H_\nu^{(1)}(\rho) \\ H_\nu^{(2)}(\rho)\end{array}\right\}e^{i\nu\theta}\,. \end{aligned}$$

Even through the eigenvalue k^2 of the operator $-\nabla^2$ is infinitely degenerate, the eigenvalues of L_θ in the equation

$$\frac{1}{i}\,\frac{\partial}{\partial\theta}\left\{\begin{array}{c} H_\nu^{(1)}(\rho) \\ H_\nu^{(2)}(\rho)\end{array}\right\}e^{i\nu\theta} = \nu\left\{\begin{array}{c} H_\nu^{(1)}(\rho) \\ H_\nu^{(2)}(\rho)\end{array}\right\}e^{i\nu\theta}\,,$$

serve to distinguish the elements of the degenerate set.

Property 6 (Cylinder waves)
The domain of a *cylinder harmonic* is the r and θ coordinatized transverse *cross section* of a cylinder. A cylinder harmonic itself is the r and θ dependent part of a *cylinder wave*

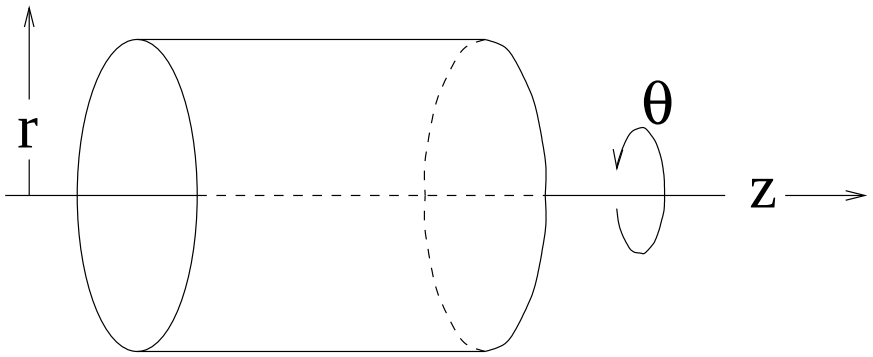


Figure 5.5: The domain of a cylinder harmonic is the perpendicular cross section of a cylindrical configuration.

$$\psi = H_\nu(kr)e^{i\nu\theta}e^{ik_z z}e^{-i\omega t} \quad ,$$

which satisfies the wave equation

$$\left[\frac{1}{r}\,\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\right]\psi = 0$$

whenever the constants k, k_z and ω satisfy the dispersion relation

$$\frac{\omega^2}{c^2} = k^2 + k_z^2\,.$$

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Property 7 (Two kinds of Hankel functions)
The $\rho = kr$ dependent factors of these cylinder harmonics,

$$H_\nu^{(1)}(\rho) = c_1\int_{C_1} e^{i\rho\cos\alpha + i\nu\alpha}d\alpha$$
$$H_\nu^{(2)}(\rho) = c_2\int_{C_2} e^{i\rho\cos\alpha + i\nu\alpha}d\alpha \quad ,$$

are called *Hankel functions* of the first and second *kind* of (complex) *order* ν. The constants c_1 and c_2 are not arbitrary. Their values, as shown in Properties 8 and 11, are equal and are given by Eq.(5.22). Thus one has

$$H_\nu^{(1)}(\rho) = \frac{e^{-i\nu\pi/2}}{\pi} \int_{C_1} e^{i\rho\cos\alpha + i\nu\alpha} d\alpha \tag{5.18}$$

$$H_\nu^{(2)}(\rho) = \frac{e^{-i\nu\pi/2}}{\pi} \int_{C_2} e^{i\rho\cos\alpha + i\nu\alpha} d\alpha \quad . \tag{5.19}$$

It is worthwhile to reemphasize that the integral representations of $H_\nu^{(1)}$ and $H_\nu^{(2)}$ converge and are well defined for *any complex number* ν.

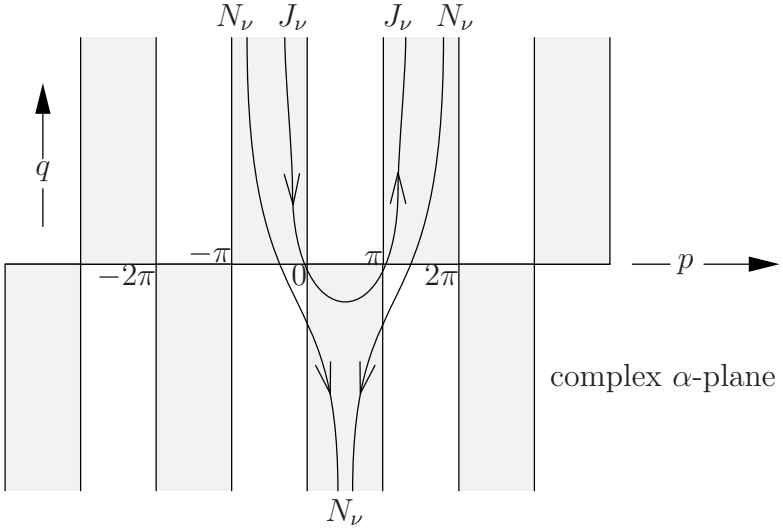
Property 8 (Bessel function)
Having equal normalization constants,

$$c_1 = c_2 \equiv c(\nu)\,.$$

the two Hankel functions, Eqs. (5.14) and (5.15), determine the *Bessel function* of (complex) *order* ν,

$$J_\nu(\rho) = \frac{1}{2}[H_\nu^{(1)}(\rho) + H_\nu^{(2)}(\rho)]\,. \tag{5.20}$$

One arrives at this definition by means of the *union* of the two paths C_1 and C_2 which define $H_\nu^{(1)}$ and $H_\nu^{(2)}$. By the Cauchy-Goursat theorem these paths can be deformed into a *single* path as depicted in Figure 5.7



If c_1 were not equal to c_2, then the contributions to the complex integral from the lower parts of C_1 and C_2 would not cancel.

Figure 5.6: Contour integration paths for the Bessel function J_ν. The integration contour for the Neumann function N_ν is the union of the two indicated paths.

Property 9 (Neumann function)
Their difference

$$N_\nu(\rho) \equiv Y_\nu(\rho) \equiv \frac{1}{2i}[H_\nu^{(1)}(\rho) - H_\nu^{(2)}(\rho)] \tag{5.21}$$

is the *Neumann function* of (complex) *order* ν.

Its integral representation requires the two integration contours depicted in Figure 5.6.

Property 10 (Analogue to trigonometric and exponential functions)
The Hankel functions are the analogues of the *exponential functions*in trigonometry. In fact, as we shall see, one has the following scheme

$$\begin{array}{cccc} e^{ix}\,; & e^{-ix}\,; & \cos x\,; & \sin x \\ H_\nu^{(1)}(\rho)\,; & H_\nu^{(2)}(\rho)\,; & J_\nu(\rho)\,; & N_\nu(\rho)\,. \end{array}$$

$$\cong \quad \sqrt{\tfrac{2}{\pi\rho}}e^{i\left[\rho-\left(\nu+\frac{1}{2}\right)\pi/2\right]}\,; \quad \sqrt{\tfrac{2}{\pi\rho}}e^{-i[\cdots]}\,; \quad \sqrt{\tfrac{2}{\pi\rho}}\cos[\cdots]\,; \quad \sqrt{\tfrac{2}{\pi\rho}}\sin[\cdots]$$

for large ρ, as we shall see later.

The next property asks and answers the the question: How do the Bessel and the Neuman functions depend on their complex order ν? With the universally agreed-upon value for the normalization constant $c(\nu)$, the answer could be no simpler: For real ν these functions are real and for complex ν these functions are their analytic extensions into the complex domain. More precisely, we have

Property 11 (Reflection Principle)

1. If the *order* ν is *real* then their sum (the "Bessel function")

   (a) $J_\nu(\rho) = \frac{1}{2}[H_\nu^{(1)}(\rho) + H_\nu^{(2)}(\rho)]$ is *real* when ν is real and

   (b) $J_0(0) = 1$

provided the normalization constant $c(\nu)$ is

$$c_1 = c_2 \equiv c(\nu) = \frac{e^{-i\nu\pi/2}}{\pi}\,. \tag{5.22}$$

2. If ν is *complex*, then, for fixed positive ρ, both $J_\nu(\rho)$ and $N_\nu(\rho)$ are analytic functions of their order ν. Furthermore, they obey the *reflection principle:*
$$\overline{J_\nu(\rho)} = J_{\overline{\nu}}(\rho) \text{ and } \overline{N_\nu(\rho)} = N_{\overline{\nu}}(\rho)$$

That the Bessel and the Neumann functions are analytic in their order ν follows from their defining integral representations and the form of the normalization constant $c(\nu)$.

The reflection principle is a general property which analytic functions enjoy whenever their values are real on the real (ν) axis. It is shown below that the form of the normalization constant $c(\nu)$ guarantees this. Indeed, for the Bessel function $J_\nu(\rho)$ the proof consists of three steps below. (We delay the application of the reflection principle to the Neumann function until after we have exhibited the complex conjugation property applied to the two Hankel functions on page 341.)

Step 1: Deform the integration path into straight lines. The result is

$$J_\nu(\rho) = \frac{c(\nu)}{2}\left[\int_{-\frac{\pi}{2}+i\infty}^{-\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} + \int_{\frac{3\pi}{2}}^{\frac{3\pi}{2}+i\infty}\right] e^{i\rho\cos\alpha+i\nu\alpha}d\alpha\,.$$

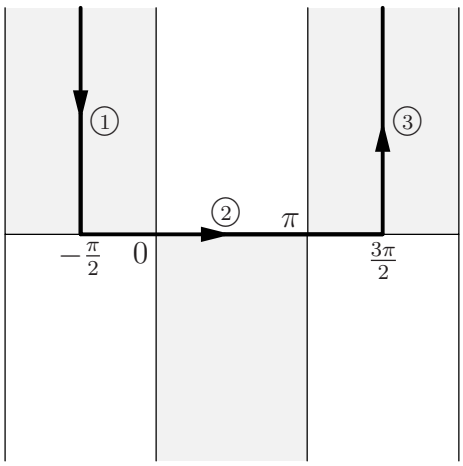


Figure 5.7: Three-part integration contour for the Bessel function.

Step 2: Symmetrize the integrals by shifting the integration limits to the left. This is achieved by introducing the new dummy variable

$$\beta = \alpha - \frac{\pi}{2}\,, \qquad\qquad \alpha = \beta + \frac{\pi}{2}\quad .$$

The result is

$$J_\nu(\rho) = \frac{c(\nu)}{2}\left[\overbrace{\int_{-\pi+i\infty}^{-\pi}}^{①} + \overbrace{\int_{-\pi}^{\pi}}^{②} + \overbrace{\int_{\pi}^{\pi+i\infty}}^{③}\right] e^{i\rho\sin\beta}e^{i\nu\beta}d\beta\; e^{i\nu\pi/2}\,.$$

Reminder: We have not shifted the path of integration. Instead, we have only altered the coordinate labelling used to describe that path in the complex plane.
Step 3:

1. Fix the normalization constant $c(\nu)$ by requiring that J_ν be real when ν is real. This is achieved by setting $c(\nu) = \frac{1}{\pi}e^{-i\nu\pi/2}$. This cancels out the last factor.

2. To bring this reality of J_ν to light, combine the first and third integral by introducing

   $$\left.\begin{array}{ll}\beta = -\pi + i\gamma & \text{for } ① \\ \beta = \pi + i\overline{\gamma} & \text{for } ③\end{array}\right\} \quad \begin{array}{l}\text{to make the}\\ \text{integration limits}\\ \text{equal!}\end{array}$$

   The result, after dropping the bar, is

$$\begin{aligned} J_\nu(\rho) &= \frac{1}{2\pi}\int_0^{\gamma=\infty} e^{-\rho\sinh\gamma-\nu\gamma}d\gamma\,\underbrace{[\overbrace{(-i)e^{-i\nu\pi}}^{①} + \overbrace{ie^{i\nu\pi}}^{③}]}_{-2\sin\nu\pi} \\ &+ \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\rho\sin\beta+i\nu\beta}d\beta}_{\frac{1}{\pi}\int_0^{\pi}\cos(\rho\sin\beta-\nu\beta)d\beta} \end{aligned}$$

Conclusion: When ν is real, then

$$\begin{aligned} &J_\nu(\rho) \text{ is real} \\ &J_0(0) = 1\,, \end{aligned}$$

also, if ν is complex, then by inpection one finds

$$\overline{J_\nu(\rho)} = J_{\overline{\nu}}(\rho) \quad ,$$

which is what we set out to show.

Property 12 (Bessel function of integral order)
The Bessel functions of integral order ($\nu = m = 0, 1, 2, \dots$) is given by

$$\begin{aligned} J_m(\rho) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\rho\sin\beta + im\beta} d\beta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\rho\sin\beta - im\beta} d\beta \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos(\rho\sin\beta - m\beta) d\beta\,. \end{aligned}$$

Furthermore, the set $\{J_m, J_{-m}\}$ forms a linearly dependent set. Indeed,

$$J_{-m}(\rho) = (-1)^m J_m(\rho) \qquad\qquad m = 0, 1, 2 \dots\,. \tag{5.23}$$

This equation is the result of changing the integration variable β. Letting $\beta = \pi - \overline{\beta}$, one obtains

$$\begin{aligned} J_m(\rho) &= \frac{1}{2\pi}\int_{2\pi}^{0} e^{-i\rho\sin\overline{\beta} - im\overline{\beta}} (-1)^m (-) d\overline{\beta} \\ &= (-1)^m \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\rho\sin\overline{\beta} - im\overline{\beta}} d\overline{\beta} \\ &= (-1)^m J_{-m}(\rho)\,. \end{aligned}$$

Exercise 5.2.1 (HANKEL AND NEUMANN FUNCTIONS OF INTEGRAL ORDER)
Show that

$$\begin{aligned} H^{(1)}_{-n}(\rho) &= (-1)^n H^{(1)}_n(\rho) \\ H^{(2)}_{-n}(\rho) &= (-1)^n H^{(2)}_n(\rho) \\ N_{-n}(\rho) &= (-1)^n N_n(\rho) \end{aligned}$$

Lecture 42

Property 13 (Power series)
The Bessel function $J_\nu(\rho)$ of complex order and for real ρ has the following Frobenious expansion around the origin

$$\begin{aligned} J_\nu(\rho) &= \left(\frac{\rho}{2}\right)^\nu \left[\frac{1}{\Gamma(\nu+1)} - \frac{1}{1!\Gamma(\nu+2)}\left(\frac{\rho}{2}\right)^2 + \frac{1}{2!\Gamma(\nu+3)}\left(\frac{\rho}{2}\right)^4 - \cdots\right] \\ &= \left(\frac{\rho}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-\rho/2)^{2k}}{\Gamma(1+\nu+k)k!} \end{aligned} \tag{5.24}$$

Remark: Near $\rho = 0$ the dominant behaviour of $J_\nu(\rho)$ is given by

$$\boxed{J_\nu(\rho) = \frac{1}{\Gamma(1+\nu)}\left(\frac{\rho}{2}\right)^\nu \qquad \rho \ll 1\ .}$$

The power series, together with its normalization constant, follows from the integral representation

$$J_\nu(\rho) = \frac{e^{-i\nu\pi/2}}{2}\int_C e^{i\rho\cos\alpha + i\nu\alpha} d\alpha\,,$$

where C is the integration contour indicated in Figure 5.7. Indeed, introduce the new variable of integration

$$z = \frac{\rho}{2}e^{-i(\alpha - 3\pi/2)}, \quad d\alpha = i\frac{dz}{z}, \quad e^{i\alpha} = \frac{\rho}{2}\frac{1}{z}e^{i3\pi/2} \quad .$$

Under this change, the new integration contour is the one depicted in Figure 5.8, which is based on the following scheme:

$$\begin{array}{c|c|c|c|c|c|c} \alpha = & i\infty - \frac{\pi}{2} & -\frac{\pi}{2} & 0 & \frac{\pi}{2} & \frac{3\pi}{2} & i\infty + \frac{\pi}{2} \\ \hline z = & \infty & \frac{\rho}{2} & \frac{\rho}{2}\exp\frac{3\pi i}{2} & \frac{\rho}{2}\exp\pi i & \frac{\rho}{2}\exp\frac{\pi i}{2} & \infty \end{array}\ .$$

The integral becomes

$$J_\nu(\rho) = -\frac{e^{i\nu\pi}}{2\pi i}\left(\frac{\rho}{2}\right)^\nu \int_{Z_0} e^{-z + \frac{\rho^2}{4z}} z^{-\nu-1} dz \quad .$$

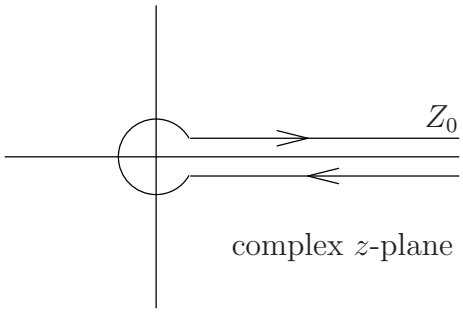


Figure 5.8: The transformed integration contour in the complex z-plane surrounds and touches the branch cut of the multiple-valued function $z^{-(\nu+1)}$.

By expanding the exponential $e^{\frac{\rho^2}{4z}}$ in a Taylor series one obtains Eq.(5.24), provided one sets

$$\frac{1}{\Gamma(\nu+1)} = \frac{e^{i\pi(\nu+1)}}{2\pi i}\int_{Z_0} e^{-z}z^{-(\nu+1)}dz \quad ,$$

which is one of the definitions of the gamma function. This contour integral is meaningless unless one specifies the branch of the multiple-valued function $z^{-(\nu+1)}$. The branch is dictated by the requirement that

$$\frac{1}{\Gamma(m+1)} = \frac{1}{m!} \ .$$

For this branch the domain is restricted to $0 < \arg z < 2\pi$ whenever the cut for this branch is the positive x-axis, as in Figure 5.8.

Let us look at the solutions to Bessel's equation from the viewpoint of linear algebra. The solution space is two dimensional. There are two important spanning sets. The first one,

$$\left\{ J_\nu \simeq \frac{(\rho/2)^\nu}{\Gamma(1+\nu)},\ J_{-\nu} \simeq \frac{(\rho/2)^{-\nu}}{\Gamma(1-\nu)} \right\}$$

is simple whenever $\rho \ll 1$. By contrast, the second one

$$\left\{ H_\nu^{(1)} \simeq \sqrt{\frac{2}{\pi\rho}}e^{i[\rho-(\nu+\frac{1}{2})\frac{\pi}{2}]},\ H_\nu^{(2)} \simeq \sqrt{\frac{2}{\pi\rho}}e^{-i[\rho-(\nu+\frac{1}{2})\frac{\pi}{2}]} \right\}$$

is simple whenever $1 \ll \rho$. However, we know that these two bases are related by a linear transformation.

$$\left[\begin{array}{cc} H_\nu^{(1)} & H_\nu^{(2)} \end{array} \right] = \left[\begin{array}{cc} J_\nu & J_{-\nu} \end{array} \right] \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$$

The question is: what is this linear transformation? The answer is provided by the following

Property 14 (Hankel-Bessel relation)
When the order of a Bessel function is not an integer ($\nu \neq m$), then the set of Bessel functions $\{J_\nu, J_{-\nu}\}$ form an independent set. Moreover, one has

$$H_\nu^{(1)}(\rho) = \frac{e^{-i\nu\pi} J_\nu(\rho) - J_{-\nu}(\rho)}{-i \sin \pi\nu} \tag{5.25}$$

$$H_\nu^{(2)}(\rho) = \frac{e^{i\nu\pi} J_\nu(\rho) - J_{-\nu}(\rho)}{i \sin \pi\nu} \,. \tag{5.26}$$

Once one recalls the defining equation for the Bessel equation, Eq.(5.20), i.e. that $H_\nu^{(2)} = 2J_\nu - H_\nu^{(1)}$, one finds that Eq.(5.25) is a mere consequence of Eq.(5.26). Thus one's primary concern is the validity of that second equation. However, before validating it, let us identify some consequences of the fundamental identities, Eqs.(5.25) and (5.26).

Property 15 (Complex conjugation property)
If the Hankel functions are of real order ν (and $\rho = |\rho|$), then

$$\overline{H_\nu^{(2)}(\rho)} = H_\nu^{(1)}(\rho) \,, \qquad \left(\begin{array}{l} \text{This is analogous to} \\ \overline{e^{-ix}} = e^{ix} \ \text{ for} \\ \text{real } x \end{array} \right)$$

i.e., they are complex conjugates of each other. This follows from equations 5.25 and 5.26 of Property 13.

Remark: There are three additional consequences:
First of all, it follows from Property 12 that if ν is complex, then

$$\overline{J_\nu(\rho)} = J_{\overline{\nu}}(\rho) \Rightarrow \overline{H_\nu^{(2)}(\rho)} = H_{\overline{\nu}}^{(1)}(\rho) \,.$$

Second, apply this complex conjugation property to the defining Eq.(5.21) and obtain the reflection principle applied to the Neumann functions

$$\overline{N_\nu(\rho)} = N_{\overline{\nu}}(\rho) \ .$$

Third, if ρ is complex also, then

$$\overline{H_\nu^{(2)}(\rho)} = H_{\overline{\nu}}^{(1)}(\overline{\rho}) \ .$$

Returning to the validation of the Hankel-Bessel identities, one finds that this process consists of four steps. They consist primarily of manipulating the intergration paths of the integral representations of J_ν and $J_{-\nu}$.
Step 1. Recall the definition of $J_{-\nu}(\rho)$:

$$\begin{aligned} 2\pi e^{-i\nu\pi/2} J_{-\nu}(\rho) &= \int_{C_0} e^{i\rho\cos\alpha - i\nu\alpha} d\alpha \\ &= -\int_{\overline{C}} e^{i\rho\cos\alpha + i\alpha} d\overline{\alpha} \qquad \overline{\alpha} = -\alpha \ , \text{and then drop the "bar"} \end{aligned}$$

Here the $\overline{C}$ is the path inversion symmetric to C_0. It is obtained by drawing a straight line through the origin and extending it by an equal amount to the corresponding old point on $\overline{C}$.

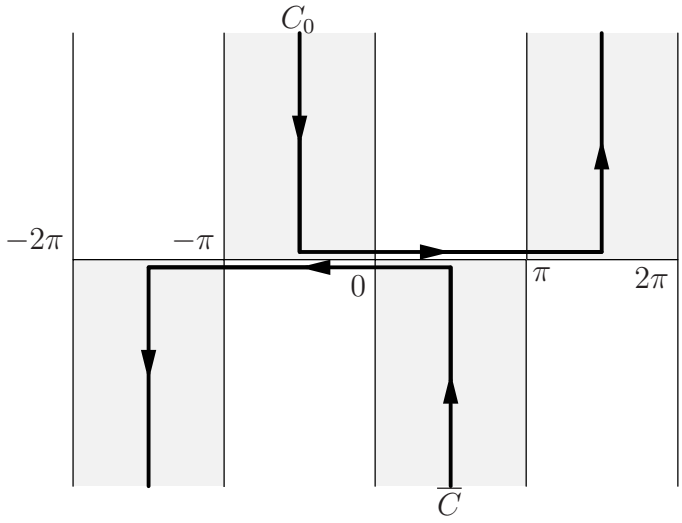


Figure 5.9: Integration path C_0 and $\overline{C}$, its inversion symmetric twin.

The rest of the proof consists of
Step 2. Subtract this from an analogous expression for J_ν.
Step 3. Deform the contour and reexpress the r.h.s. in terms of $H_\nu^{(2)}$. This yields the desired equation.

Step 4. Use Property 8 to obtain the expression for $H_\nu^{(1)}$.

The details of these remaining steps are

Step 2. Subtract the expression in Step 1 from the analogous expression for J_ν. After a slight deformation of the two respective integration path, obtain

$$\begin{aligned} 2\pi(e^{i\nu\pi/2}J_\nu - e^{-i\nu\pi/2}J_{-\nu}) &= \left[\int_{C_0} + \int_{\overline{C}}\right] e^{i\rho\cos\alpha + i\nu\alpha} d\alpha \\ &= \left[\int_{C_2} + \int_{C_2'}\right] e^{i\rho\cos\alpha + i\nu\alpha} d\alpha \,. \end{aligned}$$

Here C_2 is the integration contour for $H_\nu^{(2)}$ and C_2' is $-C_2$ shifted by 2π to the left.

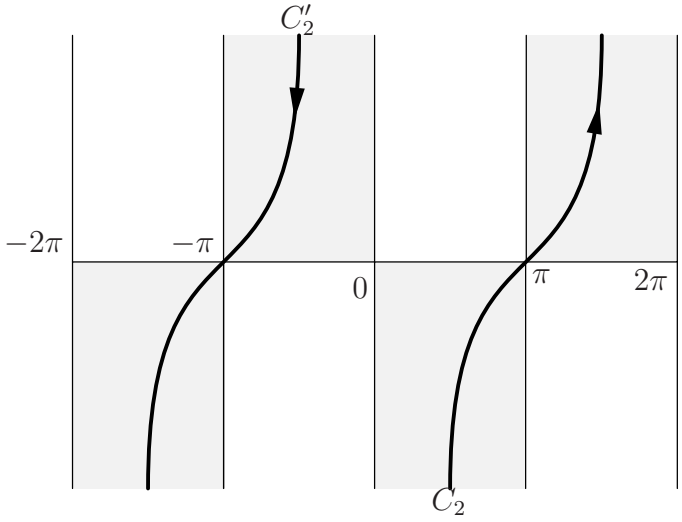


Figure 5.10: The two paths in Figure 5.9 have been deformed into C_2' and C_2.

Step 3.

1. Recall from Property 7 that

$$\int_{C_2} e^{i\rho\cos\alpha + i\nu\alpha} d\alpha = \pi e^{i\nu\pi/2} H_\nu^2(\rho) \,.$$

2. In addition, we have

$$\begin{aligned}\int_{C_2'} e^{i\rho\cos\alpha+i\nu\alpha}d\alpha &= \int_{-\pi+\varepsilon+i\infty}^{-\pi-\varepsilon-i\infty} e^{i\rho\cos\alpha+i\nu\alpha}d\alpha \qquad \overline{\alpha}=\alpha+2\pi \\ &= \int_{\pi-\varepsilon+i\infty}^{\pi-\varepsilon-i\infty} e^{i\rho\cos\overline{\alpha}+i\nu\overline{\alpha}}d\overline{\alpha}e^{-i\nu 2\pi} \\ &= (-)\pi e^{i\nu\pi/2}H_\nu^{(2)}(\rho)e^{-i\nu 2\pi}\,.\end{aligned}$$

3. Introduce the results of 1. and 2. into the last expression in Step 2, and obtain

$$2\pi(e^{i\nu\pi/2}J_\nu - e^{-i\nu\pi/2}J_{-\nu}) = \pi(e^{i\nu\pi/2}-e^{-i3\pi\nu/2})H_\nu^{(2)}\,.$$

Using

$$e^{\nu\pi/2}-e^{-3\pi\nu/2} = 2ie^{-i\pi\nu/2}\sin\nu\pi\,,$$

and solving for $H_\nu^{(2)}$ one obtains

$$H_\nu^{(2)} = \frac{e^{i\nu\pi}J_\nu - J_{-\nu}}{i\sin\nu\pi}\,.$$

Step 4. Use Property 8 to obtain

$$\begin{aligned}H_\nu^{(1)} &= 2J_\nu - H_\nu^{(2)} = 2J_\nu - \frac{e^{i\nu\pi}J_\nu - J_{-\nu}}{i\sin\nu\pi} \\ &= \frac{-e^{-i\nu\pi}J_\nu + J_{-\nu}}{i\sin\nu\pi}\,.\end{aligned}$$

These are the two expressions for the two kinds of Hankel functions in terms of Bessel functions of order ν and $-\nu$.

Property 16 (Contiguity relations)
Let $Z_\nu = H_\nu^{(1)}$ or $H_\nu^{(2)}$ or J_ν or $N_\nu(\equiv Y_\nu)$ be any solution to Bessel's equation of *complex order* ν. Then

$$Z_{\nu+1}(\rho) + Z_{\nu-1}(\rho) = \frac{2\nu}{\rho}Z_\nu(\rho) \tag{5.27}$$

$$Z_{\nu+1}(\rho) - Z_{\nu-1}(\rho) = -2\frac{d}{d\rho}Z_\nu(\rho)\,. \tag{5.28}$$

Proof (in two steps):

Step 1. Apply the definition to the sum and difference

$$\left.\begin{matrix} I_1 \\ I_2 \end{matrix}\right\} = \frac{\pi}{2}\{H_{\nu+1} \pm H_{\nu-1}\}$$
$$= \int_C e^{i\rho\cos\alpha} e^{i\nu\left(\alpha-\frac{\pi}{2}\right)} \frac{1}{2}[e^{i\left(\alpha-\frac{\pi}{2}\right)} \pm e^{-i\left(\alpha-\frac{\pi}{2}\right)}]d\alpha\,.$$

Step 2. Observe that

$$\frac{1}{2}[e^{i\left(\alpha-\frac{\pi}{2}\right)} \pm e^{-i\left(\alpha-\frac{\pi}{2}\right)}] = \left\{ \begin{matrix} \cos\left(\alpha-\frac{\pi}{2}\right) = \sin\alpha \\ i\sin\left(\alpha-\frac{\pi}{2}\right) = -i\cos\alpha \end{matrix}\right.$$

Consequently,

$$I_1 = \frac{-1}{i\rho}\int_C \frac{d}{d\alpha}(e^{i\rho\cos\alpha}) e^{i\nu\left(\alpha-\frac{\pi}{2}\right)} d\alpha$$
$$= \frac{-1}{i\rho}\int_C (-) e^{i\rho\cos\alpha} \frac{d}{d\alpha}(e^{i\nu\left(\alpha-\frac{\pi}{2}\right)})d\alpha = \pi\frac{\nu}{\rho}H_\nu$$
$$I_2 = -\pi\frac{d}{d\rho}H_\nu\,.$$

These are the two recursion relations (5.27) and (5.28).

These relations are quite useful. Note that by adding and subtracting the recursion relations one obtains

$$Z_{\nu+1}(\rho)e^{i(\nu+1)\theta} = -e^{i\theta}\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\,\frac{\partial}{\partial\theta}\right) Z_\nu(\rho)e^{i\nu\theta}$$
$$Z_{\nu-1}(\rho)e^{i(\nu-1)\theta} = e^{-i\theta}\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\,\frac{\partial}{\partial\theta}\right) Z_\nu(\rho)e^{i\nu\theta} \quad .$$

Let us call

$$e^{i\theta}\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\,\frac{\partial}{\partial\theta}\right) = \frac{1}{k}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \equiv L_+ \qquad \text{the } \textit{raising operator}$$

and

$$-e^{-i\theta}\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\,\frac{\partial}{\partial\theta}\right) = -\frac{1}{k}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \equiv L_- \qquad \text{the } \textit{lowering operator}\,.$$

These operators yield

Property 17 (Raising and lowering the order of Z_ν)

$$\begin{aligned} L_+ Z_\nu(\rho) e^{i\nu\theta} &= -Z_{\nu+1}(\rho) e^{i(\nu+1)\theta} \\ L_- Z_\nu(\rho) e^{i\nu\theta} &= -Z_{\nu-1}(\rho) e^{i(\nu-1)\theta} \end{aligned}$$

and

$$\begin{aligned} L_+ L_- = L_- L_+ &= -\frac{1}{k^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ &= -\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\,\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\,\frac{\partial^2}{\partial \theta^2}\right] \equiv -\frac{1}{k^2}\nabla^2 , \end{aligned} \quad (5.29)$$

which is the (rotationally and translation) invariant Laplacian operator.

Comment: (*Factorization Method for Finding Cylinder Harmonics.*)
It is difficult to exclude these raising and lowering operators as the fastest way for establishing relationships between normal modes in a cylindrical cavity. For example, suppose one knows explicitly the rotationally symmetric ($\nu = 0$) mode $J_0(\rho)e^{i0\cdot\theta}$. Then all the other modes

$$J_m(\rho)e^{im\theta} = (-1)^m (L_+)^m J_0(\rho) \quad ,$$

can be obtained by repeated application of the raising operator L_+. The lowering operator undoes the work of the raising operators

$$L_- L_+ Z_\nu e^{i\nu\theta} = Z_\nu e^{i\nu\theta} ,$$

i.e.,

$$(\nabla_\rho^2 + 1) Z_\nu e^{i\nu\theta} = 0 .$$

This feature also illustrates the fact that the 2-dimensional *Laplacian* $\nabla^2 = -L_- L_+$ (in E^2) can be factorized into a product of *first order operators*. This method is purely algebraic and its essentials are as follows:

Recalling the definition of the rotation generator $L_\theta = \frac{1}{i}\frac{\partial}{\partial\theta}$ in Section 5.1.4, notice that

$$[L_\theta, L_\pm] \equiv L_\theta L_\pm - L_\pm L_\theta = \pm L_\pm \ .$$

This commutation relation is fundamental for the following reason:

Suppose we have a solution to the Helmholtz equation

$$(\nabla^2 + k^2)\psi_m = 0 \ ,$$

and suppose that the solution is a *rotation eigenfunction*, i.e.

$$L_\theta \psi_m = m\psi_m \ .$$

Then the commutation relation implies

$$\begin{aligned} L_\theta L_+ \psi_m &= (L_+ L_\theta + L_+)\psi_m \\ &= (m+1)L_+\psi_m \ . \end{aligned}$$

In other words, $L_+\psi_m$ is another rotation eigenfunction. Furthermore,

$$\begin{aligned} (\nabla^2 + k^2)L_+\psi_m &= (-k^2 L_+ L_- + k^2)L_+\psi_m \\ &= L_+(-k^2 L_- L_+ + k^2)\psi_m \\ &= L_+(\nabla^2 + k^2)\psi_m \\ &= 0 \ , \end{aligned} \tag{5.30}$$

i.e. the new rotation eigenfunction

$$L_+\psi_m \equiv \psi_{m+1}$$

is again a solution to the Helmholtz equation. The analogous result holds for $L_-\psi_m$.

To summarize: The algebraic method for solving the Helmholtz equation is a two step process: (i) Factor the Laplacian, Eq.(5.1) into two factors L_+ and L_-, and (ii) for each eigenspace of ∇^2 construct a basis using L_+ and L_-, whose capability as raising and lowering operators is implied by the two commutation relations

$$[L_\theta, L_\pm] = \pm L_\pm \ . \tag{5.31}$$

These operators obviously commute,

$$[L_+, L_-] = 0 \ . \tag{5.32}$$

This is evident from Eq.(5.29). Furthermore, as we have seen from Eq.(5.30), the fact that

$$[\nabla^2, L_\pm] = 0 \tag{5.33}$$

acts as a guarantee that all the basis elements obtained from these raising and lowering operators lie in the same subspace characterized by the degenerate eigenvalue $-k^2$ of the Laplacian.

For illustrative purposes we compute the first few cylinder harmonics. Starting with $Z_0(\rho) = Z_0(kr)$, one obtains:

$$\begin{aligned}
Z_1 e^{i\theta} &= -e^{i\theta}\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\,\frac{\partial}{\partial\theta}\right) Z_0 \\
\Rightarrow Z_1 &= -Z_0' \qquad (5.34)\\
Z_2 e^{2i\theta} &= -e^{i\theta}\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\,\frac{\partial}{\partial\theta}\right)(-)e^{i\theta} Z_0' \\
&= e^{2i\theta}\left(Z_0'' - \frac{1}{\rho}Z_0'\right) \\
\Rightarrow Z_2 &= Z_0'' - \frac{1}{\rho}Z_0' \\
Z_3 e^{3i\theta} &= -e^{i\theta}\left[\frac{\partial}{\partial\rho} + \frac{i}{\rho}\,\frac{\partial}{\partial\theta}\right] e^{2i\theta} Z_2 \\
&= -e^{3i\theta}\left[\left(Z_0'' - \frac{1}{\rho}Z_0'\right)' - \frac{2}{\rho}\left(Z_0'' - \frac{1}{\rho}Z_0'\right)\right] \\
&= -e^{3i\theta}\left[Z_0''' - \frac{3}{\rho}Z_0'' + \frac{3}{\rho^2}Z_0'\right] \\
\Rightarrow Z_3 &= -Z_0''' + \frac{3}{\rho}Z_0'' - \frac{3}{\rho^2}Z_0' \\
Z_4 e^{4i\theta} &= -e^{i\theta}\left[\frac{\partial}{\partial\rho} + \frac{i}{\rho}\,\frac{\partial}{\partial\theta}\right] e^{3i\theta} Z_3 \\
&= -e^{4i\theta}\left[\left(-Z_0''' + \frac{3}{\rho}Z_0'' - \frac{3}{\rho^2}Z_0'\right)' - \frac{3}{\rho}\left(-Z_0''' + \frac{3}{\rho}Z_0'' - \frac{3}{\rho^2}Z_0'\right)\right] \\
&= e^{4i\theta}\left[Z_0^{iv} - \frac{6}{\rho}Z_0''' + \frac{12}{\rho^2}Z_0'' - \frac{15}{\rho^3}Z_0'\right] \\
\Rightarrow Z_4 &= Z_0^{iv} - \frac{6}{\rho}Z_0''' + \frac{12}{\rho^2}Z_0'' - \frac{15}{\rho^3}Z_0'\,. \qquad (5.35)
\end{aligned}$$

Property 18 (Plane wave as a combination of cylinder waves)
Recall that the cylinder harmonics of (complex) order ν where constructed as a linear superposition of plane wave solutions

$$\begin{aligned}\psi(r,\theta) &= H_\nu(kr)e^{i\nu\theta} \\ &= \frac{e^{-i\nu\pi/2}}{\pi}\int_C e^{ikr\cos(\alpha-\theta)}e^{i\nu\alpha}d\alpha\,.\end{aligned}$$

Let us now consider that harmonic which satisfies (i) the periodicity requirement

$$\psi(r,\theta+2\pi)=\psi(r,\theta)$$

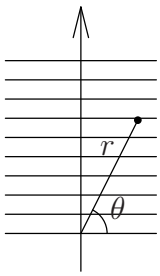
and (ii) the requirement of being finite at the origin $r=0$. These boundary conditions give rise to the cylinder waves

$$\begin{aligned}J_m(kr)e^{im\theta} &= \frac{e^{-im\pi/2}}{2\pi}\int_0^{2\pi} e^{ikr\cos\alpha}e^{im\alpha}d\alpha\; e^{im\theta} \qquad m=0,\pm1,\ldots \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{ikr\sin\overline{\alpha}}e^{-im\overline{\alpha}}d\overline{\alpha}\; e^{im\theta} \qquad \overline{\alpha}=\frac{\pi}{2}-\alpha\end{aligned}$$

We see that this is the mth Fourier coefficient of $e^{ikr\sin\theta}$ in disguise:

$$e^{ikr\sin\theta}=\sum_{m=-\infty}^{\infty} J_m(kr)e^{im\theta}\,. \tag{5.36}$$

This represents a plane wave propagating along the y axis, and it is easy to remember.



More generally, an arbitrary plane wave is also represented as a linear combination of cylinder waves.

Figure 5.11: Isograms of $Re\ e^{ikr\sin\theta}$. The arrow points into the direction of increasing phase.

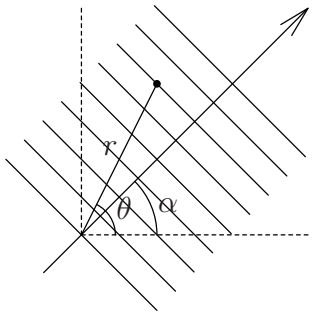


Figure 5.12: Isograms of $Re\ e^{ikr\cos(\theta-\alpha)}$.

Indeed, if one replaces θ with $\frac{\pi}{2}-(\alpha-\theta)$ in Eq.(5.36), one obtains

$$\begin{aligned} e^{i(k_x x+k_y y)} &= e^{ikr\cos(\theta-\alpha)} \\ &= \sum_{m=-\infty}^{\infty} J_m(kr) i^m e^{im(\theta-\alpha)} \quad . \end{aligned} \tag{5.37}$$

This means that any plane wave in the Euclidean plane can be represented as a linear combination of cylinder harmonics of integral order.
Remark: The plane waves $e^{ikr\sin\theta}$ and $e^{ikr\cos(\theta-\alpha)}$ are sometimes called *generating functions* of the Bessel functions of integral order. By considering appropriate derivatives and power series expansions one obtains various identities among these Bessel functions.

5.3 Applications of Hankel and Bessel Functions

Being rotation eigenfunctions, cylinder harmonics are the natural framework for the solution of a variety of boundary value problems. Their solutions are fundamental because the underlying philosophy can be readily extended to higher dimensions and other geometries.

5.3.1 Exterior Boundary Value Problem: Scattering

Let us apply the properties of the Bessel function to solve the following *exterior boundary value ("scattering") problem*:
Find that solution to the Helmholtz equation $(\nabla^2+k^2)\psi = 0$ in the Euclidean plane which satisfies

1. the Dirichlet boundary condition on the circular boundary $r = a$ and
2. the condition that its asymptotic form, as $r \to \infty$, is that of a plane wave propagating into the x-direction,

$$\psi_{inc} = e^{ikr\cos\theta}$$

plus *only outgoing* waves, if any; i.e. *no* incoming waves.

Mathematically the second condition is a type of boundary condition at infinity. It is evident that this boundary condition states that the solution consists of "plane wave + outgoing wave". The physical meaning of this condition is that it represents a scattering process.

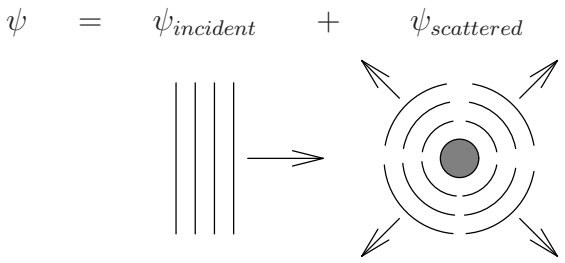


Figure 5.13: Scattering by a cylinder. An incoming plane wave $\psi_{incident}$ in the presence of a cylindrical boundary gives rise to a circular scattered wave $\psi_{scattered}$ which at large radii propagates away from the cylindrical boundary.

If the circular boundary were absent, then there would have been no scattering. The Dirichlet boundary condition 1. would have been replaced by the regularity requirement that ψ = finite at $r = 0$ while the second boundary condition 2. at $r = \infty$ would have remained the same. In that case the resulting "no scattering" solution is immediate, namely,

$$\psi = e^{ikr\cos\theta} = \sum_{m=-\infty}^{\infty} J_m(kr) i^m e^{im\theta}\,.$$

By contrast, if the circular boundary *is* present as stipulated by the problem, then this solution must be augmented so that the Dirichlet boundary conditions are satisfied,

$$\psi = e^{ikr\cos\theta} + \psi_{\text{scatt}} \quad .$$

This augmentation can be implemented with Hankel functions of the first kind, or of the second kind, or with a combination of the two. The boundary condition that the solution represent a plane wave plus a scattered wave, *outgoing* only, demands that the augmentation have the form

$$\psi_{\text{scatt}} = \sum_{m=-\infty}^{\infty} a_m H_m^{(1)}(kr) i^m e^{im\theta} \quad .$$

It expresses the requisite outgoing wave condition for $e^{-i\omega t}$ time dependence. This is because as $r \to \infty$,

$$H_m^{(1)}(kr) \simeq \sqrt{\frac{2}{\pi kr}} e^{i\left[kr - \left(m+\frac{1}{2}\right)\pi/2\right]} .$$

By contrast, at $r = a$, the Dirichlet boundary condition demands that

$$\begin{aligned} 0 &= \psi(r=a,\theta) \\ &= \sum_{m=-\infty}^{\infty} J_m(ka) i^m e^{im\theta} + a_m H_m^{(1)}(ka) i^m e^{im\theta} . \end{aligned}$$

The fact that this holds for all angles θ implies that each term (= "partial wave amplitude") in the sum must vanish. Consequently,

$$a_m = -\frac{J_m(ka)}{H_m^{(1)}(ka)} \qquad\qquad m = 0, \pm 1, \ldots .$$

It follows that the hard cylinder scattering process yields the modified plane wave

$$\begin{aligned} \psi &= e^{ikr\cos\theta} + \sum_{m=-\infty}^{\infty} (-) \frac{J_m(ka)}{H_m^{(1)}(ka)} i^m H_m^{(1)}(kr) e^{im\theta} \\ &= \sum_{m=-\infty}^{\infty} \left\{ J_m(kr) - \frac{J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(kr) \right\} i^m e^{im\theta} . \end{aligned}$$

This solution to the Helmholtz equation represents the incident plane wave plus the scattered outgoing cylinder waves.

5.3.2 Finite Interior Boundary Value Problem: Cavity Vibrations

Lecture 44

Let us extend our study of wave amplitudes from the two-dimensional Euclidean plane to three-dimensional Euclidean space plus temporal dimension as determined by the wave equation.

$$\nabla^2\psi - \frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2} = 0\,.$$

The spatial domain we consider is the interior of a finite cylinder of length L and radius a

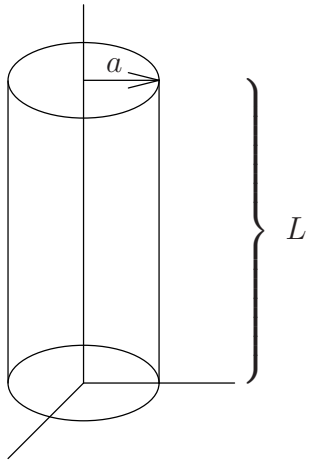


Figure 5.14: Cylindrical Cavity.

Its geometry demands that the wave equation, which governs the wave amplitude inside that cylinder for all times, be expressed relative to cylindrical coordinates,

$$\frac{1}{r}\,\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2} \;=\; 0\,. \tag{5.38}$$

The wave field ψ is finite and single valued (obviously!) inside the cylinder, and vanishes on its boundary. These observations are expressed by the fact that ψ satisfies the following three pairs of boundary conditions for *all* times.

$$\begin{aligned} \psi &= 0 \text{ at } z = 0 \\ \psi &= 0 \text{ at } z = L\,; \end{aligned} \tag{5.39}$$

$$\begin{aligned} \psi &= \text{finite at } r = 0 \\ \psi &= 0 \text{ at } r = a\,; \end{aligned} \tag{5.40}$$

$$\begin{aligned} \psi(\theta = 0) &= \psi(\theta = 2\pi) \\ \psi'(\theta = 0) &= \psi'(\theta = 2\pi)\,. \end{aligned} \tag{5.41}$$

We shall see that these homogeneous boundary conditions characterize three Sturm-Liouville eigenvalue problems, with their three sets of eigenvalues.

Suppose we know in addition the initial amplitude and velocity profiles

$$\psi(r, \theta, z, t = 0) = f(r, \theta, z)$$

and

$$\frac{\partial \psi}{\partial t}(r, \theta, z, t = 0) = g(r, \theta, z)$$

at $t = 0$. The functions f and g are called the *initial value data* for the wave equation. The problem before us is to determine, from this initial value data, $\psi(r, \theta, z, t)$, namely, the amplitude inside the cylinder for all times.

The first step is to solve the wave equation, Eq. 5.38, by the method of "separation of variables". It consists of finding those solutions which have the product form

$$\psi = R(r)\Theta(\theta)Z(z)T(t)\,.$$

Introducing it into the wave equation, dividing by the product of these four factors, one obtains

$$\frac{1}{R}\,\frac{1}{r}\,\frac{d}{dr}r\frac{dR}{dr} + \frac{1}{r^2}\,\frac{1}{\Theta}\,\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\,\frac{d^2Z}{dz^2} - \frac{1}{c^2}\,\frac{1}{T}\,\frac{d^2T}{dt^2} = 0\,.$$

Bring the z-term to the right hand side. The resulting equality holds for all r, θ, z, and t. Thus the right hand must be independent of r, θ and t, while the left hand side must be independent of z. But the two sides are equal. Thus the common quantity must be independent of r, θ, t, and z, i.e., it must be some constant. Call it k_z^2. Consequently, $Z(z)$ satisfies

$$\frac{1}{Z}\,\frac{d^2Z}{dz^2} \;=\; -k_z^2\,. \tag{5.42}$$

Next isolate the Θ term, and by the analogous argument obtain

$$\frac{1}{\Theta}\,\frac{d^2\Theta}{d\theta^2} \;=\; -\nu^2\,. \tag{5.43}$$

Similarly obtain

$$\frac{1}{R}\left[\frac{1}{r}\,\frac{d}{dr}r\frac{dR}{dr} - \frac{\nu^2}{r^2}R\right] \;=\; -k^2\,. \tag{5.44}$$

Here k_z^2, ν^2 and k^2 are three arbitrary constants. For obvious reasons they are called separation constants. Finally, the wave equation, together with these three equations, implies

$$\frac{d^2T}{dt^2} + \omega^2 T = 0$$

where

$$\omega^2 = (k_z^2 + k^2)c^2\,.$$

The initial value data f and g is nonzero at $t = 0$. Consequently, none of the four factors, whose products constitutes the solution to the wave equation, is allowed to be identically zero. Thus the boundary conditions Equations 5.39, 5.40, and 5.41, are conditions on the solutions to the differential equations, Equations 5.42, 5.43, and 5.44. There are three of each. They give rise to three Sturm-Louiville systems

1. $\frac{d^2Z}{dz^2} + k_z^2 Z = 0 \qquad Z(0) = 0 \qquad Z(L) = 0$
2. $\frac{d^2\Theta}{dz^2} + \nu^2\Theta = 0 \qquad \Theta(0) = \Theta(2\pi) \qquad \Theta'(0) = \Theta'(2\pi)$
3. $\frac{1}{r}\,\frac{d}{dr}r\frac{dR}{dr} + \left(k^2 - \frac{\nu^2}{r^2}\right)R = 0 \qquad R(0) = \text{finite} \qquad \text{R(a)=0}\,.$

Each of these three S-L eigenvalue problems determines its own eigenvalue spectrum, namely

$$\begin{aligned} k_z &= \tfrac{n\pi}{L} \quad n = 1, 2, \ldots \\ \nu &= m \quad m = 0, \pm 1, \ldots \\ k &= k_{mj} \quad J_m(ka) = 0, \quad j\text{th root of the Bessel function } J_m(x) \quad . \end{aligned}$$

For each triplet of eigenvalues there is a corrsponding amplitude profile,

$$R_{mj}(r)\Theta_m(\theta)Z_n(z) \quad .$$

The product of the first two factors,

$$R_{mj}(r)\Theta_m(\theta) \quad ,$$

is the amplitude profile in the transverse plane. The last factor, $Z_n(z)$, is the amplitude profile along the longitudinal direction.

The eigenvalue spectra are an expression of the boundary condition. Change the boundary conditions, and the eigenvalue spectra and their amplitude profiles will change. However, the boundary conditions remain fixed for all times. Consequently, the eigenvalue spectra and the corresponding amplitude profiles remain the same for all times.

Each triplet of eigenvalues $\left(k_z = \frac{2\pi}{L}\,,\ \nu = m\,,\ k = k_{mj}\right)$ determines three corresponding eigenfunctions and hence a solution to the wave equation, whose consequent reduced form is

$$\frac{1}{c^2}\frac{\partial^2 \psi_{mjn}}{\partial t^2} + c^2\left[k_{mj}^2 + \left(\frac{n\pi}{L}\right)^2\right]\psi_{mjn} = 0 \quad .$$

Here

$$\psi_{mjn}(r, \theta, z, t) = R_{mj}(r)\Theta_m(\theta)Z_n(z)T(t)\,.$$

Such a product solution, ψ_{mjn}, to the wave equation is called a *normal mode*. Normal modes have the same (oscillatory) time dependence at every point of its domain. The unique features of any particular normal mode are determined by the three integers (m, j, n) of the three eigenvalues (k_z, ν, k). This is true not only for its *spatial amplitude profile*

$$R_{mj}(r)\Theta_m(\theta)Z_n(z)\,,$$

but also for its oscillatory frequency

$$\omega = c\left(k_{mj}^2 + \left(\frac{n\pi}{L}\right)^2\right)^{1/2} \equiv \omega_{mjn}$$

which determines its oscillatory behavior as a function of time

$$\psi_{mjn}(r,\theta,z,t) = R_{mn}(r)\Theta_m(\theta)Z_n(z)[A_{mjn}\cos\omega_{mjn}t + B_{mjn}\sin\omega_{mjn}t]\,.$$

In brief, the *boundary conditions determine* the *spectrum* of *allowed oscillatory frequencies of its normal modes*. Furthermore, a cylindrical cavity illustrates a universal feature which is shared by all linear systems governed by a wave equation: *a finite system always has a discrete eigenvalue spectrum.*

Any vibratory system governed by linear wave equation obeys the *linear superposition principle*. Consequently, the general solution to the wave equation is a linear combination of normal modes

$$\psi = \sum_{m=-\infty}^{\infty}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} R_{mj}(r)\Theta_m(\theta)Z_n(z)[A_{mjn}\cos\omega_{mjn}t + B_{mjn}\sin\omega_{mjn}t]\,.$$

This is a generalized triple Fourier series. The two sets of Fourier coefficients $\{A_{mjn}\}$ and $\{B_{mjn}\}$ are determined by the initial value data $f(r,\theta,z)$ and $g(r,\theta,z)$:

$$\begin{aligned} A_{mjn} &= \int_0^a\int_0^{2\pi}\int_0^L R_{mj}(r)\Theta_m(\theta)Z_n(z)f(r,\theta,z)r\,dr\,d\theta\,dz \\ \omega_{mjn}\,B_{mjn} &= \int_0^a\int_0^{2\pi}\int_0^L R_{mj}(r)\Theta_m(\theta)Z_n(z)g(r,\theta,z)r\,dr\,d\theta\,dz\,. \end{aligned}$$

5.3.3 Infinite Interior Boundary Value Problem: Waves Propagating in a Cylindrical Pipe

Let us compare waves vibrating in a finite cylindrical cavity ($0 \le z \le L$) with waves propagating in an infinite cylinder ($-\infty < z < \infty$).

The wave equation is the same in both cases,

$$\nabla^2\psi - \frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2} = 0\,. \tag{5.45}$$

The boundary conditions along the radial and angular direction ("transverse direction") are also the same in both cases:

$$\begin{aligned} \psi(r=0,\theta,z,t) &= \text{finite} \\ \psi(r=a,\theta,z,t) &= 0 \qquad\qquad\qquad \text{(Dirichlet b.c.)} \\ \psi(r,\theta,z,t) &= \psi(r,\theta+2\pi,z,t) \qquad \text{(Periodic b.c.)} \ . \end{aligned}$$

These are the familiar two sets of boundary conditions for the two Sturm-Liouville problems on the radial and the angular domain.What do their solutions tell us?
I.) Their eigenfunctions yield the amplitude profile across any transverse cross section ($z = const.$) at any time ($t = const.$). These cross sectional profiles are determined by the two sets of eigenvalues,

$$m = 0, \pm 1, , \cdots \quad \text{and} \quad k_{mj} : J_m(k_{mj}a) = 0; j = 1, 2, \cdots$$

II.) By virtue of the wave equation (5.45) each of these transverse eigensolutions determines the properties of a wave disturbance

$$\psi = \psi_{mj}(t,z) J_m(k_{mj}r) e^{im\theta} \ .$$

propagating along the longitudinal direction. The wave equation tells us that these properties are captured by

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial z^2} + k_{mj}^2 \psi = 0 \ . \tag{5.46}$$

The mathematical behavior of its solutions is the same as that of a string imbedded in an elastic medium as discussed on page 12. If one happens to be familiar with its physical properties, one can infer the mathematical properties of wave propagation along the z-direction.

The problem of waves trapped in a cavity is similar to that of waves propagating along a pipe: both are most efficiently attacked in terms of normal modes, which satisfy

$$\frac{\partial \psi}{\partial t} = -i\omega\psi \quad \rightarrow \psi \propto \quad e^{-i\omega t} \ ,$$

However, the difference in the boundary conditions on the z-domain demands a different point of view in regard to what is given and what is to be

determined. First of all, instead of Dirichlet boundary conditions, the new condition is that for a given frequency ω the normal modes express waves travelling along the z-direction. This implies that a normal mode satisfies

$$\frac{\partial \psi}{\partial t} \mp i \frac{\omega}{k_z} \frac{\partial \psi}{\partial z} = 0$$

so that

$$\psi \propto R_{mj}(r)\Theta_m(z)e^{\pm ik_z z}e^{-i\omega t} .$$

Second, the fact that this mode satisfies the wave equation,

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} + (k_{mj}^2 + k_z^2)\psi = 0 ,$$

implies that

$$-\frac{\omega^2}{c^2} + k_{mj}^2 + k_z^2 = 0$$

or

$$k_z^2 = \frac{\omega^2}{c^2} - k_{mj}^2 .$$

Third, and finally, our viewpoint is now necessarily different. Instead of k_z being determined by an eigenvalue problem, we now take ω to be the given frequency of the wave ψ to be launched into the positive (+) or negative (−) z-direction and ask: for what values of ω will k_z be real so that ψ expresses a travelling wave

$$\psi \propto R_{mj}(r)\Theta_m(\theta)e^{\pm i|k_z|z}e^{-i\omega t}$$

and for what values of ω will k_z be *imaginary* so that ψ expresses a spatially damped (or antidamped) wave

$$\psi \propto R_{mj}(r)\Theta_m(\theta)e^{\pm |k_z|z}e^{-i\omega t} .$$

It is evident, that the answer is to be inferred from the *dispersion relation*

$$k_z = \pm \left(\frac{\omega^2}{c^2} - k_{mj}^2 \right)^{1/2} .$$

This relation between k_z and ω depends on the eigenvalues k_{mj} for the amplitude profile in the transverse plane. A wave which decays exponentially along its putative direction of propagation is called an *evanescent wave*. This

happens when the frequency of the launched wave is low enough. It is evident that there is a *critical frequency*

$$\omega_{critical} = c\ k_{mj}$$

at which the wave changes from being a propagating to being an evanescent wave. The eigenvalues k_{mj} are, of course, determined by the given Dirichlet boundary condition. For a hollow cylinder these eigenvalues are the roots of the equation

$$J_m(ka) = 0 \quad .$$

This implies that the smaller the radius of the cylindrical pipe the higher the critical frequency below which no wave can propagate. A wave which meets such a small radius pipe gets simply reflected.

Exercise 5.3.1 (AXIALLY SYMMETRIC AMPLITUDES)
The transverse amplitude of an axially symmetric wave propagating in a cylindrical pipe of radius a is determined by the following eigenvalue problem:

$$\begin{aligned} -\frac{d}{dr}\ r\ \frac{du}{dr} &= k^2 r u \qquad & 0 \le r \le a \\ u(0) &= \text{finite} \\ u(a) &= 0. \end{aligned}$$

The eigenfunctions are $u_m(r) = J_0(rk_m)$ where the boundary condition $J_0(ak_m) = 0$ determines the eigenvalues $k_m^2 \quad m = 1, 2, \ldots$.

(a) Show that $\{J_0(rk_m)\}$ is an orthogonal set of eigenfunctions on $(0, a)$.

(b) Using the problem "How to normalize an eigenfunction" on page 30, find the squared norm of $J_0(rk_m)$.

(c) Exhibit the set of orthonormalized eigenfunctions.

(d) FIND the Green's function for the above boundary value problem.

Exercise 5.3.2 (NORMAL MODES FOR A VIBRATING DRUM)
On a circular disc of radius a FIND an orthonormal set of eigenfunctions for the system defined by the eigenvalue problem

$$\begin{aligned} -\nabla^2\psi &= k^2\psi \\ \frac{\partial\psi}{\partial r}(r=a,\theta) &= 0 \qquad & \text{a = radius of disc} \\ \psi(r=0,\theta) &= \text{finite} \qquad & 0 \le \theta \le 2\pi \quad . \end{aligned}$$

Here $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$,
and EXIBIT these eigenfunctions in their optimally simple form, i.e. without referring to any derivatives.

Exercise 5.3.3 (CRITICAL FREQUENCIES FOR WAVE PROPAGATION)
Consider a wave disturbance ψ which is governed by the wave equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right]\psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}.$$

Let this wave *propagate* inside an infinitely long cylinder; in other words, it satisfied

$$\frac{\partial \psi}{\partial z} = ik_z \psi$$

where k_z is some *real* number, not equal to zero. Assume that the boundary conditions satisfied by ψ is

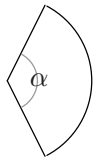
$$\begin{aligned} \psi(r=a) &= 0 \quad \text{with } a = \text{radius of cylinder} \\ \psi(r=0) &= \text{finite} \end{aligned}$$

(a)Find the "cut off" frequency, i.e. that frequency $\omega = \omega_{\text{critical}}$ below which *no propagation* in the infinite cylinder *is possible*.
(b)Note that this frequency depends on the angular integer m and the radial integer j. For fixed j, give an argument which supports the result that smaller m means smaller critical frequency.
(c)What is the smallest critical frequency, ω_{critical}, in terms of a and c to an accuracy of 2% or better?

Exercise 5.3.4 (PIE-SHAPED DRUM)
Consider the circular sector

$$\begin{aligned} S: \ 0 &\le r \le a \\ 0 &\le \theta \le \alpha \end{aligned}$$

.



(a) Exhibit the set of those normalized eigenfunctions for this sector which satisfy

$$\begin{aligned} (\nabla^2 + k^2)\psi &= 0 \\ \psi &= 0 \ \text{ on the boundary of } \ S \end{aligned}$$

(b) Compare the set of normal modes of a circular drum with the set of normal modes in Part A when $\alpha = 2\pi$

Exercise 5.3.5 (VIBRATING MEMBRANES)
Consider

(a) a circular membrane of radius $\underline{a}$

(b) a square membrane

(c) a rectangular membrane which is twice as long as it is wide.

Assume the two membranes

(i) have the *same* area.

(ii) obey the *same* wave equation $\bigtriangledown^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$

(iii) Have the *same* boundary conditions $\psi = 0$ at their boundaries

(A) TABULATE

(i) the 3 lowest frequencies for each of the two membranes

(ii) all the concomitant normal modes.

(B) For each of the normal modes of the circular membrane DRAW a picture of the nodes,
i.e. the locus of points where $\psi = 0$. LABEL each of the normal mode pictures.

(C) Do the same for the other membrane. (Caution: Watch out for degeneracies!)

Roots. λ_j is the jth root of the Bessel Functions $J_m(\lambda_j)$:

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$j=1$	2.405	3.832	5.136	6.380	7.586
$j=2$	5.520	7.016	8.417	9.760	11.064
$j=3$	8.654	10.173	11.620	13.017	14.373
$j=4$	11.792	13.324	14.796	16.223	16.223

5.4 More Properties of Hankel and Bessel Functions

Plane waves, i.e. disturbances with planar wave fronts, can be subjected to translations in the Euclidean plane. They can also be used as basis functions for the two-dimensional Fourier transform. Both of these features extend to cylinder harmonics. The first one is captured by Property 19, the second one by Eq.(5.56) of Property 21. An example of a problem which uses the translation property for cylinder harmonics is a scattering problem similar to the one on page 351:
Consider a cylindrical source of waves and some distance away from it there is a scatterer also cylindrical in shape. Given the distance between these two cylinders, find the scattered wave field.

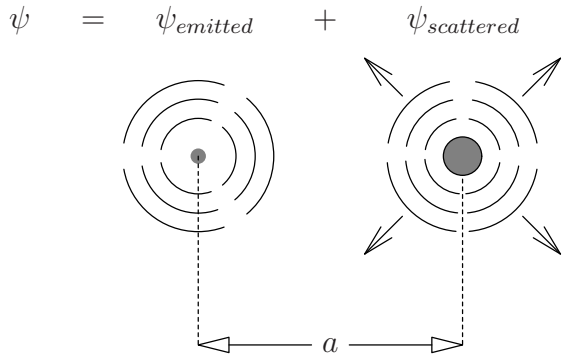


Figure 5.15: Scattering of a cylindrical disturbance by a cylinder. A cylindrical wave $\psi_{incident}$ emanating from a source on the left gives rise in the presence of a cylindrical boundary to a circular scattered wave $\psi_{scattered}$ on the right.

Property 19 (Addition theorem for cylinder harmonics)
A displaced cylinder harmonic is a linear superposition of the undisplaced cylinder harmonics. Mathematically one states this fact by the equation

$$H_\nu(kR)e^{i\nu(\Omega-\theta_0)} = \sum_{m=-\infty}^{\infty} J_m(kr_0)H_{\nu+m}(kr)e^{i(\nu+m)(\theta-\theta_0)} \quad . \tag{5.47}$$

This equation is also known as the "addition theorem" for cylinder harmonics, be they singular or non-singular at the origin $R = 0$. The geometrical meaning of this theorem is as follows: Consider a *displacement* in the Euclidean plane by the vectorial amount $\vec{x}_0$ and express this displacement in terms of polar coordinates:

$$\vec{x}_0 : \; x_0 + iy_0 = r_0 e^{i\theta_0} \quad .$$

Next, consider a *point of observation*, also expressed in terms of polar coordinates,

$$\vec{x} : \; x + iy = re^{i\theta} \quad .$$

Finally, consider this same point of observation, but relative to the displaced origin at $\vec{x}_0$. In terms of polar coordinates one has

$$\begin{aligned} \vec{X} \equiv \vec{x} - \vec{x}_0 : \; X + iY &= (x - x_0) + i(y - y_0) \\ Re^{i\Omega} &= re^{i\theta} - r_0 e^{i\theta_0} = |\vec{x} - \vec{x}_0| e^{i\Omega} \quad , \end{aligned} \tag{5.48}$$

where

$$\begin{aligned} R\cos\Omega &= r\cos\theta - r_0\cos\theta_0 \equiv x - x_0 \\ R\sin\Omega &= r\sin\theta - r_0\sin\theta_0 \equiv y - y_0 \\ R^2 &= r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0) \\ e^{2i\Omega} = \frac{Re^{i\Omega}}{Re^{-i\Omega}} &= \frac{re^{i\theta} - r_0 e^{i\theta_0}}{re^{-i\theta} - r_0 e^{-i\theta_0}} \end{aligned}$$

are the observation coordinates relative to the displaced origin.

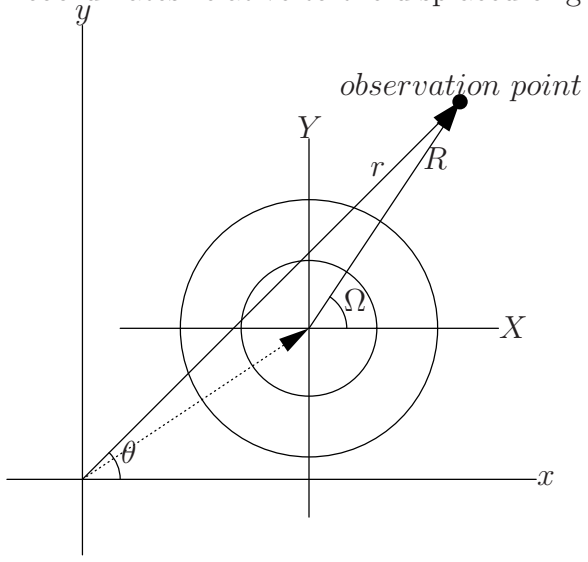


Figure 5.16: Displaced cylinder harmonic and its displaced coordinate system. The observation point is labelled ("coordinatized") in two different ways; by (r, θ) and by (R, Ω). The dotted vector is the displacement vector $\vec{x}_0 : \ r_0 e^{i\theta_0}$.

The *problem* is this: express a typical *displaced* cylinder harmonic,

$$H_\nu(kR)e^{i\nu\Omega} = H_\nu(k|\vec{x} - \vec{x}_0|)e^{i\nu\Omega}$$

a solution to the Helmholtz equation, in terms of the *undisplaced* cylinder harmonics,

$$H_\nu(kr)e^{i\nu\theta} = H_\nu(k|\vec{x}|)e^{i\nu\theta}, \quad |\vec{x}| = \sqrt{x^2 + y^2} \quad , \tag{5.49}$$

which are also solutions to the same Helmholtz equation.
The solution to this problem is given by the "addition theorem", Eq.(5.47).

It is interesting to note that both R and Ω, and hence $H_\nu(kR)e^{i\nu\Omega}$ are *periodic* functions of θ. Indeed, one notices that

$$\vec{X} = \vec{x} - \vec{x}_0$$

or, equivalently, that

$$Re^{i\Omega} = re^{i\theta} - r_0 e^{i\theta_0} \quad .$$

As a consequence, the old and the new polar coordinates are related by

$$R^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)$$

and

$$e^{2i\Omega} = \frac{Re^{i\Omega}}{Re^{-i\Omega}} = \frac{re^{i\theta} - r_0 e^{i\theta_0}}{re^{-i\theta} - r_0 e^{-i\theta_0}} \, .$$

Thus one is confronted with the problem of finding the Fourier series of the periodic function

$$H_\nu(kR)e^{i\nu(\Omega-\theta_0)} = H_\nu\left(k\sqrt{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}\right)\left(\frac{r_0 - re^{i(\theta-\theta_0)}}{r_0 - re^{-i(\theta-\theta_0)}}\right)^{\nu/2}$$

The solution to this problem is given by the "addition theorem", Eq.(5.47). We shall refrain from validating this Fourier series by a frontal assault. Instead, we give a simple three-step geometrical argument. It accomplishes the task of expressing the displaced cylinder harmonics in terms of the undisplaced cylinder harmonics

(i) Represent the displaced harmonic as a linear combination of plane waves in the usual way

$$H_\nu(kR)e^{i\nu\Omega} = \frac{e^{-i\nu\pi/2}}{\pi}\int e^{ikR\cos(\alpha-\Omega)}e^{i\nu\alpha}d\alpha \quad , \tag{5.50}$$

(ii) take each of these plane waves and reexpress them relative to the undisplaced origin:

$$e^{ikR\cos(\alpha-\Omega)} \equiv e^{i\vec{k}\cdot\vec{X}} = e^{i\vec{k}\cdot(\vec{x}-\vec{x}_0)} = e^{-i\vec{k}\cdot\vec{x}_0}e^{i\vec{k}\cdot\vec{x}}$$

The phase shift factor is a plane wave amplitude in its own right, which depends periodically on the angel θ_0, and is therefore, according to Property 18, a linear combination of Bessel harmonics

$$\begin{aligned} e^{-i\vec{k}\cdot\vec{x}_0} &\equiv e^{-ikr_0\cos(\alpha-\theta_0)} \\ &= \sum_{m=-\infty}^{\infty} J_m(kr_0)e^{im\alpha}e^{-im\left(\theta_0+\frac{\pi}{2}\right)} . \end{aligned}$$

(iii) Reintroduce the translated plane wave

$$e^{i\vec{k}\cdot\vec{x}} = e^{ikr\cos(\alpha-\theta)}$$

and its concomitant phase shift factor $e^{-i\vec{k}\cdot\vec{x}_0}$ from step (ii) into the displaced cylinder harmonic. The result is a linear sum of phase shifted cylinder harmonics, Eq.(5.50),

$$\begin{aligned} H_\nu(kR)e^{i\nu\Omega} &\equiv \frac{e^{-i\nu\pi/2}}{\pi}\int e^{i\vec{k}\cdot\vec{x}}e^{-i\vec{k}\cdot\vec{x}_0}e^{i\nu\alpha}d\alpha \\ &= \frac{e^{-i\nu\pi/2}}{\pi}\sum_{m=-\infty}^{\infty} J_m(kr_0)\int e^{ikr\cos(\alpha-\theta)}e^{i(\nu+m)\alpha}\ d\alpha\ e^{-im\left(\theta_0+\frac{\pi}{2}\right)} \\ &= \sum_{m=-\infty}^{\infty} J_m(kr_0)\underbrace{\frac{e^{-i(\nu+m)\pi/2}}{\pi}\int e^{ikr\cos(\alpha-\theta)}e^{i(\nu+m)\alpha}\ d\alpha}_{H_{\nu+m}(kr)e^{i(m+\nu)\theta}}\ e^{-im\theta_0} . \end{aligned}$$

According to the definitions, Eqs.(5.14)-(5.15), the integral is a cylinder harmonic of order $\nu + m$. Consequently, one obtains

$$H_\nu(kR)e^{i\nu\Omega} = \sum_{m=-\infty}^{\infty} J_m(kr_0)H_{\nu+m}(kr)e^{i(m+\nu)\theta}e^{-im\theta_0} .$$

Multiplying both sides by $e^{-i\nu\theta_0}$ yields the following geometrically perspicuous result:

$$H_\nu(kR)e^{i\nu(\Omega-\theta_0)} = \sum_{m=-\infty}^{\infty} J_m(kr_0)H_{\nu+m}(kr)e^{i(m+\nu)(\theta-\theta_0)}$$

Note that the left hand side is a displaced cylinder harmonic of order ν relative to the new x-axis which point along the displacement vector $\vec{x}_0$ and whose origin lies along the tip of this vector. The angle $\Omega - \theta_0$ is the new angle of observation relative to the new tilted x-axis and the new origin.

The sum on the right is composed of the cylinder harmonics of order $\nu + m$ undisplaced relative to the tilted x-axis. The angle $\theta - \theta_0$ is the old angle of observation relative to the tilted x-axis and the old origin.

The displacement formula can be summarized as follows

$$\left(\begin{array}{c}\text{displaced wave}\\ \text{of order } \nu\end{array}\right) = \sum_m J_m(kr_0) \times \left(\begin{array}{c}\text{undisplaced wave}\\ \text{of order } \nu+m\end{array}\right) .$$

Property 20 (Translations represented by cylinder harmonics)
It is amusing to specialize to the case where $\nu = n$ is an integer and $H_\nu = J_n$ is a Bessel function of integral order n. In that case the displacement formula becomes

$$\begin{aligned} J_n(kR)e^{in\Omega} &= \sum_{m=-\infty}^{\infty} \left[J_{n+m}(kr)e^{i(n+m)\theta}\right] J_m(kr_0)e^{-im\theta_0} \\ &= \sum_{m=-\infty}^{\infty} \left[J_{n-m}(kr)e^{i(n-m)\theta}\right] J_m(kr_0)e^{im(\theta_0+\pi)} \quad , \end{aligned}$$

or equivalently, after changing the summation index,

$$J_{n-\ell}(kR)e^{i(n-\ell)\Omega} = \sum_{m=-\infty}^{\infty} \left[J_{n-m}(kr)e^{i(n-m)\theta}\right] \left[J_{m-\ell}(kr_0)e^{i(m-\ell)\theta_0}\right] \quad (5.51)$$

where

$$\overline{\theta_0} = \theta_0 + \pi \quad ,$$

while Eq.(5.48) for the vector triangle becomes

$$Re^{i\Omega} = re^{i\theta} + r_0 e^{i\overline{\theta_0}} \quad . \tag{5.52}$$

Compare Eq.(5.52) with Eq.(5.51). Observe that (i) for each translation in the Euclidean plane, say $re^{i\theta}$, there is a corresponding infinite dimensional matrix

$$\{J_{n-m}(kr)e^{i(n-m)\theta} \quad : \quad n, m = 0, \pm 1, \pm 2, \cdots\}$$

and (ii) the result of successive translations, such as Eq.(5.52), is represented by the product of the corresponding matrices, Eq.(5.51).

Exercise 5.4.1 (ADDITION FORMULA FOR BESSEL FUNCTIONS)
Express $J_n(x_1 + x_2)$ as a sum of products of Bessel functions of x_1 and x_2 respectively.

Property 21 (Completeness)
The cylinder waves form a complete set. More precisely,

$$\frac{\delta(r - r_0)\delta(\theta - \theta_0)}{r} = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{\infty} kdkJ_m(kr)J_m(kr_0)e^{im(\theta-\theta_0)} \quad . \tag{5.53}$$

This relation is the cylindrical analogue of the familiar completeness relation for plane waves,

$$\begin{aligned} \delta(x - x_0)\delta(y - y_0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \frac{e^{i(k_x x + k_y y)}}{2\pi} \frac{e^{-i(k_x x_0 + k_y y_0)}}{2\pi} \\ &= \int_0^{\infty}\int_0^{2\pi} kdkd\alpha \frac{e^{ikr\cos(\alpha-\theta)}}{2\pi} \frac{e^{-ikr_0\cos(\alpha-\theta_0)}}{2\pi} \quad . \end{aligned} \tag{5.54}$$

In fact, the one for plane waves is equivalent to the one for cylinder waves. The connecting link between the two is the plane wave expansion, Eq.(5.37),

$$e^{ikr\cos(\theta-\alpha)} = \sum i^m J_m(kr)e^{im(\theta-\alpha)} \quad .$$

Introduce it into Eq.(5.54) and obtain

$$\delta(x-x_0)\delta(y-y_0) = \sum_m \int_0^\infty kdk \int_0^{2\pi} d\alpha J_m(kr) i^m \frac{e^{im(\theta-\alpha)}}{2\pi} \sum_{m'} J_{m'}(kr_0) \frac{e^{-im'(\theta_0-\alpha)}}{2\pi} (-i)^{m'} .$$

Using the orthogonality property

$$\int_0^{2\pi} d\alpha \, \frac{e^{i(m'-m)\alpha}}{2\pi} = \delta_{mm'} \quad , \tag{5.55}$$

the definition

$$\delta(x-x_0)\delta(y-y_0)dxdy = \delta(r-r_0)\delta(\theta-\theta_0)drd\theta \quad ,$$

and

$$dxdy = rdrd\theta \, ,$$

one obtains

$$\frac{\delta(r-r_0)\delta(\theta-\theta_0)}{r} = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \int_0^\infty kdk J_m(kr) J_m(kr_0) e^{im(\theta-\theta_0)} \tag{5.56}$$

the *completeness relation for the cylinder waves.*

Property 22 (Fourier-Bessel transform)
The Bessel functions $\{J_m(kr) \colon 0 \le k < \infty\}$ of fixed integral order form a complete set

$$\frac{\delta(r-r_0)}{r} = \int_0^\infty J_m(kr) J_m(kr_0) kdk \, . \tag{5.57}$$

This result is a direct consequence of Property 21. Indeed, multiply the cylinder wave completeness relation, Eq.(5.53) by $e^{-im'\theta}$, integrate over θ from 0 to 2π, again use the orthogonality property, Eq. 5.55, and cancel out the factor common factor $e^{-im'\theta_0}$ from both sides. The result is Eq.(5.57), the completeness relation for the Bessel functions on the positive r-axis.
Remark: By interchanging the roles of k and r one obtain from Eq.(5.57)

$$\frac{\delta(k-k_0)}{k} = \int_0^\infty J_m(kr) J_m(k_0 r) r dr \quad .$$

Remark: The completeness relation, Eq.(5.57), yields

$$f(r) = \int_0^\infty F(k) J_m(kr) k dk$$

where

$$F(k) = \int_0^\infty f(r) J_m(kr) r dr \, .$$

This is the Fourier-Bessel transform theorem.

It is interesting to note that the completeness relation, Eq.(5.57), is independent of the integral order of $J_m(kr)$. One therefore wonders whether Eq.(5.57) also holds true if one uses $J_\nu(kr)$, Bessel functions of *any* complex order ν. This is ideed the case.

Property 23 (Bessel transform)
The Bessel functions $\{ J_\nu(kr) : 0 < k, \infty \}$ of complex order ν form a complete set

$$\frac{\delta(r - r_0)}{r} = \int_0^\infty J_\nu(kr) J_\nu(kr_0) k dk \, . \tag{5.58}$$

This result gives rise to the transform pair

$$f(r) = \int_0^\infty F(k) J_\nu(kr) k dk \tag{5.59}$$

$$F(k) = \int_0^\infty f(r) J_\nu(kr) r dr \, . \tag{5.60}$$

and it is obvious that mathematically Property 22 is a special case of Property 23.

5.5 The Method of Steepest Descent and Stationary Phase

Lecture 46

The repeated encounter with complex integrals such as

$$H_\nu(\rho) = \int_{\alpha_1}^{\alpha_2} e^{i\rho \cos\alpha} e^{i\nu\alpha} d\alpha \quad ,$$

especially when $\rho \gg 1$, demands that we have at our disposal a systematic method for evaluating, at least approximately, integrals of the type

$$I(\rho) = \int_A^B X(z) e^{\rho f(z)} dz\,; \qquad 1 \ll \rho\,. \tag{5.61}$$

This is an integral in the complex z-plane along a curve which starts at A and terminates at B. The exponential is a rapidly changing function because $1 \ll \rho$. The function $X(z)$, by contrast, is a slowly varying function. The success of the method hinges on the following circumstance: the dominant contribution to the integral comes from only a small segment of the integration contour, and the accuracy of that dominant contribution improves with increasing ρ.

The value of the integral depends obviously on the behavior of the integrand along the integration path. However, the Cauchy-Goursat theorem implies that the integration path between the fixed limits A and B can be quite arbitrary provided that

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

is *analytic*, i.e., all its derivatives exist. This is usually, if not always, the case. Analyticity of $f(z) = f(x+iy)$ is equivalent to

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial (iy)} \\ \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= \frac{1}{i}\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\,, \end{aligned}$$

which yields the *Cauchy-Riemann* equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.$$

They imply

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \left\{ \begin{array}{c} u \\ v \end{array} \right\} = 0$$

or

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\,,$$

i.e., f is "harmonic". (*Nota bene:* a harmonic function need *not* be analytic.)

Let z_0 be an extremum of f, i.e.,

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0, \quad \text{or equivalently} \quad \frac{\partial f}{\partial z} = 0\,.$$

At such a critical point, f has neither a maximum nor a minimum, it has a *saddle point* instead, because $\frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$ prevents f from having a maximum or minimum anywhere.

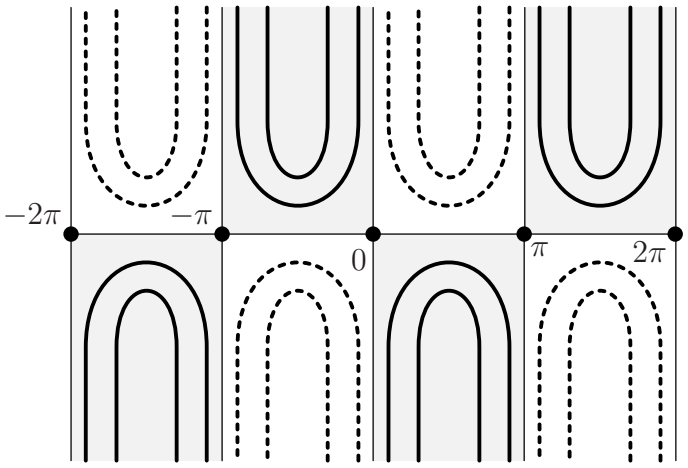


Figure 5.17: Critical points of the function $f(z) = i\rho\cos z$. The solid contours in the shaded regions ("valleys") are the isograms of $Re\ f$ which are below zero elevation, while the dashed contours in the unshaded regions ("mountain ranges") are the isograms of $Re\ f$ which are above zero elevation. Each solid dot is located in a mountain pass which connects two valleys by a path of steepest ascent and descent.

Example:

$$\begin{aligned} f(z) &= i\cos z = i\cos x\cosh y + \sin x\sinh y \\ f'(z) &= -i\sin z \\ z_0 &= n\pi\,, \quad n = 0\,, \ \pm 1\,, \ \pm 2 \qquad \text{(location of critical points)} \end{aligned}$$

The integrand of $I(\rho)$ is

$$X(z)e^{\rho f(z)} = X(z)e^{\rho u(x,y)+i\rho v(x,y)}$$

and the integration path is assumed to start and end where this integrand vanishes, i.e., where

$$\begin{aligned} u(A) &= -\infty \\ u(B) &= -\infty\,. \end{aligned}$$

This means that, in the example, points A and B would lie in different shaded strips in Figure 5.17.

The integration path between these end points can be deformed without changing the value of the integral. The method of steepest descent takes advantage of this fact by deforming the integration path so that it goes through the critical point z_0 in such a way that

$$Ref(z_0) = \text{maximun along the path}$$

and that the rate at which $Ref(z_0)$ decreases along either direction away from z_0 as rapidly as possible.

One suspects that the integral

$$\int_A^B X(z)e^{\rho u(x,y)+i\rho v(x,y)}dz$$

gets its major contribution along this path through z_0. A possible objection against such a suspicion is that along this path the integrand

$$X(z)e^{\rho f(z)} = X(z)e^{\rho u(x,y)}e^{i\rho v(x,y)}$$

might oscillate very rapidly. One might blame such a behaviour on the phase factor

$$e^{i\rho v(x,y)}\,.$$

As a consequence, one might think that the value of the integral would average to approximately zero and make its evaluation through z_0 not give the dominant contribution to the total integral. Fortunately this can never happen. Remarkably enough, the *opposite* is the case, the path of *steepest ascent and descent* is also the path of *stationary phase*. In other words, the direction along which $u(x, y)$ changes must rapidly, namely,

$$\vec{\nabla}u = \left(\frac{\partial u}{\partial x}\,,\ \frac{\partial u}{\partial y}\right) \quad (\text{“gradient of } u\text{”})$$

is also the direction along which

$$v = v(x(\tau), y(\tau))$$

is *constant*; indeed,

$$\begin{aligned} \frac{dv}{d\tau} &= \frac{dx}{d\tau}\frac{\partial v}{\partial x} + \frac{dy}{d\tau}\frac{\partial v}{\partial y} \\ &= \frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y} \\ &= \frac{\partial v}{\partial y}\,\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\,\frac{\partial v}{\partial y} = 0\,. \end{aligned}$$

Thus v is constant along the direction of the gradient of u. In still other words, the level surfaces of $u(x, y)$ and $v(x, y)$ are perpendicular to each other, a direct consequence of the Cauchy-Riemann equations.

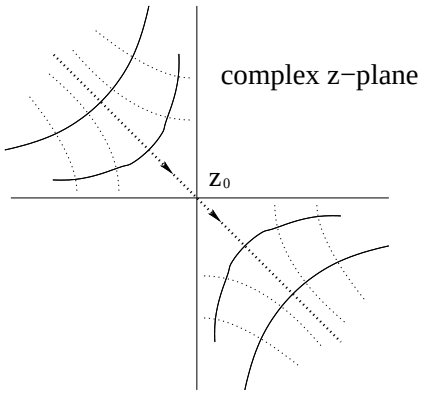


Figure 5.18: Path of steepest ascent and descent through the critical point z_0 of the function $f(z)$. The dotted lines are the isograms of $Im\ f$ $(= v(x, y))$, the locus of points where the phase of the integrand $e^{\rho f}$ is constant. Perpendicular to these are the solid lines. They are the isograms of $Re\ f$ $(= u(x, y))$. The heavily dotted directed line through z_0 is the integration path of stationary phase and steepest descent.

The important conclusion is, therefore, this:

$$e^{i\rho v(x,y)}$$

has constant phase along the direction of $\vec{\nabla}u$. It is clear that if $\vec{\nabla}u$ were not tangent to the line of constant phase, then the method of steepest descent would not work.

We now expand $f(z)$ in the neighborhood of the critical point z_0:

$$\begin{aligned} f(z) &= f(z_0) + \frac{1}{2}(z-z_0)^2 f''(z_0) + \frac{1}{3!}(z-z_0)^3 f'''(z_0) + \cdots \\ &= f(z_0) + \frac{1}{2}(z-z_0)^2 e^{i\delta_0}|f''(z_0)| + \cdots \end{aligned} \tag{5.62}$$

Here δ_0 is the phase of $f''(z_0)$. We are *assuming* that the third and higher derivative terms make a negligible contribution in controlling the asymptotic behavior of

$$e^{\rho f(z)}\,.$$

This is a good assumption provided the second derivative of $f(z)$ does not vanish at z_0,

$$f''(z_0) \neq 0\,.$$

Assuming that this is the case, we now must choose the integration path through z_0. The linear part of this path is

$$z = z_0 + e^{i\phi}\tau\,. \tag{5.63}$$

so that

$$dz = e^{i\phi}d\tau\,. \tag{5.64}$$

Here τ is the path parameter and $e^{i\phi}$ controls the direction of the path. Now comes the important step: We choose the direction of the path so that in the process of passing through z_0 the function $f(z)$ makes the integrand

$$e^{\rho f(z)}$$

rise as fast as possible to a maximum at z_0 and subsequently makes that integrand decrease as rapidly as possible. Such a path is exhibited in Figure 5.18. Along this path the function $f(z)$ has the form

$$f(z) = f(z_0) - \frac{1}{2}\tau^2|f''(z_0)| + \cdots$$

This form must coincide with Eq.(5.62) along the path. Consequently,

$$(z-z_0)^2 e^{i\delta_0} \equiv e^{2i\phi}\tau^2 e^{i\delta_0} = -\tau^2\,.$$

This condition determines the angle ϕ of the integration path.

$$e^{2i\phi}e^{i\delta_0} = -1 \ (= e^{\pm i\pi}) \, . \tag{5.65}$$

The path itself is

$$z - z_0 = e^{-i\delta_0/2}e^{\pm i\pi/2}\tau \, .$$

The $(\pm)$ ambiguity expresses the fact that the integration may proceed into the forward direction or the backward direction. The two directions obviously differ by π radians. The ambiguity is resolved by the fact that the integral $\int_A^B \cdots dz$ has its integration path along a *direct* path from A over the critical point z_0 to B. For example, the complex integrals for the Hankel functions

$$H_\nu(\rho) = \frac{e^{-i\pi\nu/2}}{\pi} \int_A^B e^{i\rho\cos z + i\nu z} dz$$

have the integrand $e^{i\rho\cos z}$ whose critical points are located at $z_0 = 0, \pm\pi, \cdots$, as in Figure 5.17. A cursory inspection of this integrand reveals quite readily through which of these critical points the *directed* integration path must pass.

In general, the ambiguity in

$$dz = e^{i(\pm\pi/2 - \delta_0/2)} d\tau$$

can only be resolved by drawing a *global picture* in which the *direct, and hence directed,* integration path connecting $A \to z_0 \to B$ is exhibited.

After the global ambiguity has been settled, the evaluation of the integral becomes straightforward. The integral, Eq.(5.61), is approximated by restricting the integration to the path segment $-\tau_1 \le \tau \le \tau_1$ centered around the saddle point:

$$I(\rho) = e^{\rho f(z_0)} \int_{-\tau_1}^{\tau_1} X(z_0 + e^{i\phi}\tau) e^{-\frac{\rho}{2}\tau^2 |f''(z_0)|} d\tau \left.\frac{dz}{d\tau}\right|_{z_0} \tag{5.66}$$

The accuracy of this approximation is determined by two seemingly irreconcilable demands. On one hand we are neglecting cubical (and higher) order terms in the exponential, and this is permitted only if

$$\tau_1^2 |f''| \gg \tau_1^3 |f'''| \quad \text{or equivalently} \quad \tau_1 \ll \left| \frac{f''}{f'''} \right| \, . \tag{5.67}$$

On the other hand, at first glance one would think that τ_1 would have to be large enough in order not to miss any contributions to the to-be evaluated

integral. However, there is no conflict. The highly localized nature of the gaussian guarantees that the integral be independent of its limts $\pm\tau_1$, even when τ_1 is small, i.e. satisfies Eq.(5.67). This is because the localized nature of the exponential is controlled by the positive parameter ρ. To make the value of the integral independent of τ_1, this parameter must be so large that

$$\tau_1 \gg \frac{2}{\sqrt{\rho|f''(z_0)|}} \left(= \begin{array}{c} \text{width of neighborhood} \\ \text{where integrand is non-neglegible} \end{array} \right) \tag{5.68}$$

Comparing Eq.(5.68) with (5.67), one finds

$$\left|\frac{f''}{f'''}\right| \gg \tau_1 \gg \frac{2}{\sqrt{\rho|f''(z_0)|}} \quad .$$

This chain of inequalities reconciles the two seemingly contradictory demands. The more the two length scales $|f''|/|f'''|$ and $2/\sqrt{\rho|f''(z_0)|}$ differ from each other the better the chain of inequalities can be satisfied., and the greater the accuracy with which the given integral Eq.(5.61) gets approximated by Eq.(5.66).

Moving forward, expand the slowly varying function X in a Taylor series and obtain

$$\begin{aligned} I(\rho) &= e^{\rho f(z_0)} \sum_{n=0}^{\infty} \left.\frac{d^n X}{dz^n}\right|_{z_0} \frac{e^{in\phi}}{n!} \int_{-\tau_1}^{\tau_1} \tau^n e^{-\frac{\rho}{2}\tau^2 |f''(z_0)|} d\tau \left.\frac{dz}{d\tau}\right|_{z_0} \\ &= \frac{e^{\rho f(z_0)}}{\sqrt{\rho|f''(z_0)|}} \left.\frac{dz}{d\tau}\right|_{z_0} \sum_{n=0}^{\infty} \left.\frac{d^n X}{dz^n}\right|_{z_0} \left(\frac{e^{i\phi}}{\sqrt{\rho|f''(z_0)|}}\right)^n \frac{1}{n!} \int_{-\tau_1\sqrt{\rho|f''|}}^{\tau_1\sqrt{\rho|f''|}} t^n e^{-\frac{1}{2}t^2} dt \end{aligned}$$

One can simplify this expression in two ways:
First of all, it is permissible to replace the integration limits by $t = \pm\infty$ whenever

$$\tau_1\sqrt{\rho|f''|} \gg \sqrt{n} \geq 1 \quad .$$

Under this condition the integral may be replaced by its limiting value,

$$\frac{1}{n!}\int_{-\infty}^{\infty} t^n e^{-t^2/2}\, dt = \sqrt{2\pi} \times \begin{cases} 0 & n = \text{odd} \\ 1 & n = 0 \\ \frac{1}{2^m m!} & n = 2m \end{cases}$$

It is obvious that the inequality is violated for sufficiently large n. However, this will not happen if the Taylor series representation of $X(z)$ can

be truncated without compromising the accuracy with which $X(z)$ is to be represented.
Secondly, one may apply Eqs.(5.65) and (5.62) to

$$\frac{e^{2i\phi}}{|f''(z_0)|} = \frac{-e^{-i\delta_0}}{|f''(z_0)|} \equiv \frac{-1}{f''(z_0)} \quad . \tag{5.69}$$

With these two simplifications the steepest descent evaluation of the contour integral Eq.(5.61) yields the following series in inverse powers of ρ:

$$\boxed{\int_A^B X(z)e^{\rho f(z)}dz = \sqrt{\frac{2\pi}{\rho}}\frac{e^{\rho f(z_0)}}{[-f''(z_0)]^{1/2}}\sum_{m=0}^{N}\left.\frac{d^{2m}X}{dz^{2m}}\right|_{z_0}\frac{(-1)^m}{m!}\left(\frac{1}{2\rho f''(z_0)}\right)^m} \tag{5.70}$$

Here N is the mandatory truncation integer, and

$$[-f''(z_0)]^{1/2} = |f''(z_0)|^{1/2}e^{-i\phi} \; (= |f''(z_0)|^{1/2}e^{i\delta_0/2}e^{\pm i\pi/2})$$

is that root which has the phase factor $e^{-i\phi}$ whose angle ϕ points along the integration path through the critical point z_0.
Example: Evaluate

$$\begin{aligned} H_\nu^{(1)} &= \frac{e^{-i\pi\nu/2}}{\pi}\int_{C_1} e^{i\rho\cos z}e^{i\nu z}dz \\ &= \frac{e^{-i\pi\nu/2}}{\pi}\int_{C_1} e^{\rho f(z)}X(z)dz\,. \end{aligned} \tag{5.71}$$

to second order accuracy in $1/\rho$. Here

$$\begin{aligned} X(z) &= e^{i\nu z} \\ f(z) &= i\cos z \\ f'(z) &= -i\sin z \\ f''(z) &= -i\cos z \end{aligned}$$

The critical points determined by $f'(z) = 0$ are

$$z = 0, \pm\pi, \cdots \; .$$

The integration limits of $H_\nu^{(1)}$ in the complex $z-$plane are indicated in Figure 5.3. They dictate that the *most direct* path of steepest descent passes through the critical point

$$z_0 = 0 \quad .$$

Consequently,

$$\begin{aligned} f''(z_0) &= -i = e^{-i\pi/2} \\ X(z_0) &= 1 \\ X''(z_0) &= -\nu^2 \end{aligned}$$

The phase angle ϕ of the integration path $z - z_0 = \tau e^{i\phi}$ is determined by the condition that

$$(z - z_0)^2 f''(z_0) = -\tau^2 |f''(z_0)| \quad .$$

Consequently, Eq.(5.65) becomes

$$e^{2i\phi} e^{-i\pi/2} = -1 \ (= e^{\pm i\pi})$$

or

$$e^{i\phi} = \pm e^{-i\pi/4}$$

The fact that the path goes from the second to the fourth quadrant (as in Figure 5.18) requires that one choose the upper sign,

$$e^{i\phi} = e^{-i\pi/4}$$

Thus, in light of Eq.(5.69, namely

$$-f''(z_0) = |f''(z_0)| e^{2i\phi} \ ,$$

one has

$$[-f''(z_0)]^{1/2} = e^{i\pi/4} \quad .$$

This is because[1] the square root of a polar representation is unique[2] It follows that the large ρ expansion of Eq.(5.71) is

$$\begin{aligned} H_\nu^{(1)}(\rho) &= \frac{e^{-i\pi\nu/2}}{\pi} \sqrt{\frac{2\pi}{\rho}} e^{i\rho} e^{-i\pi/4} \left[1 - \nu^2 \frac{-1}{1} \left(\frac{1}{2\rho(-i)} \right)^1 + \cdots \right] \\ &= \sqrt{\frac{2}{\pi\rho}} e^{i\rho - i(\nu+1/2)\frac{\pi}{2}} \left[1 + i\frac{\nu^2}{2\rho} + \cdots \right] \end{aligned} \tag{5.72}$$

[1] See the discussion surrounding Figures 4.10 and 4.11 on page 288.

[2] Indeed, another polar representation, namely,

$$-f''(z_0) = |f''(z_0)| e^{2i\phi + 2\pi i}$$

would have given the wrong result

$$[-f''(z_0)]^{1/2} = -e^{i\pi/4} \quad .$$

Exercise 5.5.1 (STEEPEST DESCENT)

(a) Using the method of steepest descent FIND an asymptotic expression for $H_\nu^{(2)}(\rho)$ and for $J_\nu(\rho)$ when $\nu << \rho$.

(b) The gamma function $\Gamma(w+1)$ which for $Re\ w > -1$ is represented by

$$\Gamma(w+1) = \int_0^\infty e^{-\tau}\tau^w d\tau$$

Using the steepest descent approach, FIND an asymptotic expression for $\Gamma(w+1)$ when $Re\ w >> 1$. Why does't it work? Try again by substituting wz for τ, and obtaining

$$\Gamma(w+1) = w^{w+1}\int_0^\infty e^{-wz}z^w dz = w^{w+1}\int_0^\infty e^{w(\ln z - z)}dz$$

5.6 Boundary Value Problems in Two Dimensions

Consider the following problem: A vibrating system has an amplitude response ψ to a source function f which is governed by the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)\psi(\vec{x}) = -f(\vec{x})\ . \tag{5.73}$$

Assume that this equation applies to a 2-dimensional region R whose boundary is designated by ∂R. Suppose that on this boundary the response amplitude satisfies the inhomogeneous mixed Dirichlet-Neumann boundary condition

$$\Big[a(\vec{x})\psi(\vec{x}) + \vec{n}\cdot\vec{\nabla}\psi(\vec{x})\Big]_{\partial R} = g(\vec{x})|_{\partial R}\ . \tag{5.74}$$

Find the response amplitude $\psi(\vec{x})$!

This problem is characterized by

1. the shape of the as-yet-unspecified region R,
2. the as-yet-unspecified inhomogeneities f and g, and

3. the as-yet-unspecified effective stiffness of the boundary, the function $a(\vec{x})$.

Thus, by omitting reference to the particular measurement of these properties, one has mentally subsumed a vast number of particular problems, which govern the response of a vast number of linear systems, into a new concept [3], an equivalence class of problems. A representative class member is characterized by Eqs.(5.73) and (5.74).

5.6.1 Solution via Green's Function

The most delightful aspect about this problem is that its solution can readily be expressed in terms of the Green's function for the given linear system.

The reasoning leading to this solution is an extension into two dimensions of the 1-dimensional problem considered in Sections 4.2 (p. 235) and 4.9 (p. 257). As in that case, the solution is easily given in terms of the associated Green's function $G(\vec{x};\vec{x}_0)$. It satisfies

$$(\nabla^2 + k^2)G(\vec{x};\vec{x}_0) = -\delta^2(\vec{x}-\vec{x}_0) \equiv \begin{cases} -\delta(x-x_0)\delta(y-y_0) & \text{in Cartesian coordinates} \\ -\dfrac{\delta(r-r_0)}{r}\delta(\theta-\theta_0) & \text{in polar coordinates} \end{cases} \tag{5.75}$$

and

$$\Big[a(\vec{x})G(\vec{x};\vec{x}_0) + \vec{n}\cdot\vec{\nabla}G(\vec{x};\vec{x}_0)\Big]_{\partial R} = 0\ . \tag{5.76}$$

The solution process to this 2-dimensional problem parallels the one for one dimension. First of all, use Lagrange's identity

$$\psi\nabla^2 G - G\nabla^2\psi = \nabla\cdot(\psi\vec{\nabla}G - G\vec{\nabla}\psi)$$

Its integral over the region R yields the Green's identity

$$\int_R\int(\psi\nabla^2 G - G\nabla^2\psi)d^2x = \oint_{\partial R}(\psi\vec{\nabla}G - G\vec{\nabla}\psi)\cdot\vec{n}\ ds$$

[3] It is worthwhile to point out that the process of measurement omission is the process by which *all* concepts are formed. This observation and the procedure for implementing this process were first spelled out by Ayn Rand in Chapters 1-2 of *Introduction to Objectivist Epistemology, 2nd Edition*, edited by H. Binswanger and L. Peikoff. Penguin Books, Inc., New York, 1990.

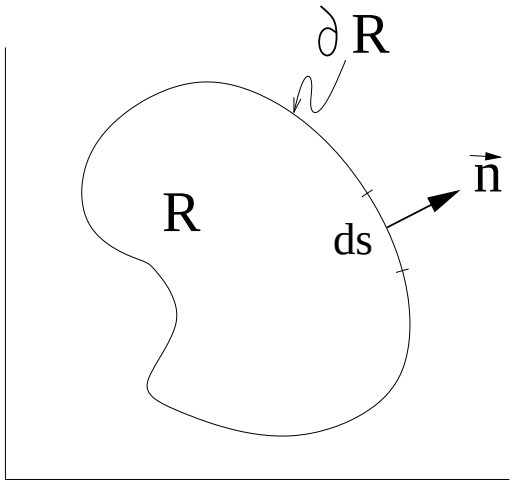


Figure 5.19: Integration region R with boundary ∂R having outward pointing normal $\vec{n}$ perpendicular to each boundary element ds.

Secondly, applying the inhomogeneous Helmholtz equation, Eq.(5.73), and Eq.(5.75) to the left hand side, one obtains

$$\int_R \int \left[\psi(\vec{x})(-)\delta^2(\vec{x}-\vec{x}_0) + G(\vec{x};\vec{x}_0)f(\vec{x})\right] d^2x = \oint_{\partial R} (\psi \vec{\nabla} G - G\vec{\nabla}\psi)\cdot \vec{n}\; ds \; .$$

Finally, substituting the two boundary conditions, Eqs.(5.74) and (5.76) into the right hand side, one finds that

$$\psi(\vec{x}_0) = \int_R \int f(\vec{x}) G(\vec{x};\vec{x}_0) d^2x - \oint_{\partial R} g(\vec{x}) G(\vec{x};\vec{x}_0)|_{\partial R}\, ds \; .$$

Thus, knowledge of the Green's function G, automatically yields the response amplitude ψ in terms of its values on the boundary ∂R and in terms of the source distribution f in the region R.

5.6.2 Green's Function via Dimensional Reduction

To find the Green's function whose domain dimension is two or higher, introduce a technique whose virtue is that it reduces the problem to a Green's function problem in just one dimension. The potency of this technique is a consequence of the fact that it leads to success whenever the Helmholtz

equation is separable relative to the curvilinear coordinate system[4] induced by the boundarie(s) of the given domain.

Polar coordinates is a case in point. It is illustrated by the following

Problem (Green's Function for Radiation in the Euclidean Plane) The equation for the Green's function relative to polar coordinates is

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+k^2\right]G_k(\vec{x};\vec{x}_0)=-\frac{\delta(r-r_0)}{r}\delta(\theta-\theta_0)\ . \tag{5.77}$$

Let the homogeneous boundary conditions for G be

(i) Sommerfeld's outgoing radiation condition

$$0=\sqrt{r}\left(a(\vec{x})G_k(\vec{x};\vec{x}_0)+\vec{n}\cdot\vec{\nabla}G(\vec{x};\vec{x}_0)\right)\Big|_{\partial R}\equiv\lim_{r\to\infty}\sqrt{r}\left((-)ikG+\frac{\partial}{\partial r}G\right)$$

(ii) $G_k(\vec{x};\vec{x}_0)$ is finite at $r=0$, where θ is undefined.

Solution

This problem is solved by expanding the Green's function as a Fourier series on $[0,2\pi]$:

$$G_k(\vec{x};\vec{x}_0)=\sum_{-\infty}^{\infty}g_m(r;r_0)\frac{e^{im(\theta-\theta_0)}}{2\pi} \tag{5.78}$$

with to-be-determined Fourier coefficients. The method of dimensional reduction consists of establishing that each of them satisfies a 1-dimensional Green's function problem. Next one constructs its solution using formula Eq.(4.38) on page 253. Finally one introduces this solution into the Fourier series expansion. This yields the desired 2-dimensional Green's function. As an additional benefit one finds that this expansion can be summed into a closed form expression given by a familiar function.

The details of this four step procedure are as follows: Introduce the Fourier expansion into the 2-dimensional Green's function equation and obtain

$$\sum_{-\infty}^{\infty}\frac{e^{im(\theta-\theta_0)}}{2\pi}\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}+\left(k^2-\frac{m^2}{r^2}\right)\right]g_m(r;r_0)=-\frac{\delta(r-r_0)}{r}\delta(\theta-\theta_0)\ . \tag{5.79}$$

[4]In three dimensions the Helmholtz equation is separable in eleven coordinate systems. They are listed and depicted at the end of chapter five of reference [3]

To isolate the equation obeyed by the each of the coefficient functions $g_m(r;r_0)$ introduce the Fourier representation of the Dirac delta function restricted to $[0,2\pi]$:

$$\delta(\theta-\theta_0)=\sum_{-\infty}^{\infty}\frac{e^{im(\theta-\theta_0)}}{2\pi}\ ,$$

and make use of the linear independence of the functions $e^{im(\theta-\theta_0)}$. Alternatively, multiply both sides of Eq.(5.79) by $e^{-im'(\theta-\theta_0)}$, integrate $\int e^{-im'(\theta-\theta_0)}(\cdots)d\theta$, make use of orthogonality, and finally drop the prime to obtain

$$\left[\frac{d}{dr}r\frac{d}{dr}+\left(rk^2-\frac{m^2}{r}\right)\right]g_m(r;r_0)\ ,=-\delta(r-r_0)$$

the equation for the 1-dimensional Green's function. The boundary conditions for G imply that the solution

$$g_m(r;r_0)=\frac{-1}{c}\begin{cases}u_1(r)u_2(r_0) & \text{for}\quad r<r_0\\ u_1(r_0)u_2(r) & \text{for}\quad r_0<r\end{cases}$$

satisfies

$$\begin{aligned} g_m(r=0;r_0) &= finite \\ g_m(r;r_0) &\sim \frac{e^{ikr}}{\sqrt{r}}\ \text{for very large } r\end{aligned}$$

This is a set of 1-dimensional Green's function problems whose solutions yield the 2-d Green's funtion, Eq.(5.78). The two functions which satisfy the homogeneous differential equation and the respective boundary conditions are

$$u_1=J_m(kr)$$

and

$$u_2=H_m^{(1)}(kr)\ ,$$

while their Wronskian is

$$u_1u_2'-u_1'u_2=\frac{2i}{\pi r}\ .$$

Consequently, the 1-dimensional Green's function is

$$g_m(r;r_0) = \begin{cases} \dfrac{\pi i}{2} J_m(kr) H_m^{(1)}(kr_0) & r \le r_0 \\ \dfrac{\pi i}{2} J_m(kr_0) H_m^{(1)}(kr) & r_0 \le r \end{cases}$$

The 2-dimensional Green's function, Eq.(5.78), for outgoing radiation in the Euclidean plane is therefore

$$G_k(\vec{x};\vec{x}_0) = \sum_{-\infty}^{\infty} \frac{i}{4} e^{im(\theta-\theta_0)} J_m(kr_<) H_m^{(1)}(kr_>) \ .$$

This expression can be simplified by means of the *displacement formula* for cylinder modes, Property 19, on page 363,

$$H_\nu(k|\vec{x}-\vec{x}_0|)e^{i\nu(\Omega-\theta_0)} = \sum_{m=-\infty}^{\infty} e^{i(\nu+m)(\theta-\theta_0)} J_m(kr_0) H_{\nu+m}(kr)$$

Set $\nu = 0$, compare the Green's function with the right hand side of the displacement formala, and conclude that

$$G_k(\vec{x};\vec{x}_0) = \frac{i}{4} H_0(k|\vec{x}-\vec{x}_0|) \ ; \tag{5.80}$$

in other words,

$$\boxed{\left(\nabla^2 + k^2\right) \frac{i}{4} H_0(k|\vec{x}-\vec{x}_0|) = -\delta^2(\vec{x}-\vec{x}_0)} \tag{5.81}$$

Thus one has obtained an expression for the 2-dimensional Green's function which exhibits the rotational and translational symmetry of the linear system. It represents an asymtotically (large $|\vec{x}|$!) outgoing wave whose source is located at $|\vec{x}_0|$. This is the amplitude profile of a wave that you make when you stick your wiggling finger at $|\vec{x}_0|$ into an otherise motionless pond.

5.6.3 Green's Function: 2-D Laplace vs. (Limit of) 2-D Helmholtz

It is instructive to attempt to solve Eq.(5.77) and its boundary conditions by simply computing the limit of the solution, Eq.(5.80),

$$\lim_{k\to 0} \frac{i}{4} H_0(kR); \qquad R = |\vec{x}-\vec{x}_0| \ ,$$

and compare this limit with the result obtained by directly solving the boundary value

Problem (Green's Function for the Potential of an Isolated Source) Setting $k = 0$ in the previous problem, find the Green's function which satisfies

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] G_0(\vec{x};\vec{x}_0) = -\frac{\delta(r-r_0)}{r}\delta(\theta-\theta_0) \ . \tag{5.82}$$

together with

(i)

$$0 = \sqrt{r}\vec{n}\cdot\vec{\nabla}G_0(\vec{x};\vec{x}_0)\Big|_{\partial R} \equiv \lim_{r\to\infty}\sqrt{r}\frac{\partial}{\partial r}G_0$$

and

(ii) $G_0(\vec{x};\vec{x}_0)$ is finite at $r = 0$, where θ is undefined.

Solution: Using the method of dimensional reduction, one again starts with

$$G_0(\vec{x};\vec{x}_0) = \sum_{-\infty}^{\infty} g_m(r;r_0)\frac{e^{im(\theta-\theta_0)}}{2\pi} \tag{5.83}$$

and the implied boundary conditions

$$\begin{aligned} g_m(r=0;r_0) &= finite \\ \lim_{r\to\infty} &= \sqrt{r}\frac{\partial g_m}{\partial r} = 0 \ . \end{aligned}$$

The solution can be summed into the closed form

$$G_0(\vec{x};\vec{x}_0) = -\frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0| \text{ whenever } r \neq r_0 \tag{5.84}$$

How does this Green's function compare with the asymptotic limit of the Helholtz Green's function, Eq.(5.80),

$$\lim_{k\to 0} G_k(\vec{x};\vec{x}_0) \ ?$$

The answer is contained in a comparison with the following

Problem (Potential Green's Function as an Asymtotic Limit) Exhibit the limiting form of Eq.(5.80) as $k \to 0$.

Solution
Using Property 14 on page 341,

$$H_\nu^{(1)}(kR) = e^{-i\nu\pi/2}\ \frac{e^{-i\nu\pi/2}J_\nu(kR) - e^{i\nu\pi/2}J_{-\nu}(kR)}{-i\sin\pi\nu}\ ,$$

one needs to calculate

$$\lim_{k\to 0} H_0^{(1)}(kR) = \lim_{k\to 0}\lim_{\nu\to 0} H_\nu^{(1)}(kR)$$

This is a double limit which one obtains by evaluating

$$\lim_{\nu\to 0} H_\nu^{(1)}(kR)$$

for $kR \ll 1$, and then by examining the behaviour of the resulting expression as $k \to 0$.

The evaluation yields

$$\lim_{\nu\to 0} H_\nu^{(1)}(kR) = \frac{0}{0}\ .$$

Consequently, l'Hospital's rule

$$\begin{aligned}\lim_{\nu\to 0} H_\nu^{(1)}(kR) &= \lim_{\nu\to 0}\frac{\dfrac{d}{d\nu}\left(e^{-i\nu\pi/2}J_\nu(kR) - e^{i\nu\pi/2}J_{-\nu}(kR)\right)}{\dfrac{d}{d\nu}\left(-i\sin\pi\nu\right)} \qquad (5.85)\\ &= \lim_{\nu\to 0}\frac{e^{-i\nu\pi/2}\left[\frac{-i\pi}{2}J_\nu(kR) + \dfrac{dJ_\nu}{d\nu}\right] - e^{i\nu\pi/2}\left[\frac{i\pi}{2}J_{-\nu}(kR) + \dfrac{dJ_{-\nu}}{d\nu}\right]}{-i\pi\cos\pi\nu}\end{aligned}$$

must be used. Taking advantage of the asymptotic small-k expansion of $J_\nu(kR)$, Eq.(5.24),

$$\begin{aligned}J_\nu(kR) &\approx \left(\frac{kR}{2}\right)^\nu \frac{1}{\Gamma(1+\nu)}\ ,\\ \frac{d}{d\nu}\left(\frac{kR}{2}\right)^\nu &= \left(\frac{kR}{2}\right)\ln\left(\frac{kR}{2}\right)\ ,\end{aligned}$$

and

$$\begin{aligned}\lim_{\nu\to 0}\frac{d}{d\nu}\left(\frac{1}{\Gamma(1+\nu)}\right) &= \lim_{\nu\to 0}\frac{-1}{\Gamma(1+\nu)}\times\frac{\Gamma'(1+\nu)}{\Gamma(1+\nu)}\\ &= -1\times(-)C\ ;\quad C=.5772\ \text{(Euler-Mascheroni constant)}\\ &\equiv \ln\gamma\ ;\quad \gamma=1.781\end{aligned}$$

one finds that

$$\begin{aligned}\frac{dJ_\nu(kR)}{d\nu} &= \left(\frac{kR}{2}\right)^\nu\ln\left(\frac{kR}{2}\right)+\ln\gamma\\ &= \left(\frac{kR}{2}\right)^\nu\ln\left(\frac{\gamma kR}{2}\right)\end{aligned}$$

Consequently, l'Hospital's rule tells us that

$$\begin{aligned}\lim_{\nu\to 0}H_\nu^{(1)}(kR) &= \frac{\left[\dfrac{-i\pi}{2}+\ln\dfrac{\gamma kR}{2}\right]\times 2}{-i\pi}\\ &= 1+\frac{2i}{\pi}\ln\frac{\gamma kR}{2}\end{aligned}$$

Thus the small-k form of the 2-D helmholtz Green's function, Eq.(5.80), is

$$\begin{aligned}G_k(\vec{x};\vec{x}_0) &= \frac{i}{4}H_0(k|\vec{x}-\vec{x}_0|) \qquad (5.86)\\ &= -\frac{1}{2\pi}\ln|\vec{x}-\vec{x}_0|+\left(\frac{i}{4}-\frac{1}{2\pi}\ln\frac{\gamma kR}{2}\right)\text{whenever } k\ll 1.\end{aligned}$$

This is the solution to the problem and it expresses the amplitude profile of a membrane responding to a unit force applied at $\vec{x}_0$. This membrane is imbedded in an elastic medium whose local force of restitution (per unit area) is proportional to k^2:

$$G_k\times k^2\Delta(\text{area})=\Delta(\text{force of restitution})$$

Thus k^2 is the Young's modulus of the elastic medium in which the membrane is imbedded. As $k^2\to 0$ there is nothing to push the membrane back towards its zero-amplitude equilibrium. Consequently, the smaller that Young's modulus is, the further the membrane gets pushed away from this equilibrium

by the Dirac delta function force density. This is why $G_k(\vec{x};\vec{x}_0) \to \infty$ as $k^2 \to 0$. Equation (5.86) expresses this fact quantitatively.

By contrast $G_0(\vec{x};\vec{x}_0)$ as given by Eq.(5.84) does not presume any elastic medium. The asymptotic Neumann boundary condition that went into G_0 forbids it being interpreted as the amplitude of any membrane. Instead G_0 expresses the potential due to an electrostatic charge.

5.7 Wave Equation for Spherically Symmetric Systems

Lecture 47

The formulation of linear wave phenomenon in terms of the wave equation, the Helmholtz equation, and its solutions in terms of orthonormal function on the Euclidean plane can be extended readily to three dimensional Euclidean space. For this space the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2} = 0$$

can be solved relative to various orthogonal coordinate systems (there are at least eleven of them). The choice of coordinates is invariably dictated by symmetry and boundary conditions. This means that the coordinates are usually chosen so one or more of the coordinate surfaces mold themselves onto boundaries where the boundary conditions are specified. In terms of ubiquity, the three most important coordinate systems are the rectangular, cylindrical, and the spherical coordinates.

We shall now consider the wave equation relative to spherical coordinates given by

$$\begin{aligned} x &= r\sin\theta\cos\varphi \quad 0 \le \theta \le \pi\,, \quad 0 \le \varphi \le 2\pi \\ y &= r\sin\theta\sin\varphi \quad 0 \le r < \infty \\ z &= r\cos\theta \end{aligned}$$

The angles φ and θ are called the *azimuthal* and the *polar* angle respectively. Relative to these coordinates, the Laplace operator has the form

$$\nabla^2\psi = \frac{1}{r^2}\,\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2}\right]\psi\,.$$

A useful observation, rather valuable as we shall see momentarily, is the fact that the first term can be written in the form

$$\frac{1}{r^2}\,\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\psi = \left(\frac{1}{r}\,\frac{\partial}{\partial r}r\right)^2\psi \equiv \frac{1}{r}\,\frac{\partial^2}{\partial r^2}r\psi\ .$$

Another useful observation is that the second term is easy to remember. Indeed, for small θ ($\theta \ll 1$) one has

$$\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2} \approx \frac{1}{\theta}\,\frac{\partial}{\partial\theta}\theta\frac{\partial}{\partial\theta} + \frac{1}{\theta^2}\,\frac{\partial^2}{\partial\varphi^2}\ ,$$

the familiar two-dimensional Laplacian, Eq.(5.1) on P317, for the Euclidean plane. This is as it should be: around the north pole of a sphere the spherical coordinates reduce to the polar coordinates of the Euclidean plane around the origin.

The physically and mathematically most revealing solutions are the normal modes. They are *time translation eigenfunctions* and, as we have already learned from Section 5.1.4, they satisfy the equation

$$\frac{\partial}{\partial t}\Psi = i\omega\Psi\,.$$

A normal mode has the form

$$\Psi(r,\theta,\varphi,t) = \psi(r,\theta,\varphi)e^{i\omega t}\,.$$

Here $\psi(r,\theta,\varphi)$ is the spatial amplitude profile which satisfies the Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0\,, \qquad\qquad k^2 = \frac{\omega^2}{c^2}$$

or

$$\left\{\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2}\right] + k^2\right\}\psi = 0\,,$$

relative to spherical coordinates. This partial differential equation lends itself to being separated into a system of ordinary differential equations. Letting $\psi = j(r)Y(\theta,\varphi)$, dividing by $j(r)$, multiplying by r^2, and tranferring the r-dependent term to the right hand side, one finds that the r.h.s. is independent of θ and ϕ, while the l.h.s. is independent of r. But these two sides are equal and hence are independent of all three variables. Thus both the l.h.s. and

the r.h.s. are equal to the same constant, the *separation constant*, say $-\lambda$. This yields two equations for $Y(\theta,\phi)$ and $j(r)$ respectively. Explicitly one has

$$\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\sin\theta\frac{\partial Y}{\partial\theta}+\frac{1}{\sin^2\theta}\,\frac{\partial^2 Y}{\partial\varphi^2} \;=\; -\lambda Y\,; \tag{5.87}$$

and

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\,\frac{d}{dr}+\left(k^2-\frac{\lambda}{r^2}\right)\right\}j(r)=0 \tag{5.88}$$

These are two eigenvalue equations. The separation constant λ is the eigenvalue determined by the boundary conditions on the angular function, and k^2 is the eigenvalue determined by the boundary conditions on the radial function $j(r)$. One of the allowed eigenvalues for λ expresses the circumstance where the amplitude profile is spherically symmetric, i.e. is independent of the angles θ and ϕ. For this circumstance the solutions to the Helmholtz equation can be found immediately.

5.7.1 Spherically Symmetric Solutions

Spherically symmetric solutions are those whose form is characterized by $Y(\theta,\phi)\equiv$const. so that $\lambda=0$. Consequently, the radial part $j(r)$ of the solution obeys

$$\left[\frac{1}{r}\,\frac{d^2}{dr^2}r+k^2\right]j=0$$

so that

$$j(r)=\frac{e^{ikr}}{r},\frac{e^{-ikr}}{r},\frac{\sin kr}{r},\frac{\cos kr}{r}\,,$$

or any of their linear combinations. Which one it is, and what the allowed values of k are, depends entirely on the given boundary conditions. The concomitant spherically symmetric normal modes have the form

$$\Psi=e^{-i\omega t}\frac{e^{ikr}}{r},e^{-i\omega t}\frac{e^{-ikr}}{r},\text{etc.}$$

For example, the amplitudes of the spherically symmetric normal modes confined to the interior of a hard sphere of radius a are

$$\Psi=e^{-i\omega t}\frac{\sin n\pi r/a}{r}\quad\omega=\frac{n\pi}{ca},\quad n=1,2,\cdots$$

A pure sound note in a spherical resonator is an example of such a normal mode. It vibrates with frequency $\frac{\omega}{2\pi} = \frac{n}{2ca}$.

5.7.2 Factorization Method for Solving a Partial Differential Equation: Spherical Harmonics

Now back to the remaining eigenvalues, $\lambda \neq 0$. The radial equation can be changed into a familiar one by letting

$$j = \frac{J}{\sqrt{r}}\,.$$

This results in

$$\frac{d^2J}{dr^2} + \frac{1}{r}\,\frac{dJ}{dr} + \left(k^2 - \frac{\lambda + \frac{1}{4}}{r^2}\right) J = 0$$

which is the familiar Bessel equation of order $\sqrt{\lambda + \frac{1}{4}}$. Its solutions are $H^{(1)}_{\sqrt{\lambda+\frac{1}{4}}}(kr)$ and $H^{(2)}_{\sqrt{\lambda+\frac{1}{4}}}(kr)$, where λ is to be determined.

Lecture 48

The value of λ is not arbitrary. It is one of the (degenerate) eigenvalues of Eq.(5.87), the two-dimensional Helmholtz equation on the unit two-sphere. As already observed on page 390, for $\theta \ll 1$ this equation reduces to Helmholtz's equation on the Euclidean plane. This observation is very useful for several reasons. One of them is that it implies, as shown on page 346, that there is a simple algebraic way of generating a complete basis for each degenerate eigenspace of

$$\nabla^2 Y = -\lambda Y\ .$$

We shall now extend this algebraic method from the eigenfunctions of ∇^2 on the two-dimensional Euclidean plane to those of ∇^2 on the two-dimensional surface of a unit sphere.

The factorization method of solving a partial (or ordinary) differential equation is remarkable. This method differs from a power series or a numerical approach in that one solves a calculus problem without the use of calculus: one obtains the linear algebraic aspects of the problem (eigenvalues, all normalized eigenvectors, their properties, etc.) in one fell swoop without ever having to determine explicitly the detailed functional form (i.e. local

behaviour) of the solutions. To be sure, one can readily determine and exhibit these solutions in explicit form in terms of Legendre and associated Legendre polynomials, and we shall do so. However, this is a straight forward, but secondary, calculus exercise which is an implied but not an integral part of the algebraic approach.

Important Reminder: Throughout the ensuing discussion an *eigenfunction* on the unit sphere refers to a function which is *square-integrable*, i.e.

$$0 < \langle Y, Y \rangle \equiv \int_0^{\pi} \int_0^{2\pi} |Y(\theta, \varphi)|^2 \sin\theta d\theta d\varphi < \infty \ . \tag{5.89}$$

One will see that the very existence of the eigenvalue spectrum of ∇^2 on the unit sphere hinges on this fact. For this reason, the extension of this algebraic method is considerably more powerful. It yields not only the basis for each eigenspace of ∇^2, but also the actual value for each allowed degenerate eigenvalue.

Global Analysis: Algebra

Global analysis deals with the solutions of a differential equation "wholesale". It characterizes them in relationship to one another without specifying their individual behaviour on their domain of definition. Thus one focusses via algebra, linear or otherwise, on "the space of solutions", its subspaces, bases etc.

Local analysis (next subsubsection), by contrast, deals with the solutions of a differential equation "retail". Using differential calculus, numerical analysis, one zooms in on individual functions and characterizes them by their local values, slopes, location of zeroes, etc.

1. Factorization

The algebraic method depends on factoring

$$\nabla^2 = \frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

into a pair of first order operators which are adjoints of each other. The method is analogous to factoring a quadratic polynomial, except that here one has differential operators $\partial/\partial\theta$ and $\partial/\partial\varphi$ instead of the variables x and y. Taking our cue from *Properties 16* and *17*, one attempts

$$\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \stackrel{?}{=} e^{i\phi}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)e^{-i\phi}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

However, one immediately finds that this factorization yields $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$ for a cross term. This is incorrect. What one needs instead is $\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}$. This leads us to consider

$$\begin{aligned} & e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)e^{-i\phi}\left(\frac{\partial}{\partial\theta}-i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\ &= \frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{i}{\sin^2\theta}(1-\cos^2\theta)\frac{\partial}{\partial\varphi}+\frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \\ &= \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-\frac{\partial^2}{\partial\varphi^2}-\frac{1}{i}\frac{\partial}{\partial\varphi} \\ &\equiv \qquad\qquad \nabla^2 \qquad\qquad + L_\varphi^2 \quad - L_\varphi \end{aligned} \tag{5.90}$$

Here we have introduced the self-adjoint operator

$$L_\varphi = \frac{1}{i}\frac{\partial}{\partial\varphi}\ .$$

It generates rotations around the polar axis of a sphere. This operator, together with the two mutually adjoint operators

$$L_\pm = \pm e^{\pm i\phi}\left(\frac{\partial}{\partial\theta}\pm i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

are of fundamental importance to the factorization method of solving the given differential equation. In terms of them the factorized Eq.(5.90) and its complex conjugate have the form

$$L_\pm L_\mp = -\nabla^2 - L_\varphi^2 \pm L_\varphi\ . \tag{5.91}$$

This differs from Eq.(5.29), (*Property 17* on page 346), the factored Laplacian on the Euclidean plane.

2. Fundamental Relations

In spite of this difference, the commutation relations corresponding to Eqs.(5.31), (5.32), and (5.33) are all the same, except one. Thus, instead of Eq.(5.32), for a sphere one has

$$[L_+, L_-] = 2L_\varphi\ . \tag{5.92}$$

This is obtained by subtracting the two Eqs.(5.91). However, the commutation relations corresponding to the other two equations remain the same. Indeed, a little algebraic computation yields

$$L_\varphi L_\mp = L_\pm L_\varphi \pm L_\varphi\ ,$$

or

$$[L_\varphi, L_\pm] = \pm L_\pm \ . \tag{5.93}$$

Furthermore, using Eq.(5.91) one finds

$$\begin{aligned}
[\nabla^2, L_+] &= [-L_+L_- - L_\varphi^2 + L_\varphi, L_+] \\
&= -[L_+L_-, L_+] - [L_\varphi^2, L_+] + [L_\varphi, L_+] \\
&= -L_+(L_-L_+ - L_+L_-) \\
&\quad - L_\varphi(L_\varphi L_+ - L_+L_\varphi) - (L_\varphi L_+ - L_+L_\varphi)L_\varphi \\
&\quad + (L_\varphi L_+ - L_+L_\varphi) \\
&= 0 \ .
\end{aligned} \tag{5.94}$$

The last equality was obtained with the help of Eqs.(5.92) and (5.93). Together with the complex conjugate of this equation, one has therefore

$$[\nabla^2, L_\pm] = 0 \ . \tag{5.95}$$

In addition, one has quite trivially

$$[\nabla^2, L_\varphi] = 0 \tag{5.96}$$

The three algebraic relations, Eqs.(5.92)-(5.93) and their consequences, Eq.(5.95)-(5.96), are the fundamental equations from which one deduces the allowed degenerate eigenvalues of Eq.(5.87) as well as the corresponding *normalized* eigenfunctions.

3. The Eigenfunctions

One starts by considering a function Y_λ^m which is a simultaneous solution to the two eigenvalue equations

$$\begin{aligned}
L_\varphi Y_\lambda^m &= mY_\lambda^m \\
\nabla^2 Y_\lambda^m &= -\lambda Y_\lambda^m \ .
\end{aligned}$$

This is a consistent system, and it is best to postpone until later the easy task of actually exhibiting non-zero solutions to it. First we deduce three properties of any given solution Y_λ^m.

The first property is obtained by applying the operator L_+ to this solution. One finds that

$$\begin{aligned}
L_\varphi(L_+Y_\lambda^m) &= (L_+L_\varphi + L_+)Y_\lambda^m \\
&= (m+1)(L_+Y_\lambda^m)
\end{aligned}$$

Similarly one finds

$$L_\varphi(L_- Y_\lambda^m) = (m-1)(L_- Y_\lambda^m) \ .$$

Thus $L_+ Y_\lambda^m$ and $L_- Y_\lambda^m$ are again eigenfunctions of L_φ, but having eigenvalues $m+1$ and $m-1$. One is, therefore, justified in calling L_+ and L_- *raising* and *lowering* operators. The "raised" and "lowered" functions $L_\pm Y_\lambda^m$ have the additional property that they are still eigenfunctions of ∇^2 belonging to the same eigenvalue λ. Indeed, with the help of Eq.(5.95) one finds

$$\nabla^2 L_\pm Y_\lambda^m = L_\pm \nabla^2 Y_\lambda^m = -\lambda L_\pm Y_\lambda^m \ .$$

Thus, *if Y_λ^m belongs to the eigenspace of λ, then so do $L_+ Y_\lambda^m$ and $L_- Y_\lambda^m$.*
4. Normalization and the Eigenvalues
The second and third properties concern the normalization of $L_\pm Y_\lambda^m$ and the allowed values of λ. One obtains them by examining the sequence of squared norms of the sequence of eigenfunctions

$$L_\pm^k Y_\lambda^m \quad , \qquad k = 0, 1, 2, \cdots \quad .$$

All of them are square-integrable. Hence their norms are non-negative. In particular, for $k = 1$ one has

$$\begin{aligned} 0 \leq \int_0^\pi \int_0^{2\pi} |L_\pm Y_\lambda^m(\theta,\varphi)|^2 \sin\theta d\theta d\varphi &\equiv \langle L_\pm Y_\lambda^m, L_\pm Y_\lambda^m \rangle \\ &= \langle Y_\lambda^m, L_\mp L_\pm Y_\lambda^m \rangle \\ &= \langle Y_\lambda^m, (-)(\nabla^2 + L_\varphi^2 \pm L_\varphi) Y_\lambda^m \rangle \\ &= [\lambda - m(m \pm 1)] \langle Y_\lambda^m, Y_\lambda^m \rangle \quad (5.97) \end{aligned}$$

This is the second property. It is a powerful result for two reasons:
First of all, if Y_λ^m has been normalized to unity, then so will be

$$\frac{1}{[\lambda - m(m \pm 1)]^{1/2}} L_\pm Y_\lambda^m(\theta,\varphi) \equiv Y_\lambda^{m\pm 1}(\theta,\varphi) \qquad (5.98)$$

This means that once the normalization integral has been worked out for any one of the Y_λ^m's, the already normalized $Y_\lambda^{m\pm 1}$ are given by Eq.(5.98); no additional normalization integrals need to be evaluated. By repeatedly applying the operator $L_\pm$ one can extend this result to $Y_\lambda^{m\pm 2}$, $Y_\lambda^{m\pm 3}$, etc. They all are already normalized if Y_λ^m is. *No extra work is necessary.*

Secondly, repeated use of the relation (5.97) yields

$$\langle L_\pm^k Y_\lambda^m, L_\pm^k Y_\lambda^m\rangle = [\lambda - (m \pm (k-1))\,(m \pm k)]\cdots[\lambda - m(m\pm 1)]\langle Y_\lambda^m, Y_\lambda^m\rangle$$

This relation implies that for sufficiently large integer k the leading factor in square brackets must vanish. If it did not, the squared norm of $L_\pm^k Y_\lambda^m$ would become negative. To prevent this from happening, λ must have very special values. This is the third property: The only allowed values of λ are necessarily

$$\lambda = \ell(\ell+1) \qquad \ell = 0, 1, 2, \cdots \ .$$

(Note that $\ell = -1, -2, \cdots$ would give nothing new.) Any other value for λ would yield a contradiction, namely a negative norm for some integer k. As a consequence, one has the result that for each allowed eigenvalue there is a sequence of eigenfunctions

$$Y_\ell^m(\theta,\varphi) \qquad m = 0, \pm 1, \pm 2, \cdots \tag{5.99}$$

(*Nota bene:* Note that these eigenfunctions are now labelled by the non-negative integer ℓ instead of the corresponding eigenvalue λ.) Of particular interest are the two eigenfunctions Y_ℓ^ℓ and $Y_\ell^{-\ell}$. The squared norm of $L_+ Y_\ell^\ell$,

$$\|L_+ Y_\ell^\ell\|^2 = [\ell(\ell+1) - \ell(\ell+1)]\|Y_\ell^\ell\|^2$$

is not positive. It vanishes. This implies that

$$Y_\ell^{\ell+1} \propto L_+ Y_\ell^\ell = 0 \ . \tag{5.100}$$

In other words, $Y_\ell^{\ell+1}$ and all subsequent members of the above sequence, Eq.(5.99) vanish, i.e. they do not exist. Similarly one finds that

$$Y_\ell^{-\ell-1} \propto L_- Y_\ell^{-\ell} = 0 \ . \tag{5.101}$$

Thus members of the sequence below $Y_\ell^{-\ell}$ do not exist either. It follows that the sequence of eigenfunctions corresponding to $\ell(\ell+1)$ is finite. The sequence has only $2\ell+1$ members, namely

$$Y_\ell^{-\ell}(\theta,\varphi), Y_\ell^{-\ell+1}(\theta,\varphi), \cdots, Y_\ell^{-1}(\theta,\varphi), Y_\ell^0(\theta,\varphi), Y_\ell^1(\theta,\varphi), \cdots, Y_\ell^{\ell-1}(\theta,\varphi), Y_\ell^\ell(\theta,\varphi)$$

for each integer ℓ. The union of these sequences forms a semi-infinite lattice in the (ℓ, m) as shown in Figure 5.20.

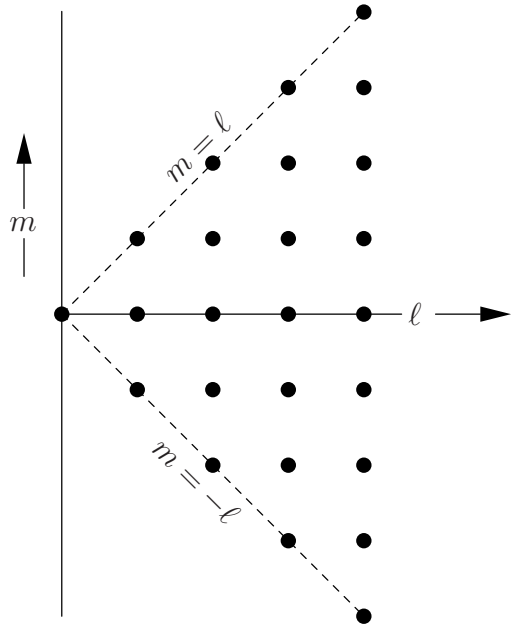


Figure 5.20: Lattice of eigenfunctions (spherical hamonics) labelled by the angular integers ℓ and m. Application of the raising operator L_+ increases m by 1, until one comes to the top of each vertical sequence (fixed ℓ). The lowering operator L_- decreases m by 1, until one reaches the bottom. In between there are exactly $2\ell + 1$ lattice points, which express the $(2\ell + 1)$-fold degeneracy of the eigenvalue $\ell(\ell + 1)$. There do not exist any harmonics above or below the dashed boundaries.

For obvious reasons it is appropriate to refer to this sequence as a *ladder* with $2\ell+1$ elements, and to call Y_ℓ^ℓ the *top*, and $Y_\ell^{-\ell}$ the *bottom* of the ladder. The raising and lowering operators $L_\pm$ are the ladder operators which take us up and down the $(2\ell + 1)$-element ladder. It is easy to determine the elements $Y_\ell^{\pm\ell}$ at the top and the bottom, and to use the ladder operators to generate any element in between.

5. Orthonormality and Completeness

The operators$\{\nabla^2, L_\phi\}$ form a *complete set of commuting operators*. This means that their eigenvalues (ℓ, m) serve as sufficient labels to uniquely identify each of their (common) eigenbasis elements for the vector space of solutions to the Hermholtz equation

$$[\nabla^2 + \ell(\ell + 1)]Y_\ell^m(\theta, \varphi) = 0$$

on the two-sphere. No additional labels are necessary. The fact that these operators are self-adjoint relative to the inner product, Eq.(5.89), implies that these eigenvectors (a.k.a spherical harmonics) are *orthonormal*:

$$\langle Y_\ell^m, Y_{\ell'}^{m'} \rangle = \delta_{\ell\ell'}\delta_{mm'}$$

The semi-infinite set $\{Y_\ell^m(\theta,\varphi) : \ -\ell \le m \le \ell;\ \ell = 0, 1, \cdots\}$ is a basis for the vector space of functions square-integrable on the unit two-sphere. Let $g(\theta,\varphi)$ be any such function. Then

$$\begin{aligned} g(\theta,\varphi) &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi)\langle Y_\ell^m, g\rangle \\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi)\int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\varphi' \overline{Y_\ell^m(\theta',\varphi')} g(\theta',\varphi') \end{aligned}$$

In other words, the spherical harmonics are the basis elements for a generalized double Fourier series representation of the function $g(\theta,\varphi)$. If one leaves this function unspecified, then this *completeness relation* can be restated in the equivalent form

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi)\overline{Y_\ell^m(\theta',\varphi')} = \frac{\delta(\theta-\theta')}{\sin\theta}\delta(\varphi-\varphi')$$

in terms of the Dirac delta functions on the compact domains $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$.

Local Analysis: Calculus

What is the formula for a harmonics $Y_l^m(\theta,\phi)$? An explicit functional form determines the graph, the location of its zeroes, and other aspects of its local behaviour.

1. Spherical Harmonics: Top and Bottom of the Ladder

Each member of the ladder sequence satisfies the differential equation

$$L_\varphi Y_\ell^m \equiv \frac{1}{i}\frac{\partial}{\partial\varphi} Y_\ell^m(\theta,\varphi) = m Y_\ell^m(\theta,\varphi)\ .$$

Consequently, all eigenfunctions have the form

$$Y_\ell^m(\theta,\varphi) = c_{\ell m} P_\ell^m(\theta)\frac{e^{im\varphi}}{\sqrt{2\pi}}\ . \tag{5.102}$$

Here $c_{\ell m}$ is a normalization factor. The two eigenfunctions Y_ℓ^ℓ and $Y_\ell^{-\ell}$ at the top and the bottom of the ladder satisfy Eqs.(5.100) and (5.101) respectively, namely

$$L_\pm Y_\ell^{\pm\ell} \equiv \pm e^{i\phi}\left(\frac{\partial}{\partial\theta} \pm i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) P_\ell^{\pm\ell}(\theta)e^{\pm i\ell\varphi} = 0 \tag{5.103}$$

It is easy to see that their solutions are

$$\begin{aligned} Y_\ell^\ell &= c_\ell \sin^\ell\theta e^{i\ell\varphi} \\ Y_\ell^{-\ell} &= c_\ell \sin^\ell\theta e^{-i\ell\varphi} \ . \end{aligned}$$

The normalization condition

$$\int_0^\pi \int_0^{2\pi} |Y_\ell^{\pm\ell}(\theta,\varphi)|^2 \sin\theta d\theta d\varphi = 1$$

implies that

$$Y_\ell^\ell(\theta,\phi) = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\sin^\ell\theta e^{i\ell\phi}. \tag{5.104}$$

The phase factor $(-1)^\ell$ is not determined by the normalization. Its form is chosen so as to simplify the to-be-derived formula for the Legendre polynomials, Eq.(5.108).

2. Spherical harmonics: Legendre and Associated Legendre polynomials

The functions $Y_\ell^m(\theta,\varphi)$ are obtained by applying the lowering operator L_- to $Y_\ell^\ell(\theta,\varphi)$. A systematic way of doing this is first to apply repeatedly the lowering relation

$$\begin{aligned} &Y_\ell^{m-1}(\theta,\varphi) = \\ &= \frac{1}{\sqrt{\ell(\ell+1)-m(m-1)}}\ L_- Y_\ell^m(\theta,\varphi) \\ &= \frac{1}{\sqrt{\ell^2-m^2+\ell+m}}(-1)e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) Y_\ell^m(\theta,\varphi) \\ &= \frac{-1}{\sqrt{(\ell+m)(\ell-m+1)}}e^{-i\varphi}\left(\frac{\partial}{\partial\theta} + m\frac{\cos\theta}{\sin\theta}\right) Y_\ell^m(\theta,\varphi) \\ &= \frac{1}{\sqrt{(\ell+m)(\ell-m+1)}}\frac{1}{\sin^{m-1}\theta}\frac{\partial}{\partial(\cos\theta)}\ \sin^m\theta e^{-i\varphi}Y_\ell^m \end{aligned} \tag{5.105}$$

to $Y_\ell^\ell(\theta,\varphi)$ until one obtains the azimuthally invariant harmonic $Y_\ell^0(\theta,\varphi) = Y_\ell^0(\theta)$. Then continue applying this lowering relation, or alternatively the raising relation

$$\begin{aligned} Y_\ell^m(\theta,\varphi) &= \\ &= \frac{1}{\sqrt{\ell(\ell+1)-m(m-1)}}\, L_+ Y_\ell^{m-1}(\theta,\varphi) \\ &= \frac{-1}{\sqrt{(\ell+m)(\ell-m+1)}} \sin^m\theta \frac{\partial}{\partial(\cos\theta)}\, \frac{1}{\sin^{m-1}\theta} e^{i\varphi} Y_\ell^{m-1} \end{aligned} \tag{5.106}$$

until one obtains the desired harmionic $Y_\ell^m(\theta,\varphi)$. The execution of this two-step algorithm reads as follows:
Step 1: Letting $m = |m|$, apply Eq.(5.105) m times and obtain

$$\begin{aligned} Y_\ell^0(\theta,\varphi) &= \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \underbrace{L_- L_- \cdots L_-}_{m \text{ times}} Y_\ell^m(\theta,\varphi) \\ &= \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \frac{\partial^m}{\partial(\cos\theta)^m}\, \sin^m\theta e^{-im\varphi} Y_\ell^m(\theta,\varphi), \end{aligned}$$

which, because of Eq.(5.102), is independent of φ. Now let $m = \ell$, use Eq.(5.104), and obtain

$$\begin{aligned} Y_\ell^0(\theta,\varphi) &= \frac{(-1)^\ell}{2^\ell \ell!} \sqrt{\frac{(2\ell+1)}{4\pi}} \frac{\partial^\ell}{\partial(\cos\theta)^\ell} \sin^{2\ell}\theta \\ &\equiv \sqrt{\frac{(2\ell+1)}{4\pi}} P_\ell(\cos\theta). \end{aligned} \tag{5.107}$$

The polynomials in the variable $x = \cos\theta$

$$P_\ell(x) \equiv \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell \tag{5.108}$$

are called the *Legendre polynomials*. They have the property that at the North pole they have the common value unity, while at the South pole their value is $+1$ whenever $P_\ell(x)$ is an even polynomial and -1 whenever it is odd:

$$P_\ell(x = \pm 1) = (\pm 1)^\ell.$$

Step 2: To obtain the harmonics having positive azimuthal integer m, apply the raising operator L_+ m times to Y_ℓ^0. With the help of Eq.(5.106) one obtains (for $m = |m|$)

$$\begin{aligned} Y_\ell^m(\theta,\varphi) &= \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\underbrace{L_+L_+\cdots L_+}_{m \text{ times}}Y_\ell^0(\theta,\varphi) \\ &= \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}(-1)^m\sin^m\theta\frac{\partial^m}{\partial(\cos\theta)^m}\ e^{im\varphi}Y_\ell^0(\theta,\varphi) \\ &= \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\frac{(-1)^{\ell+m}}{2^\ell\ell!}\sqrt{\frac{2\ell+1}{4\pi}}\sin^m\theta\frac{\partial^{\ell+m}}{\partial(\cos\theta)^{\ell+m}}\sin^{2\ell}\theta e^{im\varphi} \\ &= \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\sqrt{\frac{2\ell+1}{4\pi}}\ P_\ell^m(\cos\theta)\ e^{im\varphi} \end{aligned} \tag{5.109}$$

The polynomials in the variable $x = \cos\theta$

$$P_\ell^m(x) \equiv \frac{(-1)^m}{2^\ell\ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \tag{5.110}$$

are called the *associated Legendre polynomials*. Inserting Eq.(5.109) into Eq.(5.87), one finds that they satisfy the differential equation

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P_\ell^m(\cos\theta) = 0\ .$$

Also note that $P_\ell^{-m}(\cos\theta)$ satisfies the same differential equation. In other words, P_ℓ^{-m} and P_ℓ^m must be proportional to each other. (Why?) Indeed,

$$P_\ell^{-m}(\cos\theta) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(\cos\theta)\ . \tag{5.111}$$

This one sees by comparing the right hand side of Eq.(5.109) with the right

hand side of

$$\begin{aligned} Y_\ell^{|m|}(\theta,\varphi) &= \sqrt{\frac{(\ell+|m|)!}{(\ell-|m|)!}}\underbrace{L_-L_-\cdots L_-}_{\ell-|m|\ \text{times}}Y_\ell^\ell(\theta,\varphi) \\ &= \sqrt{\frac{(\ell+|m|)!}{(\ell-|m|)!}}\frac{1}{\sin^{|m|}\theta}\frac{\partial^{\ell-|m|}}{\partial(\cos\theta)^{\ell-|m|}}\sin^\ell\theta\ e^{-i(\ell-|m|)\varphi}\ Y_\ell^\ell(\theta,\varphi) \\ &= \sqrt{\frac{(\ell+|m|)!}{(\ell-|m|)!}}\frac{1}{\sin^{|m|}\theta}\frac{\partial^{\ell-|m|}}{\partial(\cos\theta)^{\ell-|m|}}\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{2\ell+1}{4\pi}}\sin^{2\ell}\theta e^{i|m|\varphi} \\ &= \sqrt{\frac{(\ell+|m|)!}{(\ell-|m|)!}}\sqrt{\frac{2\ell+1}{4\pi}}(-1)^m P_\ell^{-|m|}(\cos\theta)\ e^{i|m|\varphi} \quad . \end{aligned} \tag{5.112}$$

This validates Eq.(5.111), but only for $m = |m|$. One sees, however, that this formula is also true for $m = -|m|$. Could it be that formulas Eqs.(5.109) and (5.110) are also true whenever $m = -|m|$? The answer is 'yes'. This follows from considering the $m = -|m|$ harmonics. They are obtained by using Eq.(5.105) $|m|$ times starting with Y_ℓ^0:

$$\begin{aligned} Y_\ell^{-|m|}(\theta,\varphi) &= \sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}\underbrace{L_-L_-\cdots L_-}_{|m|\ \text{times}}Y_\ell^0(\theta,\varphi) \\ &= \sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}\sin^{|m|}\theta\frac{\partial^{|m|}}{\partial(\cos\theta)^{|m|}}\ e^{-i|m|\varphi}Y_\ell^0(\theta,\varphi) \\ &= \sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{2\ell+1}{4\pi}}\sin^{|m|}\frac{\partial^{\ell+|m|}}{\partial(\cos\theta)^{\ell+|m|}}\sin^{2\ell}\theta e^{-i|m|\varphi} \\ &= \sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}\sqrt{\frac{2\ell+1}{4\pi}}\ (-1)^m P_\ell^{|m|}(\cos\theta)\ e^{-i|m|\varphi} \qquad (5.113) \\ &= \sqrt{\frac{(\ell+|m|)!}{(\ell-|m|)!}}\sqrt{\frac{2\ell+1}{4\pi}}\ P_\ell^{-|m|}(\cos\theta)\ e^{-i|m|\varphi} \qquad (5.114) \end{aligned}$$

(*Nota bene:* The first line was obtained by using Eq.(5.105) and letting $m = -1, -2, \cdots, -|m|$, the third, fourth, and fifth line used Eqs.(5.107), (5.110), and (5.111), respectively.) Comparison with Eq.(5.109) verifies

that the spherical harmonic

$$\boxed{Y_\ell^m(\theta,\varphi) = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\sqrt{\frac{2\ell+1}{4\pi}}\; P_\ell^m(\cos\theta)\; e^{im\varphi}} \tag{5.115}$$

is indeed correct for all positive and negative integers m that satisfy $-\ell < m < \ell$.

A second result is obtained by comparing Eq.(5.113) with Eq.(5.109). This comparison yields the complex conjugation formula

$$\overline{Y_\ell^m(\theta,\varphi)} = (-1)^m Y_\ell^{-m}(\theta,\varphi),$$

which holds for both for positive and negative azimuthal integers m.

5.8 Static Solutions

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Solutions to the wave equation which are static, are characterized by $\frac{\partial^2\psi}{\partial t^2} = 0$, and hence by $k^2 = 0$. The governing equation becomes

$$\nabla^2\psi = 0\,.$$

Relative to spherical coordinates this equation reads

$$\nabla^2\psi = \left\{\frac{1}{r}\,\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2}\right)\right\}\psi = 0\,.$$

Its solution is a superposition of solutions having the simple product form,

$$\psi_{\ell m}(t,\theta,\varphi) = R_\ell(r)Y_\ell^m(\theta,\varphi)\,.$$

They give rise to the two ordinary differential equations

$$\left\{\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2} + \lambda\right\}Y_\ell^m(\theta,\varphi)\,,$$

where

$$\lambda = \ell(\ell+1)\,, \qquad\qquad \ell = 0,1,2,\ldots$$

and hence

$$\frac{1}{r}\left\{\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right\} rR_\ell(r) = 0\,.$$

This equation is Euler's differential equation whose solutions are r^ℓ and $r^{-(\ell+1)}$. Thus, the static solution is a superposition

$$\psi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_{\ell m} r^\ell + B_{\ell m} r^{-(\ell+1)}) Y_\ell^m(\theta, \varphi)$$

whose coefficients, the A's and the B's, are determined by the given boundary conditions. These coefficients are called the *multipole moments* of the source of the field.

5.8.1 Static Multipole Field

The manner in which static multipole moments of a source give rise to a multipole field is illustrated by the following

Problem (Multipole field of an asymmetric static source).

Given:

1. The potential inside and outside a sphere of radius $r = r_0$ satisfies the Laplace equation
$$\nabla^2 \psi = 0\,.$$
2. The value of the potential on the sphere is
$$\psi(r_0, \theta, \varphi) = \frac{g(\theta, \varphi)}{r_0}\,.$$

Find the potential $\psi(r, \theta, \varphi)$ inside $(r < r_0)$ and outside $(r_0 < r)$ the sphere.

The potential may be an electrical potential, in which case its value on the sphere is determined by the charge distribution on sphere. By contrast, if the potential is a gravitational potential, its value on the sphere is determined by the mass distribution.

In either case, the governing equation would be

$$\nabla^2 \psi = -\frac{\delta(r - r_0)}{r^2} g(\theta, \varphi)\,,$$

as one can verify after we have found the solution to the given problem.

There are two boundary conditions implicit in the given problem, namely

$$\psi(r=0,\theta,\varphi) = \text{finite}$$

and

$$\psi(r=\infty,\theta,\varphi) = 0\,.$$

The second boundary condition expresses the fact that there are no masses (or charges) distributed at very large r. The two boundary conditions demand that the radial part of the potential be

$$R_\ell(r) \propto \begin{cases} \left(\frac{r}{r_0}\right)^\ell & r < r_0 \\ \left(\frac{r_0}{r}\right)^{\ell+1} & r_0 < r \end{cases}$$

The total solution is, therefore,

$$\psi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \left\{ \begin{array}{l} \frac{1}{r_0}\left(\frac{r}{r_0}\right)^\ell \\ \frac{1}{r_0}\left(\frac{r_0}{r}\right)^{\ell+1} \end{array} \right\} \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi)\langle Y_\ell^m, g\rangle \quad \begin{array}{l} \text{INSIDE} \\ \\ \text{OUTSIDE}\,, \end{array}$$

where

$$\langle Y_\ell^m, g\rangle = \int_0^\pi \int_0^{2\pi} \overline{Y}_\ell^m(\theta,\varphi) g(\theta,\varphi) \sin\theta d\theta d\varphi\,.$$

Let us exhibit explicitly the *exterior* potential. It is a superposition of various "multipole" potential fields,

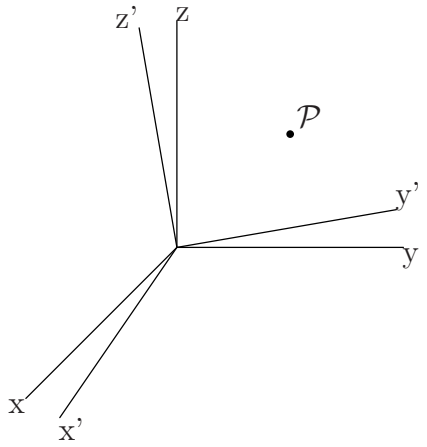
$$\begin{aligned} \psi \;\; = \;\; & Y_0^0(\theta,\varphi)\frac{\langle Y_0^0, g\rangle}{r} + \sum_{m=-1}^{1} Y_1^m(\theta,\varphi)\frac{\langle Y_1^m, g\rangle r_0}{r^2} \\ & + \sum_{m=-2}^{2} Y_2^m(\theta,\varphi)\frac{\langle Y_2^m, g\rangle r_0^2}{r^3} + \cdots \end{aligned}$$

They are called the *monopole, dipole, quadrupole, ... and* 2^ℓ*-pole fields* respectively. The constant numerators express the source strengths of the fields. These numerators are called the monopole moment, dipole moment (which has three components), quadrupole moment (which has five components), ... and 2^ℓ-pole moment (which has $2\ell+1$ components). Each one of them is an example of a *multipole moment.*

Analogous descriptive names hold for the interior field.

5.8.2 Addition Theorem for Spherical Harmonics

One can use the static multipole solution in order to infer the behavior of the spherical harmonics under rotation. We consider an arbitrary orthogonal coordinate rotation



For any fixed point $\mathcal{P}$ the effect of this change is given

$$\begin{aligned}(x(\mathcal{P}), y(\mathcal{P}), z(\mathcal{P})) &\rightsquigarrow (x'(\mathcal{P}), y'(\mathcal{P}), z'(\mathcal{P}))\\ (\theta, \varphi) &\rightsquigarrow (\theta', \varphi')\end{aligned}$$

but the radius and the Laplacian remain fixed;

$$\begin{aligned}\sqrt{x^2+y^2+z^2} &= r = \sqrt{x'^2+y'^2+z'^2}\\ \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\,.\end{aligned}$$

Thus for any fixed point $\mathcal{P}$ the potential ψ determined relative to the rotated coordinate system is the same as that relative to the unrotated one,

$$\psi(r(\mathcal{P}), \theta(\mathcal{P}), \varphi(\mathcal{P})) = \psi(r'(\mathcal{P}), \theta'(\mathcal{P}), \varphi'(\mathcal{P}))\,.$$

In other words, relabelling the coordinates of a fixed point has no effect on the (measured) potential at this point. This equality holds for *all* radii r.

It follows that the coresponding 2^ℓ-pole fields are equal for each integral. Thus

$$\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi)\langle Y_\ell^m, g\rangle = \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta',\varphi')\langle Y_\ell^m, g\rangle\,.$$

This equality also holds for *all* boundary value functions g. Consequently,

$$\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi)\overline{Y}_\ell^m(\theta_0,\varphi_0) = \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta',\varphi')Y_\ell^m(\theta'_0,\varphi'_0)\,,$$

$\ell = 0, 1, 2, \ldots$. This is a remarkable statement: it says that this particular sum of products is unchanged under a rotation. It is a *scalar*, even though each individual factor does get altered.

Question: What is this rotationally invariant sum equal to?

To find out, orient the z'-axis so that it passes through the source point $\mathcal{P}_0$, which now becomes the new North pole. Relative to the new (x', y', z') frame the spherical coordinates of the source point $\mathcal{P}_0$ and the observation point $\mathcal{P}$ are given by

$$\begin{aligned} \text{Source point } \mathcal{P}_0 \text{ (“new North Pole”)} \quad &: \quad \begin{cases} \theta'_0 = 0 \\ \varphi'_0 = \text{indeterminate} \end{cases} \\ \text{Observation point } \mathcal{P} \quad &: \quad \begin{cases} \theta' = \Theta \\ \varphi' = \Phi \end{cases} \end{aligned}$$

Suppose we reexpress the spherical harmonics in terms of the associated Legendre polynomials $P_\ell^m(\cos\theta)$. Then

$$Y_\ell^m(\theta,\varphi) = \sqrt{\frac{2\ell+1}{2}}\,\sqrt{\frac{\ell-m!}{\ell+m!}}(-1)^m P_\ell^m(\cos\theta)\frac{e^{im\varphi}}{\sqrt{2\pi}}\,.$$

These polynomials satisfy

$$P_\ell^m(\cos 0) = 0 \qquad\qquad m \neq 0$$

and

$$P_\ell^0(\cos 0) = 1\,.$$

Consequently,

$$\begin{aligned} \overline{Y}_\ell^m(0,\varphi'_0) &= \delta_{m0}\sqrt{\frac{2\ell+1}{4\pi}} \\ Y_\ell^0(\Theta,\Phi) &= \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\Theta)\,. \end{aligned}$$

The rotationally invariant sum simplifies into

$$P_\ell(\cos\Theta) = \sum_{m=-\ell}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta) P_\ell^m(\cos\theta_0) e^{im(\varphi-\varphi_0)}\,.$$

We conclude that $P_\ell(\cos\Theta)$, which is an amplitude pattern azimuthally symmetric around the new North pole (θ_0, φ_0), is a finite linear combination of amplitude patterns centered around the old North pole.

5.8.3 The Bessel-Legendre Connection

The spherical harmonics of this section constitute the main elements arising from the application of spherical symmetry to the Helmholtz equation. We have already learned in the previous section that the same is true about the cylinder harmonics arising from the application of rotational and translational symmetry applied to that same equation

Recall from page 390 that, upon letting $\theta \to 0$ and $\ell \to \infty$ in such a way that $\theta\ell$ remains finite, the associated Legendre equation

$$\left[\frac{1}{\sin\theta}\,\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + \ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] P_\ell^m = 0$$

becomes

$$\left[\frac{1}{\theta}\,\frac{d}{d\theta}\theta\frac{d}{d\theta} + \ell^2 - \frac{m^2}{\theta^2}\right] P_\ell^m = 0\,.$$

Consequently,

$$P_\ell^m \quad\to\quad J_m(\ell\theta) \equiv J_m(kr)\,. \tag{5.116}$$

Furthermore, recall the expression for the translated cylinder wave, Eq.(5.47) on page 363,

$$H_\nu(k|\vec{x}-\vec{x}_0|)e^{i\nu(\phi-\varphi_0)} = \sum_{m=-\infty}^{\infty} H_{\nu+m}(kr)J_m(kr_0)e^{i(\nu+m)(\varphi-\varphi_0)}\,.$$

Specialize to the case where the wave is rotationally symmetric around the point $\vec{x}_0 \colon (r_0, \varphi_0)$. Consequently, $\nu = 0$. This, together with the requirement that the wave amplitudes be finite everywhere leads to

$$J_0(k|\vec{x}-\vec{x}_0|) = \sum_{m=-\infty}^{\infty} J_m(kr)J_m(kr_0)e^{im(\varphi-\varphi_0)}\,.$$

It is evident that, with the help of Equation 5.116, this is the "small θ, large ℓ" asymptotic limit of the spherical addition theorem

$$P_\ell(\cos\Theta) = \sum_{m=-\ell}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta) P_\ell^m(\cos\theta_0) e^{im(\varphi-\varphi_0)} \,.$$

These equations illustrate the relational nature of our knowledge: In the limit of large ℓ the spherical harmonics becomes indistinguishable from the Bessel hamonics. Learning about one allows us more readily to grasp the other.

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Chapter 6

Partial Differential Equations

Linear algebra is the mathematical guide of choice for implementing the principle of unit-economy[1] applied to partial differential equations. The present chapter considers two kinds:

1. Single linear partial differential equations corresponding to the linear system
$$A\vec{u} = \vec{0}\,.$$
However, instead of merely exhibiting general solutions to such a system, we shall take seriously the dictum which says that *a differential equation is never solved until "boundary" conditions have been imposed on its solution.* As identified in the ensuing section, these conditions are not arbitrary. Instead, they fall into three archetypical classes determined by the nature of the physical system which the partial differential equation conceptualizes.

2. Systems of pde's corresponding to an over-determined system
$$A\vec{u} = \vec{b}\,. \tag{6.1}$$
The idea for solving it takes advantage of the *fundamental subspaces*[2]of A [5]. Let A be a 4×4 matrix having rank 3. Such a matrix, we recall,

[1]As identified in the footnote on Page 192.

[2]Besides the domain and the target space, there are four of them:

(i) the *column space* of A, denoted by $\mathcal{R}(A)$,

(ii) the *nullspace* (or *kernel*) of A, denoted by $\mathcal{N}(A)$, or also by $Ker(A)$,

(iii) the *row space* of A, denoted by $\mathcal{R}(A^T)$, and

has a vector $\vec{u}_r$ which satisfies $A\vec{u}_r = \vec{0}$, or, to be more precise

$$A\vec{u}_r c_0 = \vec{0}\,, \tag{6.2}$$

where c_0 is any non-zero scalar. Thus $\vec{u}_r$ spans A's one-dimensional nullspace

$$\mathcal{N}(A) = span\{\vec{u}_r\}\ .$$

This expresses the fact that the columnes of A are linearly dependent.

In addition we recall that the four rows of A are linearly dependent also, a fact which is expressed by the existence of a vector $\vec{u}_\ell$ which satisfies

$$\vec{u}_\ell^{\,T} A = \vec{0}\,, \tag{6.3}$$

and which therefore spans A's one-dimensional left nullspace

$$\mathcal{N}(A^T) = span\{\vec{u}_\ell^{\,T}\}\ .$$

In general there does not exist a solution to the over-determined system Eq.(6.1). However, a solution obviously does exist if and only if $\vec{b}$ satisfies

$$\vec{u}_\ell^{\,T}\vec{b} = 0\ .$$

Under such a circumstance there are infinitely many solutions, each one differing from any other merely by a multiple of the null vector $\vec{u}_r$. The most direct path towards these solutions is via eigenvectors.

One of them is, of course, the vector $\vec{u}_r$ in Eq.(6.2). The other three, which (for the A under consideration) are linearly independent, satisfy $A\vec{v}_i = \lambda_i\vec{v}_i$ with $\lambda_i \neq 0$, or, in the interest of greater precision (which is needed in Section 6.2.3),

$$A\vec{v}_1 c_1 = \lambda_1\vec{v}_1 c_1 \tag{6.4}$$
$$A\vec{v}_2 c_2 = \lambda_2\vec{v}_2 c_2 \tag{6.5}$$
$$A\vec{v}_3 c_3 = \lambda_3\vec{v}_3 c_3 \tag{6.6}$$

where, like c_0, the c_i's are any non-zero scalars. Because of the simplicity of $\vec{u}_\ell^{\,T}$ for the A under consideration one can find the eigenvectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$,

(iv) the *left nullspace* of A, denoted by $\mathcal{N}(A^T)$.

and hence their eigenvalues, by a process of inspection. These vectors span the range of A,

$$\mathcal{R}(A) = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \ ,$$

and therefore determine those vectors $\vec{b}$ for which there exists a solution to Eq.(6.1). Such vectors belong to $\mathcal{R}$ and thus have the form

$$\vec{b} = \vec{v}_1 b_1 + \vec{v}_2 b_2 + \vec{v}_3 b_3 \ .$$

These eigenvectors also serve to represent the corresponding solution,

$$\vec{u} = \vec{u}_r c_0 + \vec{v}_1 c_1 + \vec{v}_2 c_2 + \vec{v}_3 c_3 \ .$$

This, the fact that $A\vec{u} = \vec{b}$, and the linear independence of the $\vec{v}_i's$ imply that the scalars c_i satisfy the three equations

$$\lambda_1 c_1 = b_1 \tag{6.7}$$

$$\lambda_2 c_2 = b_2 \tag{6.8}$$

$$\lambda_3 c_3 = b_3 \ . \tag{6.9}$$

As expected, the contribution c_4 along the direction of the nullspace element is left indeterminate. These ideas from linear algebra and their application to solving a system, such as Eq.(6.1), can be extended to corresponding systems of partial differential equations. The Maxwell field equations, which we shall analalyze using linear algebra, is a premier example. In this extension the scalar entries of A and $\vec{u}_\ell^T$ get replaced by differential operators, the vectors $\vec{u}$ and $\vec{b}$ by vector fields, the scalars b_i and c_i by scalar fields, the eigenvalues λ_i by a second order wave operator, and the three Eqs.(6.7)-(6.9) by three inhomogeneous scalar wave equations corresponding to what in physics and engineering are called

- *transverse electric* (TE),
- *transverse magnetic* (TM), and
- *transverse electric magnetic* (TEM),

modes respectively.

Lecture 50

6.1 Single Partial Differential Equations: Their Origin

There are many phenomena in nature, which, even though occuring over finite regions of space and time, can be described in terms of properties that prevail at each point of space and time separately. This description originated with Newton, who with the aid of his differential calculus showed us how to grasp a global phenomenon, for example, the elliptic orbit of a planet, by means of a locally applied law, for example $F = ma$.

This manner of making nature comprehensible has been extended from the motion of single point particles to the behavior of other forms of matter and energy, be it in the form of gasses, fluids, light, heat, electricity, signals traveling along optical fibers and neurons, or even gravitation.

This extension consists of formulating or stating a partial differential equation governing the phenomenon, and then solving that differential equation for the purpose of predicting measurable properties of the phenomenon.

There exist many partial differential equations, but from the view point of mathematics, there are basically only *three types* of partial differential equations.

They are exemplified by

1. Laplaces equation
$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0\,,$$
which governs electrostatic and magnetic fields as well as the velocity potential of an incompressible fluid, by

2. the wave equation
$$\nabla^2\psi - \frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2} = 0$$
for electromagnetic or sound vibrations, and by
$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2} = 0$$
for the vibrations of a simple string, and by

3. the diffusion equation
$$\nabla^2\psi - \frac{1}{k}\,\frac{\partial\psi}{\partial t} = 0$$

for the temperature in three dimensional space and in time, or by

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{k}\,\frac{\partial\psi}{\partial t} = 0$$

for the temperature along a uniform rod.

6.1.1 Boundary Conditions of a Typical Partial Differential Equation in Two Dimensions

For the purpose of simplicity, we shall start our consideration with partial differential equations in only two variables and linear in the second derivatives. Such equations have the general form

$$A(x,y)\frac{\partial^2\psi}{\partial x^2} + 2B(x,y)\frac{\partial^2\psi}{\partial x\partial y} + C(x,y)\frac{\partial^2\psi}{\partial y^2} = \Phi\left(x, y, \psi, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right).$$

Such an equation is called a quasilinear second order partial differential equation. If the expression Φ where linear in ψ, i.e., if

$$\Phi = D(x,y)\frac{\partial\psi}{\partial x} + E(x,y)\frac{\partial\psi}{\partial y} + F(x,y)\psi + G(x,y)\,,$$

then the equation would be a linear p.d.e., but this need not be the case.

The equation has a nondenumerable infinity of solution. In order to single out a unique solution, the to-be-found function $\psi(x,y)$ must satisfy additional conditions. They are usually specified at the boundary of the domain of the p.d.e.

In three dimensional space, this boundary is a *surface*, but in our two dimensional case, we have a *boundary line* which can be specified by the parametrized curve

$$\begin{aligned} x &= \xi(s) \\ y &= \eta(s)\,, \end{aligned}$$

where s is the arclength parameter

$$s = \int ds = \int \sqrt{dx^2 + dy^2}\,.$$

The tangent to this curve has components

$$\left(\frac{d\xi}{ds}\,,\ \frac{d\eta}{ds}\right)\,.$$

They satisfy

$$\left(\frac{d\xi}{ds}\right)^2+\left(\frac{d\eta}{ds}\right)^2=1\ .$$

The normal to this boundary curve has components

$$\left(\frac{d\eta}{ds}\,,-\frac{d\xi}{ds}\right)=\vec{n}\,.$$

We assume that $\vec{n}$ points towards the *interior* of the domain where the solution is to be found. If this is not the case, we reverse the signs of the components of it.

The additional conditions which the to-be-found solution ψ is to satisfy are imposed at this boundary curve, and they are conditions on the partial derivatives and the value of the function ψ evaluated at the curve.

The boundary curve accomodates three important types of boundary conditions.

1. *Dirichlet conditions:* $\psi(s)$ is specified at each point of the boundary.

2. *Neumann conditions:* $\frac{d\psi}{dn}(s)=\vec{n}\cdot\nabla\psi$, the normal componet of the graident of ψ is specified at each point of the boundary.

3. *Cauchy conditions:* $\psi(s)$ and $\frac{d\psi}{dn}(s)$ are specified at each point of the boundary. The parameter s is usually a time parameter. Consequently, Cauchy conditions are also called *intial value conditions* or *initial value data* or simply *Cauchy data*.

There exists also the *mixed Dirichlet-Neumann* conditions. They are intermediate between the Dirichlet and the Neumann boundary conditions, and they are given by

$$\alpha(s)\psi(s)+\beta(s)\frac{d\psi}{dn}(s)=f(s)\,.$$

Here $\alpha(s)$, $\beta(s)$, and $f(s)$ are understood to be given on the boundary.

We recall that in the theory of ordinary second order differential equations, a unique solution was obtained once the solution and its derivative were specified at a point. The generalization of this condition to partial differential equations consists of the Cauchy boundary conditions.

Consequently, we now inquire whether the solution of the partial differential equation is uniquely determined by specifying Cauchy boundary conditions on the boundary $(\xi(s), \eta(s))$.

6.1.2 Cauchy Problem and Characteristics

In order to compute the function $\psi(x, y)$ at points off the boundary curve, we resort to the Taylor series on two dimensions;

$$\begin{aligned}\psi(x,y) &= \psi(\xi,\eta) + (x-\xi)\frac{\partial\psi}{\partial x} + (y-\eta)\frac{\partial\psi}{\partial y} \\ &\quad + \frac{1}{2!}\left[(x-\xi)^2\frac{\partial^2\psi}{\partial x^2} + 2(x-\xi)(y-\eta)\,\frac{\partial^2\psi}{\partial x\partial y} + (y-\xi)^2\frac{\partial^2\psi}{\partial y^2}\right] + \cdots .\end{aligned}$$

Here the derivatives are to be evaluated on the boundary.

The problem we are confronted with is this:

Determine all partial derivatives, starting with the first partials on up from the given Cauchy boundary conditions, the given boundary, and the given partial differential equation!

We shall do this first for the first derivatives.

From the Cauchy data we obtain two equations

$$\left.\begin{aligned}\frac{d\psi(s)}{dn} &= \frac{d\eta}{ds}\,\frac{\partial\psi}{\partial x} - \frac{d\xi}{ds}\,\frac{d\psi}{dy} \\ \frac{d\psi(s)}{ds} &= \frac{d\xi}{ds}\,\frac{\partial\psi}{\partial x} + \frac{d\eta}{ds}\,\frac{d\psi}{dy}\end{aligned}\right\} \quad \text{at} \quad (x,y) = (\xi(s),\eta(s))\,. \tag{6.10}$$

From these we obtain the first partial derivatives of ψ evaluates on the boundary

$$\begin{aligned}\left(\frac{\partial\psi}{\partial x}\right)_{(\xi,\eta)} &= \frac{d\psi(s)}{dn}\,\frac{d\eta}{ds} + \frac{d\psi}{ds}\,\frac{d\xi}{ds} \\ \left(\frac{\partial\psi}{\partial y}\right)_{(\xi,\eta)} &= -\frac{d\psi(s)}{dn}\,\frac{d\eta}{ds} + \frac{d\psi}{ds}\,\frac{d\xi}{ds}\,.\end{aligned} \tag{6.11}$$

The procurement of the second derivatives is more interesting. We differentiate the (known) first derivatives along the boundary. Together with the

given p.d.e. we have

$$\begin{aligned}
\frac{d}{ds}\left(\frac{\partial\psi}{\partial x}\right) &= \frac{d\xi}{ds}\,\frac{\partial^2\psi}{\partial x^2} + \frac{d\eta}{ds}\,\frac{\partial^2\psi}{\partial y\partial x} \\
\frac{d}{ds}\left(\frac{\partial\psi}{\partial y}\right) &= \frac{d\xi}{ds}\,\frac{\partial^2\psi}{\partial x\partial y} + \frac{d\eta}{ds}\,\frac{\partial^2\psi}{\partial y^2} \\
\Phi &= A\frac{\partial^2\psi}{\partial x^2} + 2B\frac{\partial^2\psi}{\partial x\partial y} + C\frac{\partial^2\psi}{\partial y^2}\,.
\end{aligned}$$

The left hand sides of these three equations are known along the whole boundary. So are the coefficients of the three unknown partial derivatives on the right hand side. One can solve for these partial derivatives unless

$$\begin{vmatrix} \dfrac{d\xi}{ds} & \dfrac{d\eta}{ds} & 0 \\ 0 & \dfrac{d\xi}{ds} & \dfrac{d\eta}{ds} \\ A & 2B & C \end{vmatrix} = 0$$

or

$$A\left(\frac{d\eta}{ds}\right)^2 - 2B\frac{d\eta}{ds}\,\frac{d\xi}{ds} + C\left(\frac{d\xi}{ds}\right)^2 = 0\,.$$

If this determinant does not vanish, one can solve for the second derivatives evaluated on the boundary. Differentiating along the boundary yields

$$\begin{aligned}
\frac{d}{ds}\psi_{xx} &= \frac{d\xi}{ds}\psi_{xxx} + \frac{d\eta}{ds}\psi_{yxx} \\
\frac{d}{ds}\psi_{xy} &= \frac{d\xi}{ds}\psi_{xxy} + \frac{d\eta}{ds}\psi_{yxy} \\
\Phi_x + \cdots &= A\psi_{xxx} + 2B\psi_{xyx} + C\psi_{xyy}\,.
\end{aligned}$$

Subscripts refer to partial derivatives. The last equation was obtained differentiating the given p.d.e. with respect to x. The left hand side contains only lower order derivatives, which are known on the boundary.

We see that one can solve for

$$\psi_{xxx}\,,\ \psi_{yxx}\,,\ \psi_{xyy}$$

on the boundary unless the determinant, the same one as before, vanishes. It is evident that one can continue the process of solving for the other higher

order derivatives, provided the determinant of the system does not vanish. We are led to the conclusion that one can expand $\psi(x, y)$ in a Taylor series at every point of the boundary and that the coefficients of the series are uniquely determined by the Cauchy boundary conditions on the given boundary.

We must now examine the vanishing of the system determinant

$$A(x,y)\left(\frac{dy}{ds}\right)^2 - 2B(x,y)\frac{dy}{ds}\,\frac{dx}{ds} + C(x,y)\left(\frac{dx}{ds}\right)^2 = 0 \tag{6.12}$$

at every point of the domain of the partial differential equation.

Depending on the coefficients A, B, and C, this quadratic form determines two characteristic curves, $\lambda(x, y) = const.$ and $\mu(x, y) = const.$, through each point (x, y). We distinguish between three cases:

1. $AC - B^2 > 0$: *elliptic type* in which the two characteristics λ and μ are complex conjugates of each other.

2. $AC - B^2 < 0$: *hyperbolic type* in which case for each (x, y) the characteristics λ and μ are real. They sre two curves intersecting at (x, y). As one varies (x, y) one obtains two distinct families.

3. $AC - B^2 = 0$: *parabolic type* in which there is only one family of characteristics.

These three cases imply three different types of differential equations. By utilizing the characteristic, one can introduce new coordinates relative to which a differential equation of each type assumes a standard normal form. Let the new coordinate surfaces be

$$\lambda(x, y) = \text{const} \qquad \mu(x, y) = \text{const}\,.$$

Then the coordinate transformation

$$u + iv = \lambda \qquad \text{and} \qquad u - iv = \mu$$

yields a normal form of the *elliptic type*,

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} = \Phi\left(u, v, \psi, \frac{\partial\psi}{\partial u}\,,\ \frac{\partial\psi}{\partial v}\right)\,.$$

By contrast the coordinate transformation

$$\lambda = \lambda(x, y) \qquad \text{and} \qquad \mu = \mu(x, y)$$

yields a normal form of the *hyperbolic type*,

$$\frac{\partial^2\psi}{\partial\lambda\partial\mu} = \Phi\left(\lambda, \mu, \psi, \frac{\partial\psi}{\partial\lambda}, \frac{\partial\psi}{\partial\mu}\right). \tag{6.13}$$

Finally, the coordinate transformation

$$\lambda = \lambda(x, y) = \mu(x, y), \quad x = x$$

yields a normal form of the *parabolic type*,

$$\frac{\partial^2\psi}{\partial\lambda^2} = \Phi\left(x, \lambda, \psi, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial\lambda}\right).$$

We recognize that *elliptic* partial differential equations express an *equilibrium* or a static potential phenomenon.

By introducing the standard coordinates

$$t = \lambda + \mu \qquad \text{and} \qquad z = \lambda - \mu$$

in terms of which

$$\lambda = \frac{1}{2}(t + z) \qquad \text{and} \qquad \mu = \frac{1}{2}(t - z),$$

one finds that

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial z^2} = \Phi'\left(t, z, \psi, \frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial z}\right),$$

the wave equation of a general vibrating string. We, therefore, recognize that a *hyperbolic* p.d. equation expresses the phenomenon of a *propagating wave* or *disturbance*.

Finally, a parabolic p.d. equation expresses a diffusion process. In fact, the two dimensional *Laplace equation*, the *equation for a vibrating sting*, and the *heat conduction equation* are the simplest possible examples of *elliptic, hyperoblic*, and *parabolic equations*.

6.1.3 Hyperbolic Equations

The quadratic form, Eq.(6.12), determined by the coefficients A, B, and C of the given p.d.e. can be factored into two ordinary differential equation

$$A\,dy = (B + \sqrt{B^2 - AC})\,dx \qquad \text{and} \qquad A\,dy = (B - \sqrt{B^2 - AC})\,dx.$$

These are the equations for the *two families of characteristic curves* of the given p.d.e.

Their significance, we recall, is this: if the boundary line coincides with one of them, then specifying Cauchy data on it *will not* yield a unique solution. If, however, the boundary line intersects each family only once, then the Cauchy data *will* yields a unique solution.

This point becomes particularly transparent if one introduces the curvilinear coordinates λ and μ relative to which the given p.d.e. assumes its standard form, Eq.(6.13). We shall consider the hyperbolic case by assuming that

$$B^2(x,y) - A(x,y)C(x,y) > 0$$

throughout the (x,y) domain.

We shall demand the new coordinates λ and μ – the *characteristic coordinates* – have the property that their isograms ("loci of points of constant values") contain the characteristic lines $(x(s), y(s))$, i.e.,

$$\lambda(x(s), y(s)) = \text{const} \qquad \text{and} \qquad \mu(x(s), y(s)) = \text{const}$$

for all s. This implies that

$$\lambda_x \frac{dx}{ds} + \lambda_y \frac{dy}{ds} = 0 \qquad \text{and} \qquad \mu_x \frac{dx}{ds} + \mu_y \frac{dy}{ds} = 0$$

where, as usual

$$\lambda_x = \frac{\partial \lambda}{\partial x}\,, \text{etc.}$$

Substituting these equations into Eq.(6.12), the equation for the characteristic directions, one obtains

$$A\left(\frac{\partial \lambda}{\partial x}\right)^2 + 2B\frac{\partial \lambda}{\partial x}\,\frac{\partial \lambda}{\partial y} + C\left(\frac{\partial \lambda}{\partial y}\right)^2 = 0\,. \tag{6.14}$$

An equation with the same coefficients is obtained for the other function $\mu(x,y)$. The two solutions $\lambda(x,y)$ and $\mu(x,y)$ are real valued functions. Their isograms, the *characteristics* of the hyperbolic equation, give us the new curvilinear coordinate system

$$\lambda = \lambda(x,y) \qquad\qquad \mu = \mu(x,y)\,.$$

The partial derivatives of the given differential equation are now as follows

$$\begin{aligned}
\frac{\partial^2\psi}{\partial x^2} &= \frac{\partial^2\psi}{\partial\lambda^2}(\lambda_x)^2 + 2\frac{\partial^2\psi}{\partial\lambda\partial\mu}\lambda_x\mu_x + \frac{\partial^2\psi}{\partial\mu^2}(\mu_x)^2 + \cdots \\
\frac{\partial^2\psi}{\partial x\partial y} &= \frac{\partial^2\psi}{\partial\lambda^2}\lambda_x\lambda_y + \frac{\partial^2\psi}{\partial\lambda\partial\mu}(\lambda_x\mu_y + \mu_x\lambda_y) + \frac{\partial^2\psi}{\partial\mu^2}\mu_x\mu_y + \cdots \\
\frac{\partial^2\psi}{\partial y^2} &= \frac{\partial^2\psi}{\partial\lambda^2}\lambda_y^2 + 2\frac{\partial^2\psi}{\partial\lambda\partial\mu}\lambda_y\mu_y + \frac{\partial^2\psi}{\partial\mu^2}\mu_y^2 + \cdots .
\end{aligned}$$

Here $+\cdots$ refers to additional terms involving only the first partial derivatives of ψ. Inserting these expressions into the given p.d. equation, one obtains

$$\begin{aligned}
[A\lambda_x^2 + 2B\lambda_x\lambda_y + C\lambda_y^2]\frac{\partial^2\psi}{\partial\lambda^2} &+ [2A\lambda_x\mu_x + B(\lambda_x\mu_y + \mu_x\lambda_y) + 2C\lambda_y\mu_y]\frac{\partial^2\psi}{\partial\lambda\partial\mu} \\
&+ [A\mu_x^2 + 2B\mu_x\mu_y + C\mu_y^2]\frac{\partial^2\psi}{\partial\mu^2} \\
&= \Phi'\left(\lambda, \mu, \psi, \frac{\partial\psi}{\partial\lambda}, \frac{\partial\psi}{\partial\mu}\right) . \qquad (6.15)
\end{aligned}$$

It follows from Equation 6.14 that the coefficients of $\psi_{\lambda\lambda}$ and $\psi_{\mu\mu}$ vanish. Solving for $\frac{\partial^2\psi}{\partial\lambda\partial\mu}$ yields Equation 6.13, the hyperbolic equation in normal form.

The coordinates λ and μ, whose surfaces contain the characteristic lines, are called the *characteristic coordinates* or *null coordinates* of the hyperbolic equation.

These coordinates are important for at least two reasons. First of all, they are boundaries across which a solution can be nonanalytic. If $\lambda(x, y) = \lambda_0$ is one of the isograms ("locus of points where λ has constant value") of the solution to Eq.(6.14), then the first term of the p.d. Eq.(6.15)

$$[A\lambda_x^2 + 2B\lambda_x\lambda_y + C\lambda_y^2]\frac{\partial^2\psi}{\partial\lambda^2} = finite$$

even if $\frac{\partial^2\psi}{\partial\lambda^2} \to \infty$ as $\lambda \to \lambda_0$. In other words, *there are solutions to Eq.(6.15) for which the first derivative* $\frac{\partial\psi}{\partial\lambda}$ *has a discontinuity across the characteristic* $\lambda(x, y) = \lambda_0$. Similarly, there exist solutions to Eq.(6.15) whose first derivative $\frac{\partial\psi}{\partial\mu}$ has a discontinuity across $\mu(x, y) = \mu_0$ whenever $\mu(x, y)$ satisfies Eq.(6.14) with λ replaced by μ.

Secondly, these coordinates depict the history of a moving disturbance. The simple string illustrates the issue involved.

Example: The Simple string The governing equation is

$$\frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2} = 0\,.$$

Its characteristic coordinates are the "retarded" and the "advanced" times

$$\lambda = ct - z \qquad \text{and} \qquad \mu = z + ct$$

and its normal form is

$$\frac{\partial^2\psi}{\partial\lambda\partial\mu} = 0\,.$$

The solution is

$$\psi = f(\lambda) + g(\mu)$$

where f and g are *any* functions of λ and μ.
Next consider the initial value data at $t = 0$:

$$\begin{aligned}
\psi_0(z) \equiv \psi(t=0,z) &= f(-z) + g(z) && \text{"initial amplitude"}\\
V_0(z) \equiv \left.\frac{\partial\psi(t,z)}{\partial t}\right|_{t=0} &= \left.\frac{\partial\lambda}{\partial t}\,\frac{\partial\psi}{\partial\lambda}\right|_{\lambda=-z} + \left.\frac{\partial\mu}{\partial t}\,\frac{\partial\psi}{\partial\mu}\right|_{\mu=z} && \text{"initial velocity"}\\
&= cf'(-z) + cg'(z)\,.
\end{aligned}$$

These equations imply

$$\begin{aligned}
f(\lambda) &= \frac{1}{2}\psi_0(-\lambda) + \frac{1}{2c}\int_0^{-\lambda} V_0(z')dz'\\
g(\mu) &= \frac{1}{2}\psi_0(\mu) + \frac{1}{2c}\int_0^{\mu} V_0(z')dz'\,.
\end{aligned}$$

Consider the intersection of the two families of characteristics with the boundary line $t = 0$ as in the figure below.

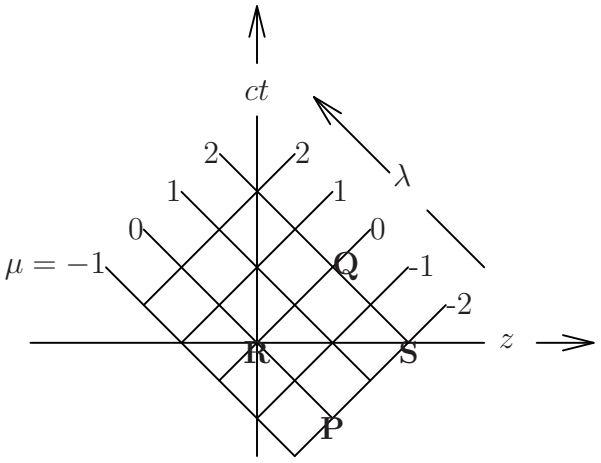

Figure 6.1: Characteristic coordinate lines μ and λ as determined by the wave equation for a simple string.

Note that f is constant along the λ characteristics (i.e., where λ =constant), while g is constant along the μ characteristics. It follows that if f is known on the boundary segment RS, then f is known along all the λ-characteristics intersecting RS. Similarly, if g is known along RS, then g is known along all the μ-characteristics intersecting RS. And this is precisely the case because the Cauchy data on RS determine the values of both f and g on that segment.

Being the sum of the two functions, the solution to the wave equation is

$$\begin{aligned} \psi(z,t) &= f(ct-z)+g(ct+z) \\ &= \frac{1}{2}\psi_0(z-ct)+\frac{1}{2}\psi_0(z+ct)+\frac{1}{2c}\int_{z-ct}^{z+ct} V_0(z')dz' \end{aligned} \quad (6.16)$$

Thus one sees that any disturbance on a string consists of two parts: one propagating to the right the other to the left. The propagation speeds are $\pm c$, the slopes of the characteristics relative to the t-z coordinate system. The idiosyncratic aspect of the simple string is that these two parts do not change their shape as they propagate along the string.

A general linear hyperbolic system does not share this feature. However, what it does share with a simple string is that its solution is uniquely determined in the common region traversed by the two sets of characteristics

which intersect RS. In fact, the Cauchy data on RS determine a unique solution $\psi(z,t)$ at every point in the region $PRQS$. This is why it is called the *domain of dependence* of RS. To justify these claims it is neccessary to construct this unique solution to a general second order linear hyperbolic differential equation.

6.1.4 Riemann's Method for Integrating the Most General 2nd Order Linear Hyperbolic Equation

In its most general form a linear second order hyperbolic equation is

$$0 = \frac{\partial^2\psi}{\partial u \partial v} + D\frac{\partial\psi}{\partial u} + E\frac{\partial\psi}{\partial v} + F\psi \equiv L(\psi)\ . \tag{6.17}$$

In compliance with standard practice one designates the characteristic coordinates by u and v. The problem to be solved is this:
Given
(a) the differential Eq.(6.17) and
(b) the initial value data (=Cauchy conditions) $\psi(s)$ and its normal derivative $\frac{d\psi(s)}{dn}$ on the given curve in Figure 6.2,
Find: the function $\psi(u,v)$ which satisfies (a) and (b).

Riemann's method of solving this problem is a three-step process whose essence parallels the Green's function method described on page 235:

1. Identify Cauchy Data in Characteristic Form

By means of the derivative $\frac{d\psi(s)}{ds}$ the Cauchy data

$$\left.\begin{array}{rcl} \dfrac{d\psi(s)}{dn} & = & \dfrac{d\eta}{ds}\dfrac{\partial\psi}{\partial x} - \dfrac{d\xi}{ds}\dfrac{d\psi}{dy} \\ \dfrac{d\psi(s)}{ds} & = & \dfrac{d\xi}{ds}\dfrac{\partial\psi}{\partial x} + \dfrac{d\eta}{ds}\dfrac{d\psi}{dy} \end{array}\right\} \quad \text{at} \quad (x,y) = (\xi(s),\eta(s))\ .$$

determines the partial derivatives $\partial\psi/\partial x$ and $\partial\psi/\partial y$. Alternatively, the known characteristics

$$\begin{array}{rcl} u & = & u(x,y) \\ v & = & v(x,y) \end{array}$$

also yield the partial derivatives $\partial\psi/\partial u$ and $\partial\psi/\partial v$ on the given curve.

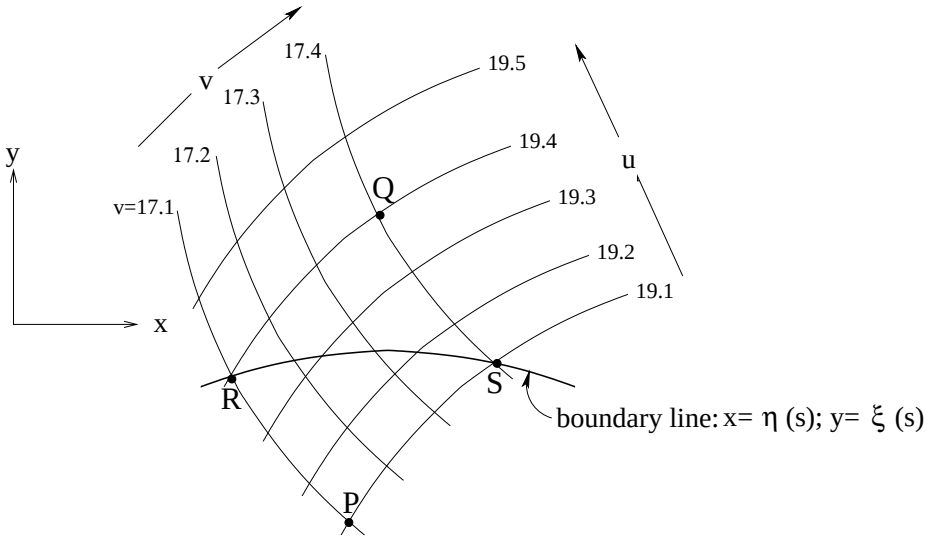

Figure 6.2: Characteristic coordinates of a hyperbolic differetial equation in two dimensions.

2. Determine Green's Identity for the Given Differential Operator

To construct in a systematic way the solution to the differential equation specified by (a) and (b) above, one introduces the *adjoint* differential equation and its solution. This philosophy is an extension of the approach already used to solve ordinary differential equations of second order.

Central to this approach was the consideration of the linear operator

$$L\psi = \frac{d^2\psi}{dx^2} + \beta\frac{d\psi}{dx} + \gamma\psi$$

and its adjoint

$$L^*\phi = \frac{d^2\phi}{dx^2} - \frac{\beta d\phi}{dx} + \gamma\phi$$

which was determined by the compatibility condition, the Lagrange identity Eq.(1.15)

$$\phi L\psi - \psi L^*\phi = \frac{d}{dx}P(\phi,\psi)\ . \tag{6.18}$$

Here the right hand side was the derivative of

$$P(\phi,\psi) = \phi\frac{d\psi}{dx} - \psi\frac{d\phi}{dx} + \beta\phi\psi\ ,$$

the "bilinear concomitant" introduced with Eq.(4.20) on page 233 for a one-dimensional domain.

The extension of Lagrange's identity to a two-dimensional domain is straight forward. Given the differential operator, Eq.(6.17), one seeks the corresponding adjoint, $L^*\phi$, which is determined by the compatibility condition

$$\boxed{\phi L\psi - \psi L^*\phi = \frac{\partial P_u}{\partial u} + \frac{\partial P_v}{\partial v}} \tag{6.19}$$

Here the right hand side is the divergence of a two-component *vectorial concomitant*. It replaces the total derivative of the scalar concomitant, Eq.(6.18).

What are $L^*\phi$ and (P_u, P_v)? Comparing the sought-after expressions in Eq.(6.19), with the known result, Eq.(6.18), one finds with a little scrutiny that

$$L^*\phi = \frac{\partial^2\phi}{\partial u\partial v} - \frac{\partial D\phi}{\partial u} - \frac{\partial E\phi}{\partial v} + F\phi \tag{6.20}$$

and

$$P_u = \frac{1}{2}\left(\phi\frac{\partial\psi}{\partial v} - \psi\frac{\partial\phi}{\partial v}\right) + D\phi\psi \tag{6.21}$$

$$P_v = \frac{1}{2}\left(\phi\frac{\partial\psi}{\partial u} - \psi\frac{\partial\phi}{\partial u}\right) + E\phi\psi \tag{6.22}$$

The boxed Eq.(6.19) is the key to success. By integrating it over a triangular region, say RSQ in Figure 6.2, one obtains Green's identity adapted to the given differential operator, Eq.(6.17). Indeed, applying Stokes' theorem to the right hand side, one obtains

$$\int_{RSQ}\int(\phi L\psi - \psi L^*\phi)du\,dv = \int_{S\to Q\to R\to S} P_u dv - P_v du \tag{6.23}$$

This equation relates a behavior of ϕ and ψ *inside* the 2-d domain RSQ to their behaviour on its boundary.

3. Apply Green's Identity

One obtains the solution $\psi(u, v)$ to the given hyperbolic equation by expressing it in terms of its Cauchy data and in terms of a simple solution to the adjoint differential equation, namely

$$L^*\phi = 0\ ,$$

subject to the boundary conditions

$$\frac{\partial\phi}{\partial u} - E\phi = 0 \qquad \text{along } SQ \text{ where } v \text{ is constant} \tag{6.24}$$

$$\frac{\partial\phi}{\partial v} - D\phi = 0 \qquad \text{along } RQ \text{ where } u \text{ is constant} , \tag{6.25}$$

and

$$\phi(u,v) = 1 \qquad \text{at } (u,v) = (u_Q, v_Q) . \tag{6.26}$$

Riemann noticed that if one can solve this adjoint boundary value problem, the solution to the given problem is obtained as follows:

Apply the fact that ψ and ϕ satisfy their respective differential equations. Green's identity, Eq.(6.23) becomes

$$0 = \int_{SQ} P_u dv - P_v du + \int_{QR} P_u dv - P_v du + \int_{RS} P_u dv - P_v du . \tag{6.27}$$

The left hand side vanishes because $L^*\phi = 0$ and because the given differential Eq.(6.17) has no source. The r.h.s. is a line integral along a closed figure, here the boundary of the triangle SQR. The goal is to infer the value of ψ at Q from (a) its values along RS and from (b) the value of ϕ along SQ and RQ.

Consider the first integral. The integration segment SQ consists of

$$SQ : \left\{ \begin{array}{rl} v & = const. \equiv v_Q \\ u_S < u & < u_Q \end{array} \right. \tag{6.28}$$

Consequently, one is left with

$$\begin{aligned}
\int_{SQ} P_u dv - P_v du &\overset{1}{=} -\int_{u_S}^{u_Q} P_v|_{v_Q} du \\
&\overset{2}{=} -\frac{1}{2}\int \left[\phi\frac{\partial\psi}{\partial u} - \psi\frac{\partial\phi}{\partial u} + 2E\phi\psi\right]_{v_Q} du \\
&\overset{3}{=} -\frac{1}{2}\int \left[\frac{\partial(\phi\psi)}{\partial u} + 2\psi\left(E\phi - \frac{\partial\phi}{\partial u}\right)\right]_{v_Q} du \\
&\overset{4}{=} -\frac{1}{2}\left[\phi\psi|_{u_S}^{u_Q}\right]_{v_Q} \\
&\overset{5}{=} -\frac{1}{2}\psi(u_Q, v_Q) + \frac{1}{2}\phi(u_S, v_S)\psi(u_S, v_S) \\
&\overset{6}{\equiv} -\frac{1}{2}\psi(Q) + \frac{1}{2}\phi(S)\psi(S) .
\end{aligned}$$

(Equality 1 uses the fact that $\int_{SQ} P_u dv = 0$ because v is constant; 2 uses Eq.(6.22); 3 adds and subtracts $\partial\phi/\partial u$; 4 uses Eq.(6.24); 5 uses Eq.(6.26).) Equality 6 introduces the short hand notation

$$\psi(Q) = \psi(u_Q, v_Q),\ \phi(S) = \phi(u_S, v_S),\ \text{etc.}$$

Similarly the second integral reduces to

$$\begin{aligned}\int_{QR} P_u dv - P_v du &= \int_{v_Q}^{v_R} P_u|_{u_Q} dv \\ &= -\frac{1}{2}\psi(Q) + \frac{1}{2}\phi(R)\psi(R)\ .\end{aligned}$$

Consequently, the vanishing of the closed line integral, Eq.(6.27), implies

$$\psi(Q) = \frac{1}{2}\phi(S)\psi(S) + \frac{1}{2}\phi(R)\psi(R) + \int_{RS} P_u dv - P_v du\ . \tag{6.29}$$

This is Riemann's representation of the solution ψ to the hyperbolic differential equation in terms of the given initial value data on the curve segment RS. A function such as $\psi(u, v)$ establishes a quantitative relationship between two sets of measurements:

1. The quantity ψ which typically expresses a measured amplitude or voltage and

2. the coordinates (u, v) which, for a hyperbolic system, indirectly express measurements of time and place, namely

$$\begin{aligned}u &= t - z \\ v &= t + z\end{aligned}$$

in terms of the familiar laboratory space and time coordinates.

Thus it is necessary to express the solution, Eq.(6.29), in terms of these coordinates. Using

$$\begin{aligned}du &= dt - dz \\ dv &= dt + dz\end{aligned}$$

and

$$\begin{aligned} \frac{\partial\psi}{\partial v} + \frac{\partial\psi}{\partial u} &= \frac{\partial\psi}{\partial t} \\ \frac{\partial\psi}{\partial v} - \frac{\partial\psi}{\partial u} &= \frac{\partial\psi}{\partial z} \end{aligned}$$

one finds that with the help of Eqs.(6.21)-(6.22) that the solution is

$$\psi(t_Q, z_Q) = \frac{1}{2}\phi(R)\psi(R) + \frac{1}{2}\phi(S)\psi(S) + \tag{6.30}$$
$$\int_R^S \left\{ \left[\frac{1}{2}\left(\phi\frac{\partial\psi}{\partial z} - \psi\frac{\partial\phi}{\partial z} \right) + (D-E)\phi\psi \right] dt + \left[\frac{1}{2}\left(\phi\frac{\partial\psi}{\partial t} - \psi\frac{\partial\phi}{\partial t} \right) + (D+E)\phi\psi \right] dz \right\}$$

Example: String Imbedded in an Elastic Medium

Let us illustrate the integration method with a simple string imbedded in an elastic medium. The governing equation is the *Klein-Gordon wave equation* in 1+1 dimensions,

$$0 = \frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial z^2} + k^2\psi\,.$$

Its solution is to satisfy at $t = 0$ the initial value conditions

$$\psi(t=0, z) = \psi_0(z) \tag{6.31}$$
$$\frac{\partial\psi}{\partial t}(t=0, z) = V_0(z)\ . \tag{6.32}$$

Here $\psi_0(z)$ and $V_0(z)$ are the given initial value data ("Cauchy data") associated with this *initial value problem*.

The equation for the characteristic coordinate functions is

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial z}\right)^2 = 0$$

Being a quadratic, this equation has two distinct *real* solutions

$$S(x,y) = \begin{cases} u(t,z) = t - z \\ v(t,z) = t + z \end{cases}$$

Its characteristic coordinate functions are

$$u = t - z \qquad \text{and} \qquad v = z + t \tag{6.33}$$

and its normal form is

$$L(\psi) = \frac{\partial^2 \psi}{\partial \lambda \partial \mu} + \frac{k^2}{4}\psi = 0\,.$$

Thus one has $D = E = 0$, which means that the hyperbolic operator is *formally self-adjoint*. Consequently, the adjoint differential equation is

$$L^*(\phi) = \frac{\partial^2 \phi}{\partial v \partial u} + \frac{k^2}{4}\phi = 0\,.$$

The adjoint boundary conditions are

$$\frac{\partial \phi}{\partial u} = 0 \qquad \text{along } SQ: \; v = v_Q \tag{6.34}$$

$$\frac{\partial \phi}{\partial v} = 0 \qquad \text{along } RQ: \; u = u_Q\,, \tag{6.35}$$

and

$$\phi(u,v) = 1 \qquad \text{at observation point Q: } (u,v) = (u_Q, v_Q)\;. \tag{6.36}$$

Remark. One can draw a very useful conclusion from Eqs.(6.34)-(6.36). The solution ϕ to the hyperbolic problem adjoint to the given one under consideration is *constant along the two characteristics through point Q*:

$$\begin{aligned} \phi(u, v_Q) &= 1 \\ \phi(u_Q, v) &= 1\;. \end{aligned}$$

Note that the two points R and S in Figure 6.2 lie on these charateristics. Consequently,

$$\phi(R) = \phi(S) = 1\;.$$

This simplifies the solution to the solution, Eq.(6.30), to the given problem considerably.

The solution to the adjoint boundary boundary value problem is achieved by recalling that the Bessel function of order zero, $J_0(x)$, satisfies

$$\frac{d^2 J_0}{dx^2} + \frac{1}{x}\frac{dJ_0}{dx} + J_0 = 0\;.$$

Letting

$$x = k\sqrt{uv}\;,$$

one finds

$$\begin{aligned}\frac{\partial J_0(k\sqrt{uv})}{\partial v} &= \frac{1}{2}k\sqrt{\frac{u}{v}}J_0'(k\sqrt{uv}) \\ \frac{\partial^2 J_0(k\sqrt{uv})}{\partial u\,\partial v} &= \frac{1}{4}k\frac{1}{\sqrt{vu}}J_0'(k\sqrt{uv}) + \frac{k^2}{4}J_0{}''(k\sqrt{uv}) \\ &= -\frac{k^2}{4}J_0(k\sqrt{uv})\ .\end{aligned}$$

Consequently,

$$\left[\frac{\partial^2}{\partial v\,\partial u} + \frac{k^2}{4}\right] J_0\left(k\sqrt{(u_Q-u)(v_Q-v)}\right) = 0$$

Furthermore, note that

$$\phi(u,v) = J_0\left(k\sqrt{(u_Q-u)(v_Q-v)}\right)$$

satisfies the three required boundary conditions

$$\begin{aligned}\left.\frac{\partial\phi(u,v)}{\partial u}\right|_{v=v_Q} &= 0 \\ \left.\frac{\partial\phi(u,v)}{\partial v}\right|_{u=u_Q} &= 0 \\ \phi(u_Q,v_Q) &= 1\ ,\end{aligned}$$

and also

$$\phi(u,v_Q) = \phi(u_Q,v) = 1\ ,$$

as required.

The solution, Eq.(6.30), to the given problem is determined by the initial value data, Eqs.(6.31)-(6.32) at $t = 0$. Substituting this data into the expression for this solution, taking note of the fact

$$\begin{aligned}R:\ (t,z) &= (0, z_Q - t_Q) \\ S:\ (t,z) &= (0, z_Q + t_Q)\ ,\end{aligned}$$

introducing (with the help of Eq.(6.33))

$$\phi = J_0\left(k\sqrt{(u_Q-u)(v_Q-v)}\right) = J_0\left(k\sqrt{(t-t_Q)^2-(z-z_Q)^2}\right)$$

into the integrand, and setting $t = 0$, one finds that the solution is

$$\begin{aligned}\psi(t_Q, z_Q) &= \frac{1}{2}\psi_0(z_Q - t_Q) + \frac{1}{2}\psi_0(z_Q + t_Q) + \\ &\frac{1}{2}\int_{z_Q - t_Q}^{z_Q + t_Q} \left[J_0\left(k\sqrt{(t - t_Q)^2 - (z - z_Q)^2}\right) V_0(z) - \right. \\ &\left. \psi_0(z)\frac{\partial}{\partial t} J_0\left(k\sqrt{(t - t_Q)^2 - (z - z_Q)^2}\right)\right]_{t=0} dz\end{aligned}$$

or in terms of standard variables,

$$\begin{aligned}\psi(t, z) &= \frac{1}{2}\psi_0(z - t) + \frac{1}{2}\psi_0(z + t) + \\ &\frac{1}{2}\int_{z-t}^{z+t} \left[J_0\left(k\sqrt{(t' - t)^2 - (z' - z)^2}\right) V_0(z') - \right. \\ &\left. \psi_0(z')\frac{\partial}{\partial t'} J_0\left(k\sqrt{(t' - t)^2 - (z' - z)^2}\right)\right]_{t'=0} dz' \ .\end{aligned} \tag{6.37}$$

Compare this result with Eq.(6.16) and observe the influence of the elastic medium on the propagation of a disturbance along the string:
In the absence of that medium an initial pulse separates into two pulses also highly localized in the same way. They move into opposite directions, but they don't change their shapes and amplitudes. The region between these pulses is a widening gap having zero amplitude.

However, the presence of an elastic medium ($k^2 \neq 0$) changes all this. An initial pulse also separates into two pulses, but each one leaves a nonzero trailing wake which fills the widening gap between them with a space-time dependent amplitude. It decreases with time in a manner dictated by the behaviour of the Bessel function in the integrand of Eq.(6.37).

Equations (6.16) and (6.37) are in perfect harmony. Indeed, the first is the $k^2 = 0$ limit of the second. This is as it must be. It is a mathematical consequence of the fact that $J_0(0) = 1$ and that $J_0'(0) = 0$ in the integrand of Eq.(6.37).

Lecture 51[3]

[3]Presentation given 10/3/2006 at the OSU Electro Science Lab.

6.2 System of Partial Differential Equations: How to Solve Maxwell's Equations Using Linear Algebra

The theme of the ensuing development is linear algebra, but the subject is an overdetermined system of partial differential equations, namely, the Maxwell field equations. The objective is to solve them via the method of eigenvectors and eigenvalues. The benefit is that the task of solving the Maxwell system of p.d. equations is reduced to solving a single inhomogeneous scalar equation[4]

$$\left(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2\right) = -4\pi S(t, \vec{x}) \ ,$$

where S is a time and space dependent source. The impatient reader will find that once this master equation, or its manifestation in another coordinate system, has been solved, the electric and magnetic fields are entirely determined as in Tables 6.1-6.9.

The starting point of the development is Maxwell's equations. There is the set of four functions, the density of charge

$$\rho = \rho(\vec{x}, t) \quad \left[\frac{(\text{charge})}{(\text{volume})}\right] \tag{6.38}$$

and the charge flux

$$\vec{J} = \vec{J}(\vec{x}, t) \quad \left[\frac{(\text{charge})}{(\text{time})(\text{area})}\right], \tag{6.39}$$

which are usually given. These space and time dependent charge distributions give rise to electric and magnetic fields, $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$. The relationship is captured by means of Maxwell's gift to twentieth century science and technology,

$$\nabla \cdot \vec{B} = 0 \qquad (\text{"No magnetic monopoles"}) \tag{6.40}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad (\text{"Faraday's law"}) \tag{6.41}$$

[4]For the purpose of putting the time derivative on the same footing as the space derivatives, we express the conventional time $t_{conv.}$ in terms of the geometrical time t, which is measured in units of length, by the equation $t = c\, t_{conv.}$.

and

$$\nabla \cdot \vec{E} = 4\pi\rho \quad (\text{"Gauss' law"}) \tag{6.42}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi\vec{J} \quad (\text{"Ampere's law"}) \ , \tag{6.43}$$

Maxwell's field equations[5].

Exercise 6.2.1 (Charge Flux-Density of an Isolated Charge)
Microscopic observations show that charged matter is composed of discrete point charges. On the other hand, macroscopic observations show that charged matter is the carrier of an electric fluid which is continuous. Dirac delta functions provide the means to grasp both attributes from a single perspective. This fact is highlighted by the following problem.
Consider the current-charge density due to an isolated moving charge,

$$\vec{J}(x,y,z,t) = q\int_{-\infty}^{\infty} \frac{d\vec{X}(\tau)}{d\tau}\delta(x - X(\tau))\delta(y - Y(\tau))\delta(z - Z(\tau))\delta(t - T(\tau))\, d\tau$$

$$\rho(x,y,z,t) = q\int_{-\infty}^{\infty} \frac{dT(\tau)}{d\tau}\delta(x - X(\tau))\delta(y - Y(\tau))\delta(z - Z(\tau))\delta(t - T(\tau))\, d\tau$$

a) Show that this current-charge density satisfies

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \ .$$

Remark. The four-vector $\left(\frac{d\vec{X}(\tau)}{d\tau}, \frac{dT(\tau)}{d\tau}\right)$ is the charge's four-velocity in spacetime. The parameter τ is the "wristwatch" time (as measured by a comoving clock) attached to this charge.
b) By taking advantage of the fact $\frac{dT(\tau)}{d\tau} > 0$, evaluate the τ-integrals, and obtain explicit expressions for the components $\vec{J}$ and ρ.
Answer:

$$\rho(x,y,z,t) = \quad q\,\delta(x - X(t))\delta(y - Y(t))\delta(z - Z(t))$$

$$\vec{J}(x,y,z,t) = \frac{d\vec{X}}{dt}\, q\,\delta(x - X(t))\delta(y - Y(t))\delta(z - Z(t))$$

---

[5]The introduction of geometrical time t is extended to the introduction of charge flux in geometrical units, $\vec{J}$ $(\frac{(charge)}{(volume)})$. Its relation to the charge flux in conventional units, $\vec{J}_{conv.}$ $(\frac{(charge)}{(time)(area)})$, is given by the equation $\vec{J} = \vec{J}_{conv.}/c$.

where

$$\vec{X}(t) = \vec{X}(\tau) \text{ evaluated at } \tau \text{ as determined by } \delta(t - T(\tau)) \ .$$

6.2.1 Maxwell Wave Equation

The first pair of Maxwell's equations, (6.40) and (6.41), imply that there exists a vector potential $\vec{A}$ and scalar potential ϕ from which one derives the electric and magnetic fields,

$$\vec{B} = \nabla \times \vec{A} \tag{6.44}$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \ . \tag{6.45}$$

Conversely, the existence of these potentials guarantees that the first pair of these equations is satisfied automatically. By applying these potentials to the differential expressions of the second pair of Maxwell's equations, (6.42)-(6.43), one obtains the mapping

$$\left[\begin{array}{c} \phi \\ \vec{A} \end{array} \right] \quad \stackrel{\mathcal{A}}{\rightsquigarrow} \mathcal{A} \left[\begin{array}{c} \phi \\ \vec{A} \end{array} \right] \ , \tag{6.46}$$

where

$$\mathcal{A} \left[\begin{array}{c} \phi \\ \vec{A} \end{array} \right] \equiv \left[\begin{array}{c} -\nabla^2\phi - \nabla \cdot \dfrac{\partial \vec{A}}{\partial t} \\ \nabla \times \nabla \times \vec{A} + \nabla \dfrac{\partial \phi}{\partial t} + \dfrac{\partial^2 \vec{A}}{\partial t^2} \end{array} \right] \ . \tag{6.47}$$

It follows that Maxwell's field equations reduce to Maxwell's four-component wave equation,

$$\left[\begin{array}{c} -\nabla^2\phi - \nabla \cdot \dfrac{\partial \vec{A}}{\partial t} \\ \nabla \times \nabla \times \vec{A} + \nabla \dfrac{\partial \phi}{\partial t} + \dfrac{\partial^2 \vec{A}}{\partial t^2} \end{array} \right] = 4\pi \left[\begin{array}{c} \rho \\ \vec{J} \end{array} \right] \ . \tag{6.48}$$

Maxwell's wave operator is the linch pin of his theory of electromagnetism. This is because it has the following properties:

1. It is a *linear* map from the space of four-vector fields into itself, i.e.

$$R^4 \xrightarrow{\mathcal{A}} R^4$$

at each point event $(t, \vec{x})$.

2. The map is singular. This means that there exist nonzero vectors $\vec{\mathcal{U}}_r$ and $\vec{\mathcal{U}}_\ell$ such that

$$\mathcal{A}\vec{\mathcal{U}}_r = \vec{0}$$

and

$$\vec{\mathcal{U}}_\ell^T \mathcal{A} = \vec{0} \ .$$

In particular, one has

(a) the fact that

$$\mathcal{A}\begin{bmatrix} -\partial_t \\ \vec{\nabla} \end{bmatrix}\Lambda = \begin{bmatrix} -\nabla^2\partial_t - \partial_t\nabla^2 \\ 0 - \partial_t\vec{\nabla}\partial_t + \partial_t^2\vec{\nabla} \end{bmatrix}\Lambda = \begin{bmatrix} 0 \\ \vec{0} \end{bmatrix} \tag{6.49}$$

for all three-times differentiable scalar fields $\Lambda(t, \vec{x})$. Thus

$$\vec{\mathcal{U}}_r \equiv \begin{bmatrix} -\partial_t \\ \vec{\nabla} \end{bmatrix} \ \in \ \mathcal{N}(\mathcal{A}) \ .$$

The null space of $\mathcal{A}$ is therefore nontrivial and 1-dimensional at each $(t, \vec{x})$.

(b) the fact that

$$\begin{bmatrix} \partial_t & \vec{\nabla}\cdot \end{bmatrix}\mathcal{A}\begin{bmatrix} \phi \\ \vec{A} \end{bmatrix} = -\partial_t\nabla^2\phi + \partial_t^2\vec{\nabla}\cdot\vec{A} + 0 + \partial_t\nabla^2\phi + \vec{\nabla}\cdot\partial_t^2\vec{A} = 0 \ ,$$

for all 4-vectors $\begin{bmatrix} \phi \\ \vec{A} \end{bmatrix}$. Thus

$$\vec{\mathcal{U}}_\ell^T \equiv \begin{bmatrix} \partial_t & \vec{\nabla}\cdot \end{bmatrix} \ \in \ \text{left null space of } \mathcal{A} \ , \tag{6.50}$$

or

$$\vec{\mathcal{U}}_\ell \ \in \ \mathcal{N}(\mathcal{A}^T) \ . \tag{6.51}$$

The left null space of $\mathcal{A}$ is therefore also 1-dimensional at each $(t, \vec{x})$.

In light of the singular nature of $\mathcal{A}$, the four-component Maxwell wave equation

$$\mathcal{A}\begin{bmatrix} \phi \\ \vec{A} \end{bmatrix} = 4\pi \begin{bmatrix} \rho \\ \vec{J} \end{bmatrix} \tag{6.52}$$

has no solution unless the source $\begin{bmatrix} \rho \\ \vec{J} \end{bmatrix}$ also satisfies

$$\vec{\mathcal{U}}_\ell^T \begin{bmatrix} \rho \\ \vec{J} \end{bmatrix} = 0 \ .$$

This is the linear algebra way of expressing

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0 \ , \tag{6.53}$$

the differential law of charge conservation. Thus Maxwell's equations apply if and only if the law of charge conservation holds. If charge conservation did not hold, then Maxwell's equations would be silent. They would not have a solution. Such silence is a mathematical way of expressing the fact that at its root theory is based on observation and established knowledge, and that arbitrary hypotheses must not contaminate the theoretical.

6.2.2 The Overdetermined System $A\vec{u} = \vec{b}$

The linear algebra aspects of Maxwell's wave operator $\mathcal{A}$ are illustrated by the following problem from linear algebra:

Solve $A\vec{u} = \vec{b}$ for $\vec{u}$, under the stipulation that

$$\begin{aligned} A: &\ R^4 \longrightarrow R^4 \\ \vec{u}_r: &\ A\vec{u}_r = \vec{0} \quad \text{so that } \mathcal{N}(A) = span\{\vec{u}_r\} \\ \vec{u}_\ell^T: &\ \vec{u}_\ell^T A = \vec{0} \quad \text{so that } \mathcal{N}(A^T) = span\{\vec{u}_\ell\} \\ \vec{b}: &\ \vec{b} \in \mathcal{R}(A) \quad \text{so that } \vec{u}_\ell^T \vec{b} = 0 \end{aligned} \tag{6.54}$$

The fact that A is singular and $\vec{b}$ belongs to the range of A makes the system over-determined but consistent. This means that there are more equations than there are unknowns.
One solves the problem in two steps.

Step I: Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be the set of eigenvectors having non-zero eigenvalues. Whatever A is, the task of finding three vectors that satisfy

$$\left.\begin{array}{l} A\vec{v}_1 c_1 = \vec{v}_1 \lambda_1 c_1 \\ A\vec{v}_2 c_2 = \vec{v}_2 \lambda_2 c_2 \\ A\vec{v}_3 c_3 = \vec{v}_3 \lambda_3 c_3 \end{array}\right\} \lambda_i \neq 0,\ c_i \text{ are scalars} \tag{6.55}$$

and

$$A\vec{u}_r c_4 = \vec{0}\ . \tag{6.56}$$

Being spanned by the three eigenvectors with non-zero eigenvalues, the range space of A,

$$\mathcal{R}(A) = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}\ ,$$

is well-determined. However, the scalars c_i are at this stage as yet undetermined.

Step II: Continuing the development, recall that quite generally

$$\begin{aligned} A\vec{u} = \vec{b} \qquad &\text{has a solution} \Leftrightarrow \vec{b} \in \mathcal{R}(A) \\ &\Leftrightarrow \vec{b} = \vec{v}_1 b_1 + \vec{v}_2 b_2 + \vec{v}_3 b_3\ , \end{aligned} \tag{6.57}$$

and that if

$$\vec{u} = \vec{v}_1 c_1 + \vec{v}_2 c_2 + \vec{v}_3 c_3 + \vec{u}_4 c_4\ ,$$

then

$$A\vec{u} = \vec{v}_1 \lambda_1 c_1 + \vec{v}_2 \lambda_2 c_2 + \vec{v}_3 \lambda_3 c_3\ . \tag{6.58}$$

It is appropriate to alert the reader that in the ensuing section the vectors $\vec{v}_i$ and the eigenvalues λ_i become differential operators which act on scalar fields c_i and that the three subscript labels will refer to the TE, TM, and TEM eletromagnetic[6] vector potentials respectively.

[6]The acronyms TE, TM, as well as TEM stand for *transverse electric, transverse magnetic*, and *transeverse electric magnetic* The justification for these apellations are given on Pages 448, 448, and 449, repectively.

Equating (6.57) and (6.58), one finds that the linear independence of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ implies the following equations for c_1, c_2, and c_3:

$$\lambda_1 c_1 = b_1 \longrightarrow c_1 = \frac{1}{\lambda_1} b_1 \tag{6.59}$$

$$\lambda_2 c_2 = b_2 \longrightarrow c_2 = \frac{1}{\lambda_2} b_2 \tag{6.60}$$

$$\lambda_3 c_3 = b_3 \longrightarrow c_3 = \frac{1}{\lambda_3} b_3 \tag{6.61}$$

Consequently, the solution is

$$\vec{u} = \vec{v}_1 \frac{1}{\lambda_1} b_1 + \vec{v}_2 \frac{1}{\lambda_2} b_2 + \vec{v}_3 \frac{1}{\lambda_3} b_3 + \vec{u}_4 c_4$$

where $\vec{u}_4 c_4$ is an indeterminate multiple of the null space vector $\vec{u}_4$.

If one represents the stated problem $A\vec{u} = \vec{b}$ ($\vec{u}$ determines $\vec{b}$) as an input-output process, as in Figure 6.3,

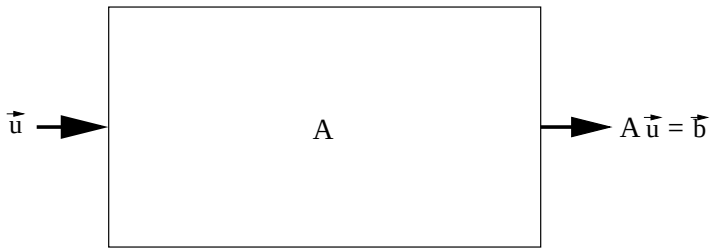

Figure 6.3: The matrix A defines an input-output process.

then its solution is represented by the inverse input-output process as in Figure 6.4.

In general, the task of finding the eigenvectors of a 4×4 matrix can be a nontrivial task. However, given the fact that the solution to

$$\vec{u}_\ell^T A = \vec{0}$$

is already known, one finds that the associated constraints on the eigenvectors,

$$\vec{u}_\ell^T \vec{v}_i = 0$$

make the task quite easy, if not trivial.

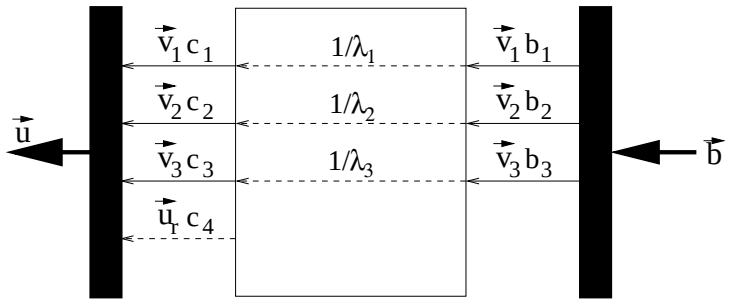

Figure 6.4: The solution to $A\vec{u} = \vec{b}$ defines an inverse input-output process.

6.2.3 Maxwell Wave Equation (continued)

The above linear algebra two-step analysis of an overdetermined (but consistent) system $A\vec{u} = \vec{b}$ is an invaluable guide in solving Maxwell's wave equation

$$\mathcal{A}\begin{bmatrix} \phi \\ A_z \\ A_x \\ A_y \end{bmatrix} = 4\pi \begin{bmatrix} \rho \\ J_z \\ J_x \\ J_y \end{bmatrix} \quad \begin{matrix} \} \text{ longitudinal components} \\ \\ \} \text{ transverse components,} \end{matrix} \tag{6.62}$$

2+2 Decomposition

We now interrupt the development to emphasize that this linear algebra analysis is not restricted to rectilinear coordinates. We shall see that the coordinates relative to which the Maxwell system can be decoupled (and solved) via the method of eigenvectors and eigenvalues are the cartesian, cylindrical, spherical, and other coordinate orthogonal coordinate systems, with time added as the fourth coordinate. For these the spacetime version of the infinitesimal interval (Pythagorean theorem) assumes the familiar form

$$\begin{aligned} ds^2 &= -dt^2 + dz^2 + dx^2 + dy^2 && \text{cartesian} \\ ds^2 &= -dt^2 + dz^2 + dr^2 + r^2 d\theta^2 && \text{cylindrical} \\ ds^2 &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) && \text{spherical} \end{aligned}$$

or more generally

$$ds^2 = g_{AB}(x^C)dx^A dx^B + R^2(x^C)(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad \begin{matrix} \text{general} \\ \text{spherical} \end{matrix}$$

Their conceptual common denominator is that the first two coordinates – the *longitudinal* coordinates – are orthogonal to the last two – the *transverse* coordinates. The longitudinal spatial direction is the propagation direction of e.m. radiation, say in the direction of a cylindrical wave guide or the radial direction of a spherical coordinate system. The two spatial transverse directions point along the cross sectional area of that wave guide or the angular directions of the concentric spheres of constant radii.

This two-plus-two decomposition applies not only to the coordinates and their differentials, but also to four-dimensional vector fields tangent to such coordinate surfaces, e.g.,

$$\left.\begin{bmatrix} \phi \\ A_z \\ A_x \\ A_y \end{bmatrix} \text{ and } \begin{bmatrix} \rho \\ J_z \\ J_x \\ J_y \end{bmatrix}\right\} \begin{array}{l} \text{longitudinal components} \\ \\ \text{transverse components,} \end{array} \tag{6.63}$$

the four-vector potential and the charge density-flux four-vector respectively.

We shall decompose the four-vector potential $\begin{bmatrix} \phi \\ \vec{A} \end{bmatrix}$ into three parts. The key finding from this decomposition is that these parts are eigenvectors of the Maxwell wave operator $\mathcal{A}$, Eq.(6.47), and that they are identified with the *transverse electric* (TE), *transverse magnetic* (TM), and *transverse electric-magnetic* (TEM) fields of Maxwell theory [7].

The eigenvector decomposition takes advantage of the fact that any two-dimensional vector field, be it longitudinal or transverse, can be decomposed uniquely into the gradient of a scalar function and into what amounts to a pure curl vector field in three dimensions. As a consequence, any four-vector such as those in Eq.(6.63) has the unique decomposition

$$\begin{bmatrix} \phi \\ A_z \\ A_x \\ A_y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \partial_y \\ -\partial_x \end{bmatrix} \Phi^{TE}}_{\begin{array}{c}\text{transverse}\\ \text{curl}\end{array}} + \underbrace{\begin{bmatrix} -\partial_z \\ \partial_t \\ 0 \\ 0 \end{bmatrix} \Phi^{TM}}_{\begin{array}{c}\text{longitudinal}\\ \text{curl}\end{array}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \partial_x \\ \partial_y \end{bmatrix} \Psi}_{\begin{array}{c}\text{transverse}\\ \text{gradient}\end{array}} + \underbrace{\begin{bmatrix} -\partial_t \\ \partial_z \\ 0 \\ 0 \end{bmatrix} \Phi}_{\begin{array}{c}\text{longitudinal}\\ \text{gradient}\end{array}} . \tag{6.64}$$

[7] The justification for these apellations are given on on Pages 448, 448, and 449, repectively.

This 2+2 decomposition establishes a one-to-one correspondence between four-vector fields and the scalar fields Φ, Φ^{TE} and Ψ, Φ^{TM} in the transverse and longitudinal planes respectively.

The existence and uniqueness of this 2+2 decomposition is established in Exercise 6.2.2 on page 463.

Divergenceless Vector Fields

We now resume the development by recalling that the wave operator $\mathcal{A}$ satisfies

$$[\partial_t\ \partial_z\ \partial_x\ \partial_y]\,\mathcal{A}\begin{bmatrix}\phi\\ \vec{A}\end{bmatrix} = \vec{0} \tag{6.65}$$

for all four-vectors $\begin{bmatrix}\phi\\ \vec{A}\end{bmatrix}$, and that, as a consequence, the wave Eq.(6.62) implies that $\begin{bmatrix}\rho\\ \vec{J}\end{bmatrix}$ has zero divergence (which is an expression of charge conservation)

$$[\partial_t\ \partial_z\ \partial_x\ \partial_y]\begin{bmatrix}\rho\\ J_z\\ J_x\\ J_y\end{bmatrix} = 0 \tag{6.66}$$

whenever a solution exists. Our interest lies in the converse:
Given that the source $\begin{bmatrix}\rho\\ \vec{J}\end{bmatrix}$ satisfies charge conservation, does there exist a solution to the Maxwell wave equation?
An affirmative answer is obtained by construction. It is based on decomposing the source four-vector into a linear combination each part of which is so simple that it separately satisfies charge conservation. This decomposition is

$$\begin{bmatrix}\rho\\ J_z\\ J_x\\ J_y\end{bmatrix} = \underbrace{\begin{bmatrix}0\\ 0\\ \partial_y\\ -\partial_x\end{bmatrix}}_{\equiv\vec{\mathcal{V}}^{(1)}} S^{TE} + \underbrace{\begin{bmatrix}-\partial_z\\ \partial_t\\ 0\\ 0\end{bmatrix}}_{\equiv\vec{\mathcal{V}}^{(2)}} S^{TM} + \underbrace{\begin{bmatrix}0\\ 0\\ \partial_x\\ \partial_y\end{bmatrix} I + \begin{bmatrix}-\partial_t\\ \partial_z\\ 0\\ 0\end{bmatrix} J}_{\equiv\vec{\mathcal{W}}^{(3)}I+\vec{\mathcal{W}}^{(4)}J} \tag{6.67}$$

Charge conservation, $\partial_t + \vec{\nabla}\cdot\vec{J} = 0$, holds for all scalar fields S^{TE}, S^{TM}, I and J provided the latter two satisfy

$$0 = \left(-\partial_t^2 + \partial_z^2\right) J + \left(\partial_x^2 + \partial_y^2\right) I \quad \left(= \partial_t + \vec{\nabla}\cdot\vec{J}\right)\ . \tag{6.68}$$

The Eigenvector Fields of $\mathcal{A}$

The task of identifying and determining those eigenvector fields of $\mathcal{A}$ which have nonzero eigenvalues is facilitated by the fact that Eq.(6.65) demands that they be divergenceless. The three vector fields in Eq.(6.67) do satisfy this condition. That they span the eigenspaces in the range of $\mathcal{A}$ is verified by three explicit calculations, one for each the three eigenvector fields. Inserting them into the Maxwell wave equation, one finds that their linear independence results in the following three independent vector equations:

$$\mathcal{A}\begin{bmatrix} 0 \\ 0 \\ \partial_y \\ -\partial_x \end{bmatrix}\Phi^{TE} = \begin{bmatrix} 0 \\ 0 \\ \partial_y \\ -\partial_x \end{bmatrix}\underbrace{(-)\left(-\partial_t^2+\partial_z^2+\partial_x^2+\partial_y^2\right)}_{\lambda_1}\Phi^{TE} = 4\pi\begin{bmatrix} 0 \\ 0 \\ \partial_y \\ -\partial_x \end{bmatrix}S^{TE} \tag{6.69}$$

$$\mathcal{A}\begin{bmatrix} -\partial_z \\ \partial_t \\ 0 \\ 0 \end{bmatrix}\Phi^{TM} = \begin{bmatrix} -\partial_z \\ \partial_t \\ 0 \\ 0 \end{bmatrix}\underbrace{(-)\left(-\partial_t^2+\partial_z^2+\partial_x^2+\partial_y^2\right)}_{\lambda_2}\Phi^{TM} = 4\pi\begin{bmatrix} -\partial_z \\ \partial_t \\ 0 \\ 0 \end{bmatrix}S^{TM} \tag{6.70}$$

and

$$\mathcal{A}\begin{bmatrix} -\partial_t\Phi \\ \partial_z\Phi \\ \partial_x\Psi \\ \partial_y\Psi \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} -\partial_t \\ \partial_z \end{pmatrix}\left(\partial_x^2+\partial_y^2\right)(\Psi-\Phi) \\ \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}\left(\partial_t^2-\partial_z^2\right)(\Psi-\Phi) \end{bmatrix} = 4\pi\begin{bmatrix} \begin{pmatrix} -\partial_t \\ \partial_z \end{pmatrix}J \\ \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}I \end{bmatrix} \tag{6.71}$$

What is remarkable about these equations is that each of them is an integrable system which can be integrated by inspection. Doing so results in what in linear algebra corresponds to the three equations (6.59)-(6.61) on page 442 for the eigenvector amplitudes c_i. Here. however, the result is three scalar wave equations for the scalar fields $\Phi^{TE}, \Phi^{TM}, \Psi$ and Φ, namely[8]

$$\left(\partial_x^2+\partial_y^2+\partial_z^2-\partial_t^2\right)\Phi^{TE} = -4\pi S^{TE} \tag{6.72}$$

$$\left(\partial_x^2+\partial_y^2+\partial_z^2-\partial_t^2\right)\Phi^{TM} = -4\pi S^{TM} \tag{6.73}$$

[8]The superscripts TE, TM, as well as TEM below are acronyms which stand for *transverse electric, transverse magnetic*, and *transeverse electric magnetic* The justification for these apellations are given on Pages 448, 448, and 449, repectively.

and

$$(\partial_x^2 + \partial_y^2)(\Phi - \Psi) = -4\pi J \tag{6.74}$$

$$(\partial_z^2 - \partial_t^2)(\Phi - \Psi) = +4\pi I \ . \tag{6.75}$$

The last two equations are the TEM equations, a Poisson and a wave equation, for one and the same quantity, the *difference* $\Phi - \Psi$. These two equations are consistent, and hence integrable, because the TEM source scalars I and J are guaranteed to satisfy Eq.(6.68) on page 445. Put differently, any two of the three equations (6.74), (6.75), and (6.68) imply the third.

The Electric and the Magnetic Fields

The electric and the magnetic fields, Eqs.(6.44) and (6.45), are obtained for each type of e.m. field from the three respective vector potentials,

$$\begin{bmatrix} \phi \\ A_z \\ A_x \\ A_y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \partial_y \Phi^{TE} \\ -\partial_x \Phi^{TE} \end{bmatrix}}_{\substack{\text{Transverse} \\ \text{Electric}}}, \underbrace{\begin{bmatrix} -\partial_z \Phi^{TM} \\ \partial_t \Phi^{TM} \\ 0 \\ 0 \end{bmatrix}}_{\substack{\text{Transverse} \\ \text{Magnetic}}}, \underbrace{\begin{bmatrix} -\partial_t \Phi \\ \partial_z \Phi \\ \partial_x \Psi \\ \partial_y \Psi \end{bmatrix}}_{\substack{\text{Transverse} \\ \text{Electric} \\ \text{Magnetic}}} \ . \tag{6.76}$$

It needs to be reemphasized that for the TEM field it is unnecessary (and, in fact, uncalled for) to calculate the scalars Φ and Ψ individually. All that is necessary (and sufficient) is the difference $\Phi - \Psi$.

From the perspective of the principle of conceptual unit-economy[9] the e.m. potential is superior to the electric and magnetic field. This is because the mathematical characterization of the e.m. potential is simpler than that of the e.m. field. However, in the hierarchy of concepts, the e.m. field is much closer to the electromagnetism's foundation, namely that which is directly accessible to measurements. Thus, in order to comply with this hierarchy and prevent the e.m. potential from being a floating abstraction disconnected from reality, it is mandatory that one explicitly exhibit the e.m. field. We shall do this for TE, TM, and TEM fields relative to cartesian coordinates, and later extend the result to cylindrical and spherical coordinates.

[9] As identified in the footnote on Page 192

The TE Field: The result of deriving the e.m. field, Eqs.(6.44)-(6.45), from the TE potential in Eq.(6.76), together with the corresponding TE source, have been consolidated into Table 6.1. Any e.m. field of the type exhibited

TE Potential				TE Electric Field		
A_x	A_y	A_z	ϕ	E_x	E_y	E_z
$\frac{\partial \Phi^{TE}}{\partial y}$	$-\frac{\partial \Phi^{TE}}{\partial x}$	0	0	$-\frac{\partial}{\partial y}\frac{\partial \Phi^{TE}}{\partial t}$	$\frac{\partial}{\partial x}\frac{\partial \Phi^{TE}}{\partial t}$	0
TE Source				TE Magnetic Field		
J_x	J_y	J_z	ρ	B_x	B_y	B_z
$\frac{\partial S^{TE}}{\partial y}$	$-\frac{\partial S^{TE}}{\partial x}$	0	0	$\frac{\partial}{\partial x}\frac{\partial \Phi^{TE}}{\partial z}$	$\frac{\partial}{\partial y}\frac{\partial \Phi^{TE}}{\partial z}$	$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi^{TE}$

Table 6.1: The TE system: The components of a TE e.m. field $(\vec{E},\vec{B})$ are derived from a four-vector TE potential $(\vec{A},\phi)$, a solution the inhomogeneous Maxwell wave Eq.(6.52) on page 440. Its source is the divergenceless TE charge density-flux four-vector $(\vec{J},\rho)$.

in this table is called purely *transverse electric* (TE). This is because the electric field vector $\vec{E}$ is nonzero only in the transverse (x,y)-plane, the plane perpendicular to the longitudinal direction, the z-axis. The longitudinal electric field component,

$$E_{long.} \equiv E_z = 0 \ ,$$

of a TE electromagnetic field vanishes!

The TM Field: The result of deriving the e.m. field, Eqs.(6.44)-(6.45), from the TM potential in Eq.(6.76), together with the corresponding TM source, have been consolidated into Table 6.2. Any e.m. field of the type exhibited in this table is called purely *transverse magnetic* (TM). This is because the magnetic field vector $\vec{B}$ is nonzero only in the transverse (x,y)-plane, the plane perpendicular to the longitudinal direction, the z-axis: the longitudinal magnetic field component,

$$B_{long.} \equiv B_z = 0 \ ,$$

of a TM electromagnetic field vanishes!

TM Potential				TM Electric Field		
A_x	A_y	A_z	ϕ	E_x	E_y	E_z
0	0	$\frac{\partial \Phi^{TM}}{\partial t}$	$-\frac{\partial \Phi^{TM}}{\partial z}$	$\frac{\partial}{\partial x}\frac{\partial \Phi^{TM}}{\partial z}$	$\frac{\partial}{\partial y}\frac{\partial \Phi^{TM}}{\partial z}$	$\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}\right)\Phi^{TM}$
TM Source				TM Magnetic Field		
J_x	J_y	J_z	ρ	B_x	B_y	B_z
0	0	$\frac{\partial S^{TM}}{\partial t}$	$-\frac{\partial S^{TM}}{\partial z}$	$\frac{\partial}{\partial y}\frac{\partial \Phi^{TM}}{\partial t}$	$-\frac{\partial}{\partial x}\frac{\partial \Phi^{TM}}{\partial t}$	0

Table 6.2: The TM system: The components of a TM e.m. field $(\vec{E},\vec{B})$ are derived from a four-vector TM potential $(\vec{A},\phi)$, a solution the inhomogeneous Maxwell wave Eq.(6.52) on page 440. Its source is the divergenceless TM charge density-flux four-vector $(\vec{J},\rho)$.

Remark. Note that the TM field is the same as the TE field except that the roles of $\vec{E}$ and $\vec{B}$ are essentially reversed:

$\vec{E} \longrightarrow -\vec{B}$

$\vec{B} \longrightarrow \quad \vec{E}$ (whenever Φ^{TM} satisfies the sourceless wave equation)

TEM Potential				TEM Electric Field		
A_x	A_y	A_z	ϕ	E_x	E_y	E_z
$\frac{\partial \Psi}{\partial x}$	$\frac{\partial \Psi}{\partial y}$	$\frac{\partial \Phi}{\partial z}$	$-\frac{\partial \Phi}{\partial t}$	$\frac{\partial}{\partial x}\frac{\partial(\Phi-\Psi)}{\partial t}$	$\frac{\partial}{\partial y}\frac{\partial(\Phi-\Psi)}{\partial t}$	0
TEM Source				TEM Magnetic Field		
J_x	J_y	J_z	ρ	B_x	B_y	B_z
$\frac{\partial I^{TEM}}{\partial x}$	$\frac{\partial I^{TEM}}{\partial y}$	$\frac{\partial J}{\partial z}$	$-\frac{\partial J}{\partial t}$	$\frac{\partial}{\partial y}\frac{\partial(\Phi-\Psi)}{\partial z}$	$-\frac{\partial}{\partial x}\frac{\partial(\Phi-\Psi)}{\partial z}$	0

Table 6.3: The TEM system: The components of a TEM e.m. field $(\vec{E},\vec{B})$ are derived from a four-vector TEMpotential $(\vec{A},\phi)$, a solution the inhomogeneous Maxwell wave Eq.(6.52) on page 440. Its source is the divergenceless TEM charge flux-density four-vector $(\vec{J},\rho)$.

The TEM Field: The result of deriving the e.m. field, Eqs.(6.44)- (6.45), from the TEM potential in Eq.(6.76), together with the corresponding TEM source, have been consolidated into Table 6.3.

Any e.m. field of the type exhibited in the table is called purely *transverse electric and magnetic* (TEM). This is because both the $\vec{E}$ field

and the $\vec{B}$ field lie strictly in the trasverse (x, y)-plane. There are no longitudinal components:

$$E_{long.} \equiv E_z = 0 \quad \text{and} \quad B_{long.} \equiv B_z = 0 \; . \tag{6.77}$$

6.2.4 Cylindrical Coordinates

The benefits of the linear algebra viewpoint applied to Maxwell's equations can be extended by inspection from rectilinear cartesian to cylindrical coordinates. This is because the four-dimensional coordinate system lends itself to being decomposed into two orthogonal sets of coordinate surfaces. For cylindricals these are spanned by the *transverse* coordinates (r, θ) in the transverse plane, and the *longitudinal* coordinates (z, t) in the longitudinal plane.

The transition from a rectilinear to a cylindrical coordinate frame is based on the replacement of the following symbols:

$$dx \longrightarrow dr \quad ; \quad dy \longrightarrow r d\theta \tag{6.78}$$

$$\frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial r} \quad ; \quad \frac{\partial}{\partial y} \longrightarrow \frac{1}{r}\frac{\partial}{\partial \theta} \tag{6.79}$$

and

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \longrightarrow \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \; . \tag{6.80}$$

Such a replacement yields the vector field components relative to an orthonormal (o.n.) basis tangent to the coordinate lines. To emphasize this orthonormality, hats ($\hat{}$) are placed over the vector components.

This replacement is very powerful. It circumvents the necessity of having to repeat the previous calculations that went into exhibiting the individual components of Maxwell's TE, TM, and TEM systems of equations. We shall again take advantage of the power of this algorithm in the next section when we apply it to Maxwell's system relative to spherical coordinates.

Applying it within the context of cylindrical coordinates, one finds that the source and the vector potential four-vectors are as follows:

1. for a TE source

$$(\rho, \hat{J}_z, \hat{J}_r, \hat{J}_\theta) = \left(0, 0, \frac{1}{r}\frac{\partial S^{TE}}{\partial \theta}, -\frac{\partial S^{TE}}{\partial r}\right), \tag{6.81}$$

the solution to the Maxwell field equations has the form

$$(\phi, \hat{A}_z, \hat{A}_r, \hat{A}_\theta) = \left(0, 0, \frac{1}{r}\frac{\partial \Phi^{TE}}{\partial \theta}, -\frac{\partial \Phi^{TE}}{\partial r}\right); \tag{6.82}$$

2. for a TM source

$$(\rho, \hat{J}_z, \hat{J}_r, \hat{J}_\theta) = \left(-\frac{\partial S^{TM}}{\partial z}, \frac{\partial S^{TM}}{\partial t}, 0, 0\right), \tag{6.83}$$

the solution to the Maxwell field equations has the form

$$(\phi, \hat{A}_z, \hat{A}_r, \hat{A}_\theta) = \left(-\frac{\partial \Phi^{TM}}{\partial z}, \frac{\partial \Phi^{TM}}{\partial t}, 0, 0\right); \tag{6.84}$$

and

3. for a TEM source

$$(\rho, \hat{J}_z, \hat{J}_r, \hat{J}_\theta) = \left(-\frac{\partial J}{\partial t}, \frac{\partial J}{\partial z}, \frac{\partial I}{\partial r}, \frac{1}{r}\frac{\partial I}{\partial \theta}, \right), \tag{6.85}$$

the solution to the Maxwell field equations has the form

$$(\phi, \hat{A}_z, \hat{A}_r, \hat{A}_\theta) = \left(-\frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial z}, \frac{\partial \Psi}{\partial r}, \frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right). \tag{6.86}$$

The corresponding electromagnetic fields and their master scalar wave equations are then as follows:

The TE Field

The result of deriving the Maxwell TE electromagnetic field components Eqs.(6.44)-(6.45) from the TE potential Eq.(6.82), arising from the corresponding TE source – all relative to the o.n. cylindrical coordinate basis – have been consolidated into Table 6.4.

The TE master scalar Φ^{TE} from which this result is obtained satisfies the wave equation

$$\boxed{\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\Phi^{TE} = -4\pi S^{TE}}. \tag{6.87}$$

TE Potential			
$\hat{A}_r$	$\hat{A}_\theta$	$\hat{A}_z$	ϕ
$\frac{1}{r}\frac{\partial \Phi^{TE}}{\partial \theta}$	$-\frac{\partial \Phi^{TE}}{\partial r}$	0	0
TE Electric Field			
$\hat{E}_r$	$\hat{E}_\theta$	$\hat{E}_z$	
$-\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial \Phi^{TE}}{\partial t}$	$\frac{\partial}{\partial r}\frac{\partial \Phi^{TE}}{\partial t}$	0	
TE Magnetic Field			
$\hat{B}_r$	$\hat{B}_\theta$	$\hat{B}_z$	
$\frac{\partial}{\partial z}\frac{\partial \Phi^{TE}}{\partial r}$	$\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial \Phi^{TE}}{\partial z}$	$-\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\Phi^{TE}$	
TE Source			
$\hat{J}_r$	$\hat{J}_\theta$	$\hat{J}_z$	ρ
$\frac{1}{r}\frac{\partial S^{TE}}{\partial \theta}$	$-\frac{\partial S^{TE}}{\partial r}$	0	0

Table 6.4: The TE system: All components of any TE e.m. field $(\vec{E},\vec{B})$, as well as those of any four-vector TE potential $(\vec{A},\phi)$, are derived from a *single* master scalar function Φ^{TE}. Its source scalar S^{TE} determines the vectorial charge flux vector field, which is purely transverse, i.e. it is tangent to the set of nested cylinders.

The TM Field

The result of deriving the TM electromagnetic field components Eqs.(6.44)-(6.45) from the TM potential Eq.(6.84), arising from the corresponding TM source – all relative to the o.n. cylindrical coordinate basis – have been consolidated into Table 6.5.

The TM master scalar Φ^{TM} for these results satisfies the wave equation

$$\boxed{\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}\right)\Phi^{TM}=-4\pi S^{TM}}\,. \tag{6.88}$$

The TEM Field

The Maxwell TEM electromagnetic field components relative to the o.n. cylindrical coordinate basis, the corresponding vector potential and its source

TM Potential			
$\hat{A}_r$	$\hat{A}_\theta$	$\hat{A}_z$	ϕ
0	0	$\frac{\partial \Phi^{TM}}{\partial t}$	$-\frac{\partial \Phi^{TM}}{\partial z}$

TM Electric Field		
$\hat{E}_r$	$\hat{E}_\theta$	$\hat{E}_z$
$\frac{\partial}{\partial r}\frac{\partial \Phi^{TM}}{\partial z}$	$\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial \Phi^{TM}}{\partial z}$	$\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}\right)\Phi^{TM}$

TM Magnetic Field		
$\hat{B}_r$	$\hat{B}_\theta$	$\hat{B}_z$
$\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial \Phi^{TM}}{\partial t}$	$-\frac{\partial}{\partial r}\frac{\partial \Phi^{TM}}{\partial t}$	0

TM Source			
$\hat{J}_r$	$\hat{J}_\theta$	$\hat{J}_z$	ρ
0	0	$\frac{\partial r^2 S^{TM}}{\partial t}$	$-\frac{\partial r^2 S^{TM}}{\partial z}$

Table 6.5: The TM system: All components of any TM e.m. field $(\vec{E},\vec{B})$, as well as those of any four-vector TM potential $(\vec{A},\phi)$, are derived from a *single* master scalar function Φ^{TM}. Its source scalar S^{TM} determines the vectorial charge flux vector field, which is purely longitudinal.

have been consolidated into Table 6.6.

The underlying master scalar is the difference function $\Phi-\Psi$. It satisfies the two separate equations. The first is an equation for the two-dimensional amplitude profile in the transverse plane,

$$\boxed{\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\Phi-\Psi)=-4\pi J \quad .} \tag{6.89}$$

The second is the equation for the propagation of this profile along the z-direction,

$$\boxed{\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}\right)(\Phi-\Psi)= \quad 4\pi I \quad .} \tag{6.90}$$

these two equations are consistent because the source satisfies the charge conservation law

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)I+\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}\right)J=0 \ ,$$

TEM Potential			
$\hat{A}_r$	$\hat{A}_\theta$	$\hat{A}_z$	ϕ
$\frac{\partial \Psi}{\partial r}$	$\frac{1}{r}\frac{\partial \Psi}{\partial \theta}$	$\frac{\partial \Phi}{\partial z}$	$-\frac{\partial \Phi}{\partial t}$

TEM Electric Field		
$\hat{E}_r$	$\hat{E}_\theta$	$\hat{E}_z$
$\frac{\partial}{\partial r}\frac{\partial(\Phi-\Psi)}{\partial t}$	$\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial(\Phi-\Psi)}{\partial t}$	0

TEM Magnetic Field		
$\hat{B}_r$	$\hat{B}_\theta$	$\hat{B}_z$
$\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial(\Phi-\Psi)}{\partial z}$	$-\frac{\partial}{\partial \theta}\frac{\partial(\Phi-\Psi)}{\partial z}$	0

TEM Source			
$\hat{J}_r$	$\hat{J}_\theta$	$\hat{J}_z$	ρ
$\frac{\partial I}{\partial r}$	$\frac{1}{r}\frac{\partial I}{\partial \theta}$	$\frac{\partial J}{\partial z}$	$-\frac{\partial J}{\partial t}$

Table 6.6: The TEM system: All components of any TEM e.m. field $(\vec{E}, \vec{B})$ are derived from a *single* master scalar function, the difference $\Phi - \Psi$ between the two scalar functions. Even though both, separately, are necessary for the definition of the TEM vector potential $(\vec{A}, \phi)$, it is only their difference which is determined by an inhomogeneous Poisson equation and an inhomogeneous wave equation, Eqs.(6.89) and (6.90).

which is the polar coordinate version of Eq.(6.68).

6.2.5 Spherical Coordinates

One of the chief virtues of the linear algebra viewpoint applied to Maxwell's equations is that it directs attention to the system's fundamental vector spaces and their properties. The easiest way to identify them in a computational way happens when the underlying coordinate system permits a 2+2 decomposition into what amounts to longitudinal and transverse surfaces. Spherical coordinates provide a nontrivial example of this. There a transverse surface is a sphere spanned by (θ, ϕ), while the longitudinal coordinates are (r, t).

The distinguishing feature of spherical coordinates, as compared to recti-

linear or cylindrical coordinates, is that coordinate rectangles on successive transverse surfaces (nested spheres) are not congruent. Instead, they have areas that scale with the square of the radial distance from the origin. This scaling alters the representation of the divergence of a vector field and hence the Maxwell wave operator. Nevertheless, the eigenvalue method with its resulting TE-TM-TEM decomposition of the e.m. field readily accomodates these alterations.

Left Null Space

The key to success is to identify the divergence

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

as an element of the to-be-diagonalized Maxwell wave operator. Relative to the o.n. spherical coordinate basis $\{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\}$ one has

$$\partial_t(1 \cdot \rho) + \frac{1}{r^2}\partial_r(r^2 \cdot \hat{J}_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta \cdot \hat{J}_\theta) + \frac{1}{r\sin\theta}\partial_\varphi(1 \cdot \hat{J}_\phi) = 0 \quad (6.91)$$

or

$$\vec{\mathcal{U}}_\ell^T \begin{bmatrix} \rho \\ \hat{J}_r \\ \hat{J}_\theta \\ \hat{J}_\varphi \end{bmatrix} = 0 \ . \quad (6.92)$$

Here

$$\vec{\mathcal{U}}_\ell^T = [\partial_t \quad \frac{1}{r^2}\partial_r r^2 \quad \frac{1}{r\sin\theta}\partial_\theta \sin\theta \quad \frac{1}{r\sin\theta}\partial_\varphi] \quad (6.93)$$

is the left nullspace element of $\mathcal{A}$, the spherical representative of Eq.(6.50) on page 439. By inspecting the above four-dimensional divergence expression one readily identifies the following three divergenceless independent 4-vector

fields

$$\begin{bmatrix} \rho \\ \hat{J}_r \\ \\ \hat{J}_\theta \\ \\ \hat{J}_\varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{r\sin\theta}\frac{\partial S^{TE}}{\partial\varphi} \\ -\frac{1}{r}\frac{\partial S^{TE}}{\partial\theta} \end{bmatrix}, \begin{bmatrix} -\frac{1}{r^2}\frac{\partial\ (r^2 S^{TM})}{\partial r} \\ \\ \frac{1}{r^2}\frac{\partial\ (r^2 S^{TM})}{\partial t} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \\ 0 \\ \frac{1}{r}\frac{\partial\ I}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial I}{\partial\varphi} \end{bmatrix} + \begin{bmatrix} -\frac{1}{r^2}\frac{\partial\ (r^2 J)}{\partial t} \\ \\ \frac{1}{r^2}\frac{\partial\ (r^2 J)}{\partial r} \\ 0 \\ 0 \end{bmatrix}. \tag{6.94}$$

Here S^{TE} and S^{TM} are arbitrary scalar functions, while J and I are required to satisfy

$$\frac{\partial^2(r^2 J)}{\partial r^2} - \frac{\partial^2(r^2 J)}{\partial t^2} + \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\ I}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 I}{\partial\varphi^2}\right) = 0\ ,$$

the spherical coordinate version of Eq.(6.68)

Eigenvector Fields

According to the method of eigenvalues and eigenvectors we wish to arrive at a solution to

$$\mathcal{A}\begin{bmatrix} \phi \\ \hat{A}_r \\ \hat{A}_\theta \\ \hat{A}_\varphi \end{bmatrix} = 4\pi\begin{bmatrix} \rho \\ \hat{J}_r \\ \hat{J}_\theta \\ \hat{J}_\varphi \end{bmatrix} \tag{6.95}$$

in terms of those eigenvectors of Maxwell's wave operator $\mathcal{A}$ which are proportional to the divergenceless sources in Eq.(6.94). We achieve this goal by first validating that the vector fields

$$\begin{bmatrix} \phi \\ \hat{A}_r \\ \hat{A}_\theta \\ \hat{A}_\varphi \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{r\sin\theta}\ \partial_\varphi\Phi^{TE} \\ -\frac{1}{r}\ \partial_\theta\Phi^{TE} \end{bmatrix}}_{\equiv\vec{\mathcal{V}}^{(1)}\Phi^{TE}}, \underbrace{\begin{bmatrix} -\partial_r\ \Phi^{TM} \\ \partial_t\ \Phi^{TM} \\ 0 \\ 0 \end{bmatrix}}_{\equiv\vec{\mathcal{V}}^{(2)}\Phi^{TM}}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{r}\ \partial_\theta\ \Psi \\ \frac{1}{r\sin\theta}\ \partial_\varphi\Psi \end{bmatrix} + \begin{bmatrix} -\partial_t\ \Phi \\ \partial_r\ \Phi \\ 0 \\ 0 \end{bmatrix}}_{\equiv\vec{\mathcal{W}}^{(3)}\Psi + \vec{\mathcal{W}}^{(4)}\Phi}. \tag{6.96}$$

are invariant under $\mathcal{A}$, i.e. there exist functions F^{TE}, F^{TM}, G and H such that

$$\mathcal{A}\vec{\mathcal{V}}^{(1)}\Phi^{TE} = \alpha\vec{\mathcal{V}}^{(1)}F^{TE} \tag{6.97}$$

$$\mathcal{A}\vec{\mathcal{V}}^{(2)}\Phi^{TM} = \beta\vec{\mathcal{V}}^{(2)}F^{TM} \tag{6.98}$$

$$\mathcal{A}[\vec{\mathcal{W}}^{(3)}\Psi + \vec{\mathcal{W}}^{(4)}\Phi] = \gamma\vec{\mathcal{W}}^{(3)}G + \delta\ \vec{\mathcal{W}}^{(4)}H\ , \tag{6.99}$$

where α, β, γ, and δ are unique coefficients which are determined by $\mathcal{A}$. The verification of the form of these three equations and the determination of their scalars F^{TE}, F^{TM}, G and H in terms of $\Phi^{TE}, \Phi^{TM}, \Phi$ and Ψ is not unfamiliar. It constitutes relative to spherical coordinates what was done in Section 6.2.3 on page 446 relative to cartesian coordinates.

There, we recall, we diagonalized the operator $\mathcal{A}$ by exhibiting the TE, TM, and the TEM eigenvector fields and their respective eigenvalues, all relative to cartesian coordinates. Here we shall do the same relative to spherical coordinates.

At first sight this seems like a computationally intense task, especially because one has to calculate the curl of a curl, $\nabla \times \nabla \times \vec{A}$, in Eq.(6.47) relative to these coordinates. However, the task becomes managable, in fact, downright pleasant, if one extends to curvilinear coordinates the familiar determinantal expression for the curl,

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} .$$

Suppose one has orthogonal curvilinear coordinates (x^1, x^2, x^3) whose scale factors are $h_i(x^1, x^2, x^3),\ i = 1, 2, 3$. Then one has

$$dx^2 + dy^2 + dz^2 = h_1^2(dx^1)^2 + h_2^2(dx^2)^2 + h_3^2(dx^3)^2\ ,$$

and the determinantal expression for the curl is[10]

$$\nabla \times \vec{A} = \begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \partial_{x^1} & \partial_{x^2} & \partial_{x^3} \\ h_1\hat{A}_1 & h_2\hat{A}_2 & h_3\hat{A}_3 \end{vmatrix} \frac{1}{h_1h_2h_3}\ .$$

[10]This result is a consequence of Stoke's theorem applied to infinitesimal elements of area expressed in terms of these curvilinear coordinates.

Here $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are the o.n. basis vectors tangent to the coordinate lines, and $\hat{A}_1, \hat{A}_2, \hat{A}_3$ are the components of $\vec{A}$ relative to this o.n. basis. Relative to spherical coordinates one has therefore

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

and

$$\nabla \times \vec{A} = \left| \begin{array}{ccc} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\varphi \\ \partial_r & \partial_\theta & \partial_\varphi \\ \hat{A}_r & r\hat{A}_\theta & r\sin\theta\,\hat{A}_\varphi \end{array} \right| \frac{1}{r^2 \sin\theta} \ . \tag{6.100}$$

To exhibit the TE, TM, and TEM eigenvector fields, one inserts each of the vector potential four-vector fields, Eq.(6.96) into Eqs.(6.47) on page 438 and uses Eq.(6.100). One also uses the corresponding sources, Eq.(6.94). The result is as follows:

TE:

$$\left[\begin{array}{c} 0 \\ 0 \\ \frac{1}{r\sin\theta}\,\partial_\varphi \\ -\frac{1}{r}\,\partial_\theta \end{array} \right] \left\{ -\partial_t^2 + \partial_r^2 + \frac{1}{r^2} \left(\frac{1}{\sin\theta} \partial_\theta \sin\theta \, \partial_\theta + \frac{1}{\sin^2\theta} \right) \right\} \Phi^{TE}$$

$$= 4\pi \left[\begin{array}{c} 0 \\ 0 \\ \frac{1}{r\sin\theta}\,\partial_\varphi \\ -\frac{1}{r}\,\partial_\theta \end{array} \right] S^{TE} \ , \tag{6.101}$$

TM:

$$\frac{1}{r^2} \left[\begin{array}{c} \partial_r \\ -\partial_t \\ 0 \\ 0 \end{array} \right] r^2 \left\{ -\partial_t^2 + \partial_r^2 + \frac{1}{r^2} \left(\frac{1}{\sin\theta} \partial_\theta \sin\theta \, \partial_\theta + \frac{1}{\sin^2\theta} \right) \right\} \Phi^{TM}$$

$$= 4\pi \frac{1}{r^2} \left[\begin{array}{c} -\partial_r \\ \partial_t \\ 0 \\ 0 \end{array} \right] \left(r^2 S^{TM} \right) \ , \tag{6.102}$$

TEM:

$$\begin{bmatrix} 0 \\ 0 \\ \frac{1}{r}\,\partial_\theta \\ \frac{1}{r\sin\theta}\,\partial_\varphi \end{bmatrix} \left\{-\partial_t^2 + \partial_r^2\right\}(\Phi - \Psi) = 4\pi \begin{bmatrix} 0 \\ 0 \\ \frac{1}{r}\,\partial_\theta \\ \frac{1}{r\sin\theta}\,\partial_\varphi \end{bmatrix} I \tag{6.103}$$

$$\frac{-1}{r^2}\begin{bmatrix} -\partial_t \\ \partial_r \\ 0 \\ 0 \end{bmatrix}\left(\frac{1}{\sin\theta}\partial_\theta \sin\theta\ \partial_\theta + \frac{1}{\sin^2\theta}\right)(\Phi - \Psi) = 4\pi\frac{1}{r^2}\begin{bmatrix} -\partial_t \\ \partial_r \\ 0 \\ 0 \end{bmatrix} J\ . \tag{6.104}$$

These three systems of vector field equations constitute a step forward in one's understanding of Maxwell's equations. This is because each system, which can be integrated by inspection, yields three master wave equations which

1. are decoupled and hence independent,

2. are inhomogeneous scalar wave equations, each one with its own scalar source,

3. can be solved with the methods developed in Chpter 5 and 6,

4. have solutions from which one derives the three (TE, TM, and TEM) types e.m. fields corresponding to the three types of concomitant e.m. sources.

These three master scalar equations are

$$\textbf{TE:}\left\{-\partial_t^2+\partial_r^2+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\partial_\theta\sin\theta\;\partial_\theta+\frac{1}{\sin^2\theta}\partial_\varphi^2\right)\right\}\Phi^{TE}=-4\pi S^{TE} \tag{6.105}$$

$$\textbf{TM:}\left\{-\partial_t^2+\partial_r^2+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\partial_\theta\sin\theta\;\partial_\theta+\frac{1}{\sin^2\theta}\partial_\varphi^2\right)\right\}\Phi^{TM}=-4\pi S^{TM} \tag{6.106}$$

$$\textbf{TEM:}\qquad \left(-\partial_t^2+\partial_r^2\right)(\Phi-\Psi)=+4\pi I \tag{6.107}$$

$$\left(\frac{1}{\sin\theta}\partial_\theta\sin\theta\;\partial_\theta+\frac{1}{\sin^2\theta}\partial_\varphi^2\right)(\Phi-\Psi)=-4\pi J \tag{6.108}$$

The TEM system results in a pair coupled differential equations. However, they are integrable. Their source functions satisfy

$$\left(-\partial_t^2+\partial_r^2\right)(r^2J)=-\left(\frac{1}{\sin\theta}\partial_\theta\sin\theta\;\partial_\theta+\frac{1}{\sin^2\theta}\right)I\;. \tag{6.109}$$

This is because the left nullspace element $\vec{\mathcal{U}}_\ell^T$, Eq.(6.93), annihilates the sum of the two TEM vectors, Eqs.(6.103) and (6.104). This guarantees that one can find a function $\Phi-\Psi$ which satisfies Eqs.(6.107) *and* (6.108). The validation of this claim is consigned to Exercise 6.2.7 on page 469.

Whereas the TE and the TM systems are characterized by a single wave equation in 3+1 dimensions, the TEM system is characterized by two different problems in the form of two independent equations:

- A potential problem expressed by Poisson's equation on the transverse (θ,ϕ)-surface, and
- a wave propagation problem expressed by the wave equation on the longitudinal (r,t)-plane.

The domain of these two problems are orthogonal and independent, but their solutions are not. In fact, they are one and the same. This means that the existence of a solution $\Phi-\Psi$ implies that the source scalars J and I are not independent either. Instead, they are related so as to guarantee that the law of charge conservation $\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$, i.e. Eq.(6.109), is satisfied.

Conversely, as shown in Exercise 6.2.7, this conservation law implies the existence of a solution, $\Phi-\Psi$, to the two differential equations.

6.2.6 Electromagnetic Fields in a Spherical Coordinate System

TE Potential			
$\hat{A}_\theta$	$\hat{A}_\phi$	$\hat{A}_r$	ϕ
$\frac{1}{r\sin\theta}\frac{\partial\Phi^{TE}}{\partial\phi}$	$-\frac{1}{r}\frac{\partial\Phi^{TE}}{\partial\theta}$	0	0
TE Electric Field			
$\hat{E}_\theta$	$\hat{E}_\phi$	$\hat{E}_r$	
$-\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial\Phi^{TE}}{\partial t}$	$\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\Phi^{TE}}{\partial t}$	0	
TE Magnetic Field			
$\hat{B}_\theta$	$\hat{B}_\phi$	$\hat{B}_r$	
$\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial\Phi^{TE}}{\partial r}$	$\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial\Phi^{TE}}{\partial r}$	$-\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\Phi^{TE}$	
TE Source			
$\hat{J}_\theta$	$\hat{J}_\phi$	$\hat{J}_r$	ρ
$\frac{1}{r\sin\theta}\frac{\partial S^{TE}}{\partial\phi}$	$-\frac{1}{r}\frac{\partial S^{TE}}{\partial\theta}$	0	0

Table 6.7: The TE system: All components of any TE e.m. field $(\vec{E},\vec{B})$, as well as those of any four-vector TE potential $(\vec{A},\phi)$, are derived from a *single* master scalar function Φ^{TE}. Its source scalar S^{TE} determines the vectorial charge flux vector field. It is purely transverse: it is tangent to the set of nested two-spheres.

A spherical coordinate system induces a decomposition into a set of nested transverse manifolds (concentric spheres) spanned by the angular coordinates and a longitudinal manifold spanned by the radial and the time cordinates.

Such a coordinate decomposition induces a corresponding one in the Maxwell field equation. Following our experience with cylindrical coordinates, we make a corresponding transition to spherical coordinates according to the following heuristic replacement recipe:

$$dx \to r d\theta; \quad dy \to r\sin\theta\, d\phi; \quad dz \to dr; \quad dt \to dt$$
$$\frac{\partial}{\partial x} \to \frac{1}{r}\frac{\partial}{\partial\theta}; \quad \frac{\partial}{\partial y} \to \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}; \quad \frac{\partial}{\partial z} \to \frac{\partial}{\partial r}; \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t}$$

TM Potential			
$\hat{A}_\theta$	$\hat{A}_\phi$	$\hat{A}_r$	ϕ
0	0	$\frac{\partial \Phi^{TM}}{\partial t}$	$-\frac{\partial \Phi^{TM}}{\partial r}$
TM Electric Field			
$\hat{E}_\theta$	$\hat{E}_\phi$	$\hat{E}_r$	
$\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial \Phi^{TM}}{\partial r}$	$\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{\partial \Phi^{TM}}{\partial r}$	$\left(\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial t^2}\right)\Phi^{TM}$	
TM Magnetic Field			
$\hat{B}_\theta$	$\hat{B}_\phi$	$\hat{B}_r$	
$\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{\partial \Phi^{TM}}{\partial t}$	$-\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial \Phi^{TM}}{\partial t}$	0	
TM Source			
$\hat{J}_\theta$	$\hat{J}_\phi$	$\hat{J}_r$	ρ
0	0	$\frac{1}{r^2}\frac{\partial r^2 S^{TM}}{\partial t}$	$-\frac{1}{r^2}\frac{\partial r^2 S^{TM}}{\partial r}$

Table 6.8: The TM system: All components of any TM e.m. field $(\vec{E},\vec{B})$, as well as those of any four-vector TM potential $(\vec{A},\phi)$, are derived from a *single* master scalar function Φ^{TM}. Its source scalar S^{TM} determines the vectorial charge flux vector field, which is purely longitudinal.

and

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\rightarrow\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

As already mentioned, once the cartesian components of Maxwell's TE, TM, and TEM system have been exhibited explicitly, one can apply this recipe also to spherical coordinates. The results are given in Tables 6.7, 6.8, and 6.9 respectively.

The recipe guarantees that all electric and magnetic field components in these tables satisfy the first half, Eq.(6.40)-(6.41), of Maxwell's field equations. Furthermore, the application of this recipe to the TE, TM, and TEM cartesian master scalar Eqs.(6.72), (6.73), (6.74), and (6.75), yields Eqs.(6.105)-(6.108), the corresponding master equations relative to spherical coordinates. The above replacement recipe applies to the e.m. field and its vector potential.

However, the relation between the concentric spheres introduces the squared

TEM Potential			
$\hat{A}_\theta$	$\hat{A}_\phi$	$\hat{A}_r$	ϕ
$\frac{1}{r}\frac{\partial\Psi}{\partial\theta}$	$\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}$	$\frac{\partial\Phi}{\partial r}$	$-\frac{\partial\Phi}{\partial t}$

TEM Electric Field		
$\hat{E}_\theta$	$\hat{E}_\phi$	$\hat{E}_r$
$\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial(\Phi-\Psi)}{\partial t}$	$\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial(\Phi-\Psi)}{\partial t}$	0

TEM Magnetic Field		
$\hat{B}_\theta$	$\hat{B}_\phi$	$\hat{B}_r$
$\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial(\Phi-\Psi)}{\partial r}$	$-\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial(\Phi-\Psi)}{\partial r}$	0

TEM Source			
$\hat{J}_\theta$	$\hat{J}_\phi$	$\hat{J}_r$	ρ
$\frac{1}{r}\frac{\partial I}{\partial\theta}\frac{1}{r^2}$	$\frac{1}{r\sin\theta}\frac{\partial I}{\partial\phi}\frac{1}{r^2}$	$\frac{\partial J}{\partial r}\frac{1}{r^2}$	$-\frac{\partial J}{\partial t}\frac{1}{r^2}$

Table 6.9: The TEM system: All components of any TEM e.m. field $(\vec{E},\vec{B})$ are derived from a *single* master scalar function, the difference $\Psi-\Phi$ between the two scalar functions. Even though both, separately, are necessary for the definition of the TEM vector potential $(\vec{A},\phi)$, it is only their difference which is determined by an inhomogeneous Poisson equation and an inhomogeneous wave equation, Eqs.(6.107) and (6.108).

radius as a conformal factor between their squared elements of arclength and hence their areas. This conformal factor enters only into the the TM source and the longitudinal part of the TEM source, and hence does not seem to be under the purview of the above recipe. It is, however, taken into account by the explicit calculations that lead to Eqs.(6.101)-(6.108).

Exercise 6.2.2 (Existence and Uniqueness of the 2+2 Decomposition)

a) Exhibit the partial differential equation which each of the scalars $\Phi,\cdots,\Phi^{TM}$ satisfies, point out why each solution is unique and hence why

$$[\phi,A_z,A_x,A_y]^T\leftrightarrow(\Phi,\Phi^{TE},\Psi,\Phi^{TM}) \tag{6.110}$$

is a one-to-one mapping.

Solution: Given $[\phi, A_z, A_x, A_y]^T$, one has the following system of equations for the scalars Φ and Φ^{TE},

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi^{TE}}{\partial y} = A_x \tag{6.111}$$

$$\frac{\partial \Phi}{\partial y} - \frac{\partial \Phi^{TE}}{\partial x} = A_y \ . \tag{6.112}$$

Taking the two-dimensional curl and divergence of this system, one finds

$$\frac{\partial^2 \Phi^{TE}}{\partial x^2} + \frac{\partial^2 \Phi^{TE}}{\partial y^2} = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \tag{6.113}$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \ . \tag{6.114}$$

With appropriate boundary conditions in the x-y plane, these 2-d Poisson equations have unique scalar solutions Φ^{TE} and Φ. Similarly one obtains

$$\frac{\partial^2 \Phi^{TM}}{\partial t^2} - \frac{\partial^2 \Phi^{TM}}{\partial z^2} = \frac{\partial A_z}{\partial t} + \frac{\partial \phi}{\partial z} \tag{6.115}$$

$$\frac{\partial^2 \Psi}{\partial z^2} - \frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial A_z}{\partial z} + \frac{\partial \phi}{\partial t} \ . \tag{6.116}$$

$$\tag{6.117}$$

With appropriate initial conditions in the z-t plane, these inhomogeneous 2-d wave equations have unique scalar solutions Φ^{TM} and Ψ.

On the other hand, given the four scalar fields, Eq.(6.64) implies the unique four-vector field $(\phi, \vec{A})$. Thus Eq.(6.110) is a one-to-one mapping indeed.

b) Point out why the four vectors

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ \partial_y \\ -\partial_x \end{bmatrix} , \begin{bmatrix} -\partial_z \\ \partial_t \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} -\partial_t \\ \partial_z \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 0 \\ \partial_x \\ \partial_y \end{bmatrix} \right\} \tag{6.118}$$

form a *linearly independent* set, i.e. why the only solution to

$$\begin{bmatrix} 0 \\ 0 \\ \partial_y \\ -\partial_x \end{bmatrix} \Phi^{TE} + \begin{bmatrix} -\partial_z \\ \partial_t \\ 0 \\ 0 \end{bmatrix} \Phi^{TM} + \begin{bmatrix} -\partial_t \\ \partial_z \\ 0 \\ 0 \end{bmatrix} \Phi + \begin{bmatrix} 0 \\ 0 \\ \partial_x \\ \partial_y \end{bmatrix} \Psi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.119}$$

is the trivial one, $\Phi^{TE} = \Phi^{TM} = \Phi = \Psi \equiv 0$.

c) Show that the set of vectors

$$\left\{ \vec{\mathcal{V}}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ \partial_y \\ -\partial_x \end{bmatrix}, \ \vec{\mathcal{V}}^{(2)} = \begin{bmatrix} -\partial_z \\ \partial_t \\ 0 \\ 0 \end{bmatrix}, \ \vec{\mathcal{V}}^{(3)} = \begin{bmatrix} \partial_t c \\ -\partial_z c \\ \partial_x d \\ \partial_y d \end{bmatrix}, \ \vec{\mathcal{U}}_r = \begin{bmatrix} -\partial_t \\ \partial_z \\ \partial_x \\ \partial_y \end{bmatrix} \right\}, \tag{6.120}$$

where

$$c = \partial_x^2 + \partial_y^2 \tag{6.121}$$

and

$$d = \partial_z^2 - \partial_t^2 \ , \tag{6.122}$$

also forms a linearly independent set.

Exercise 6.2.3 (TE SCALAR WAVE EQUATION: ITS MAXWELL ORIGIN) Consider a TE e.m. potential and its source,

$$[\phi, A_z, A_x, A_y] = [0, 0, \partial_y \Phi^{TE}, -\partial_x \Phi^{TE}] \tag{6.123}$$

$$[\rho, J_z, J_x, J_y] = [0, 0, \partial_y S^{TE}, -\partial_x S^{TE}] \ . \tag{6.124}$$

a) Which two of the Maxwell field equations

$$\nabla \cdot \vec{E} = 4\pi\rho$$
$$\nabla \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{J}$$

are satisfied trivially ($0 = 0$), and which imply the nontrivial result

$$\frac{\partial}{\partial y}\{\cdots\}^{TE} = 0$$
$$\frac{\partial}{\partial x}\{\cdots\}^{TE} = 0 \ ?$$

b) What is $\{\cdots\}^{TE}$?

Solution.
a) Introducing the $\vec{E}$-field, the $\vec{B}$-field, and the charge flux-density $(\vec{J}, \rho)$ into Eqs.(6.42) and (6.43) yields the following results:

$$\nabla \cdot \vec{E} = 4\pi\rho \ : \ 0 = 0$$
$$(\nabla \times \vec{B})_z - \frac{\partial E_z}{\partial t} = 4\pi J_z : \ 0 = 0$$

$$(\nabla \times \vec{B})_x - \frac{\partial E_x}{\partial t} = 4\pi J_x \ :$$
$$\frac{\partial}{\partial y}(-)\left(\frac{\partial^2 \Phi^{TE}}{\partial x^2} + \frac{\partial^2 \Phi^{TE}}{\partial y^2}\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\frac{\partial \Phi^{TE}}{\partial z}\right)$$
$$- \frac{\partial}{\partial t}\left(\frac{\partial}{\partial y}(-)\frac{\partial \Phi^{TE}}{\partial t}\right) = 4\pi \frac{\partial S^{TE}}{\partial y}$$

or equivalently

$$\frac{\partial}{\partial y}\left\{(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2)\Phi^{TE} + 4\pi S^{TE}\right\} = 0 \ . \tag{6.125}$$

Similarly, and finally,

$$(\nabla \times \vec{B})_y - \frac{\partial E_y}{\partial t} = 4\pi J_y$$

yields

$$\frac{\partial}{\partial x}\left\{(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2)\Phi^{TE} + 4\pi S^{TE}\right\} = 0. \tag{6.126}$$

b) $\{\cdots\}^{TE} \equiv \left(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2\right)\Phi^{TE} + 4\pi S^{TE}$.

Exercise 6.2.4 (TM SCALAR WAVE EQUATION: ITS MAXWELL ORIGIN) Consider a TM e.m. potential and its source,

$$[\phi, A_z, A_x, A_y] = [0, 0, \partial_y \Phi^{TM}, -\partial_x \Phi^{TM}] \tag{6.127}$$
$$[\rho, J_z, J_x, J_y] = [0, 0, \partial_y S^{TM}, -\partial_x S^{TM}] \ . \tag{6.128}$$

a) Which two of the Maxwell field equations

$$\nabla \cdot \vec{E} = 4\pi\rho$$
$$\nabla \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{J}$$

are satisfied trivially ($0 = 0$), and which imply the nontrivial result

$$\frac{\partial}{\partial z}\{\cdots\}^{TM} = 0$$
$$\frac{\partial}{\partial t}\{\cdots\}^{TM} = 0 \ ?$$

b) What is $\{\cdots\}^{TM}$?

Solution.
a) Introducing the $\vec{E}$-field, the $\vec{B}$-field, and the charge flux-density $(\vec{J}, \rho)$ into Eqs.(6.42) and (6.43) yields the following result:

$$\nabla \cdot \vec{E} = 4\pi\rho \ : \ \frac{\partial}{\partial z}\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right\}\Phi^{TM} = -4\pi\frac{\partial S^{TM}}{\partial z}$$

or equivalently

$$\frac{\partial}{\partial z}\left\{(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2)\Phi^{TM} + 4\pi S^{TM}\right\} = 0 \ . \tag{6.129}$$

$$(\nabla \times \vec{B})_x - \frac{\partial E_x}{\partial t} = 4\pi J_x\text{:} \quad 0 = 0$$
$$(\nabla \times \vec{B})_y - \frac{\partial E_y}{\partial t} = 4\pi J_y\text{:} \quad 0 = 0$$

$$(\nabla \times \vec{B})_z - \frac{\partial E_z}{\partial t} = 4\pi J_z \ :$$
$$\frac{\partial}{\partial x}(-)\frac{\partial^2\Phi^{TM}}{\partial x\,\partial t} - \frac{\partial}{\partial y}\frac{\partial^2\Phi^{TM}}{\partial y\,\partial t} - \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\Phi^{TM} = 4\pi\frac{\partial S^{TM}}{\partial t}$$

or equivalently

$$\frac{\partial}{\partial t}\left\{(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2)\Phi^{TM} + 4\pi S^{TM}\right\} = 0 \ . \tag{6.130}$$

b) $\{\cdots\}^{TM} \equiv \left(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2\right)\Phi^{TM} + 4\pi S^{TM}$.

Exercise 6.2.5 (TEM SCALAR WAVE EQUATIONS: THEIR MAXWELL ORIGIN) Consider a TEM e.m. potential and its source,

$$[\phi, A_z, A_x, A_y] = [-\partial_t\Phi, \partial_z\Phi, \partial_x\Psi, \partial_y\Psi] \tag{6.131}$$
$$[\rho, J_z, J_x, J_y] = [-\partial_t J, \partial_z J, \partial_x I, \partial_y I] \ . \tag{6.132}$$

a) Which two of the Maxwell field equations

$$\nabla \cdot \vec{E} = 4\pi\rho$$
$$\nabla \times \vec{B} - \partial_t\vec{E} = 4\pi\vec{J}$$

imply

$$\frac{\partial}{\partial t}\{\cdots\}^{TEM} = 0$$
$$\frac{\partial}{\partial z}\{\cdots\}^{TEM} = 0\ ,$$

and which two imply

$$\frac{\partial}{\partial x}[\cdots]^{TEM} = 0$$
$$\frac{\partial}{\partial y}[\cdots]^{TEM} = 0\ ?$$

b) What is $\{\cdots\}^{TEM}$? What is $[\cdots]^{TEM}$?

Solution.
a) Introducing the $\vec{E}$-field, the $\vec{B}$-field, and the charge flux-density $(\vec{J},\rho)$ into Eqs.(6.42) and (6.43) yields the following result:

$$\nabla\cdot\vec{E} = 4\pi\rho: \qquad \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\Phi-\Psi) = 4\pi(-)\frac{\partial J}{\partial t} \tag{6.133}$$

$$(\nabla\times\vec{B})_z - \frac{\partial E_z}{\partial t} = 4\pi J_z: \ -\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\Phi-\Psi) = 4\pi\frac{\partial J}{\partial z} \tag{6.134}$$

$$(\nabla\times\vec{B})_x - \frac{\partial E_x}{\partial t} = 4\pi J_x: \qquad \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}\right)(\Phi-\Psi) = 4\pi\frac{\partial I}{\partial x} \tag{6.135}$$

$$(\nabla\times\vec{B})_y - \frac{\partial E_y}{\partial t} = 4\pi J_y: \qquad \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}\right)(\Phi-\Psi) = 4\pi\frac{\partial I}{\partial y} \tag{6.136}$$

b) $\{\cdots\}^{TEM} \equiv \left(\partial_x^2+\partial_y^2\right)(\Phi-\Psi)+4\pi J$;
$[\cdots]^{TEM} \equiv \left(\partial_z^2-\partial_t^2\right)(\Phi-\Psi)-4\pi I$.

Exercise 6.2.6 (TE, TM, AND TEM SCALAR WAVE EQUATIONS)
Point out why the previous three exercises imply

$$\left(\partial_x^2+\partial_y^2+\partial_z^2-\partial_t^2\right)\Phi^{TE} = -4\pi S^{TE} \tag{6.137}$$
$$\left(\partial_x^2+\partial_y^2+\partial_z^2-\partial_t^2\right)\Phi^{TM} = -4\pi S^{TM} \tag{6.138}$$

and

$$(\partial_x^2+\partial_y^2)(\Phi-\Psi) = -4\pi J \tag{6.139}$$
$$(\partial_z^2-\partial_t^2)(\Phi-\Psi) = +4\pi I\ . \tag{6.140}$$

Exercise 6.2.7 (TEM MASTER SCALAR SYSTEM IS INTEGRABLE)
Show that if I and J satisfy Eq.(6.68) then there exists a scalar, call it $\Phi - \Psi$, such that Eqs.(6.139) and (6.140) are satisfied.
Hint: Use Green's function.

Exercise 6.2.8 (MAGNETIC DIPOLE MOMENT AS A TE FIELD SOURCE)
The total energy of a charge flux and charge density distribution $(\vec{J}, \rho)$ interacting with the electromagnetic potential $(\vec{A}, \phi)$ is

$$W = \frac{1}{2}\int\int\int (\vec{J}\cdot\vec{A} + \rho\phi) d^3x \ .$$

(*Nota bene:* This energy is the work which an external agent expends to assemble such a distribution against the quasistatic electric and magnetic force fields generated by the distribution at any moment of time.)

a) Show that for the Transverse Electric vector potential in Eq.(6.76), this energy is

$$W = \frac{1}{2}\int\int\int S^{TE} B_z d^3x \ .$$

Comment. If one assumes that the TE source density S^{TE} is localized to such a small region that the magnetic field B_z is constant across it, then

$$W = \frac{1}{2} B_z M \ .$$

Here

$$M = \int\int\int S^{TE} d^3x$$

is called the *magnetic dipole moment* along the z-axis, and S^{TE} is the *magnetic moment density*, also known as the *magnetization* along the z-direction.

b) Let

$$\theta(x) = \begin{cases} 1 & 0 \le x \\ 0 & x < 0 \end{cases} \tag{6.141}$$

be the Heaviside unit step function so that

$$\frac{d\theta}{dx} = \delta(x)$$

is the Dirac delta function.

Consider a charge flux distribution confined to the boundary of a rectangular cylinder,

$$
\begin{aligned}
J_x &= + J(z,t)\theta(x)\theta(L_1 - x)\,[\delta(y)\theta(L_2 - y) - \theta(y)\delta(L_2 - y)] \\
J_y &= - J(z,t)\theta(y)\theta(L_2 - y)\,[\delta(x)\theta(L_1 - x) - \theta(x)\delta(L_1 - x)] \\
J_z &= 0 \\
\rho &= 0 \ .
\end{aligned}
$$

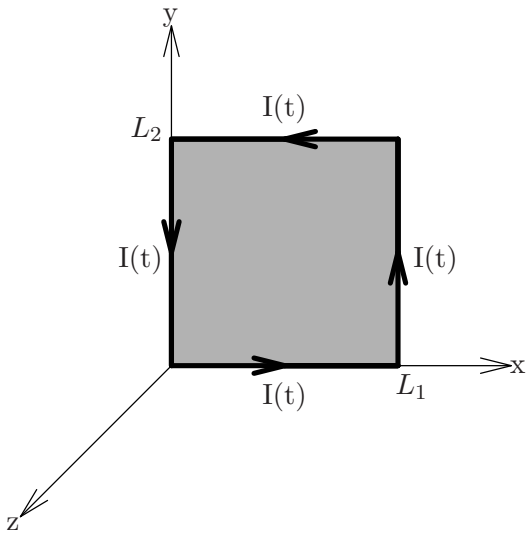

Figure 6.5: Current distributed in the form of a rectangular loop of aerea $L_1 L_2$. This loop has linear current density $J(z,t) = I(t)\,\delta(z)$.

(i) Show that it satisfies the conservation law, Eq.(6.53).

(ii) Find the magnetic dipole density S^{TE}.
Answer: $S^{TE} = J(z,t)\ \theta(x)\theta(L_1 - x)\theta(y)\theta(L_2 - y)$.

(iii) Point out why the magnetic moment is

$$M = I \times Area$$

where $Area = L_1 L_2$ and

$$I(t) = \int J(z,t)\,dz$$

is the current circulating around the rectangular boundary. The linear current density exemplified in Figure 6.5 has the form $J(z,t) = I(t)\,\delta(z)$.

Exercise 6.2.9 (ELECTRIC DIPOLE AS A SOURCE OF TM RADIATION) Consider the rate at which a given external agent does work on two charges q_1 and q_2 in order to keep them on their symmetrically placed trajectories

$$\vec{X}(t) = (0, 0, Z(t)) \tag{6.142}$$

and

$$-\vec{X}(t) = (0, 0, -Z(t)) \ . \tag{6.143}$$

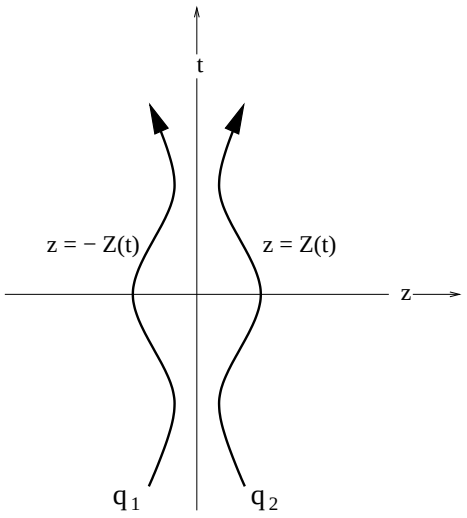

Figure 6.6: Spacetime trajectories of two charges, $q_1 = -q$ and $q_2 = q$, symmetrically placed and moving into opposite directions.Their dipole moment is $q \times (separation) \ = \ 2qZ(t)$.

Given that they move in an environment having an electric field $\vec{E}(x, y, z, t)$, the power expended by this agent is

$$\frac{d(\text{Energy})}{dt} = q_2 \frac{d\vec{X}(t)}{dt} \cdot \vec{E}(0, 0, Z(t), t) - q_1 \frac{d\vec{X}(t)}{dt} \cdot \vec{E}(0, 0, -Z(t), t) \tag{6.144}$$

$$= \int\int\int \vec{J}(x, y, z, t) \cdot \vec{E}(x, y, z, t) \, d^3x \tag{6.145}$$

where

$$\vec{J}(x, y, z, t) = \delta(x)\delta(y) \left[q_2 \, \delta\left(z - Z(t)\right) \frac{d\vec{X}}{dt} - q_1 \, \delta\left(z + Z(t)\right) \frac{d\vec{X}}{dt} \right] \ . \tag{6.146}$$

is the total charge flux vector due to these two charges.

a) Taking advantage of linear superposition, find the charge flux-density four-vector (J_x, J_y, J_z, ρ) such that it expresses the conservation of charge,

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Answer:
$$J_x = 0$$
$$J_y = 0$$
$$J_z = q_2 \frac{dZ(t)}{dt}\,\delta(x)\delta(y)\delta(z - Z(t)) - q_1 \frac{dZ(t)}{dt}\,\delta(x)\delta(y)\delta(z + Z(t))$$
$$\rho = q_2 \quad \delta(x)\delta(y)\delta(z - Z(t)) \quad + \quad q_1 \quad \delta(x)\delta(y)\delta(z + Z(t))$$

Note: Even though we shall (in compliance with physical observations) ultimately set

$$q_2 = -q_1 \equiv q \ , \tag{6.147}$$

it is somewhat easier to keep track of distinguishing contributions to the charge flux vector $\vec{J}$ by assigning correspondingly distinguishing labels to them.

b) Show that there exists a function S^{TM} such that

$$J_z = \frac{\partial S^{TM}}{\partial t} \ , \quad \rho = -\frac{\partial S^{TM}}{\partial z} \ .$$

Do this by expressing the answer in terms of the Heaviside unit step function θ, Eq.(6.141), on page 469.

Answer: $S^{TM} = -\delta(x)\delta(y)\left[q_2\,\theta\left(z - Z(t)\right) + q_1\,\theta\left(z + Z(t)\right)\right]$

c) In compliance with the observation of many cases of interest, assume that the fractional temporal rate of change of E_z is neglegibly small compared to that of $Z(t)$:

$$\frac{1}{E_z}\frac{\partial E_z}{\partial t} \ll \frac{1}{Z}\frac{\partial Z}{\partial t} = \frac{1}{S^{TM}}\frac{\partial S^{TM}}{\partial t}$$

In light of this observation, point out why the power expended by the external agent, Eq.(6.145), can be written as

$$\frac{d(\text{Energy})}{dt} = \frac{d}{dt}\int\int\int S^{TM} E_z \, d^3x \ . \tag{6.148}$$

d) Designating $\int\int\int S^{TM} E_z \, d^3x$ as the energy of the (q_1, q_2)-system, exhibit

(i) its form whenever $q_2 = -q_1 \equiv q$, Eq.(6.147), as well as

(ii) its explicit value when in addition E_z is constant on $[-Z, Z]$, the support of S^{TM}, i.e.

$$\frac{\partial E_z}{\partial z}|Z(t)| \ll E_z \ . \tag{6.149}$$

Answer:
(i) $\int\int\int S^{TM} E_z \, d^3x = \int\int\int(-)\delta(x)\delta(y)q \left[\theta\left(z - Z(t)\right) - \theta\left(z + Z(t)\right)\right] E_z \, d^3x$

(ii) $\left[\int\int\int S^{TM} \, d^3x\right] E_z(0, 0, 0, t_0)$

where $\int\int\int S^{TM} \, d^3x = 2Z(t)q$ is the "dipole moment" of the system.

Comment: The quantity $\int\int\int S^{TM} \, d^3x$ is called the *dipole moment* of the "microscopic" [as identified by the inequality (6.149)] (q_1, q_2)-system. The function

$$S^{TM}(x, y, z, t) = \frac{\text{(dipole moment)}}{\text{(volume)}}$$

is called its *dipole moment density*, and $\vec{E}_z(0, 0, 0, t_0)$ is the electric field at the location of the system.
Comment: If one has an aggregate of such systems, then their total energy is their sum. Under suitable circumstances it can be approximated by the integral

$$\int\int\int S^{TM}(x, y, z, t) E_z(x, y, z, t_0) \, d^3x \quad \left(= \int\int\int \vec{J} \cdot \vec{E} \, d^3x\right)$$

where $E_z(x, y, z, t_0)$ refers to the average electric field associated with the microscopic dipole moment centered around (x, y, z).

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APPENDIX

YOU AND YOUR RESEARCH: A stroke of genius: striving for greatness in all you do

by

Richard W. Hamming

Little has been written on managing your own research (and very little on avoiding other people managing your research); However, your research is much more under your control than you may realize.

We are concerned with great research here. Work that will get wide recognition, perhaps even win a Nobel Prize. As most people realize, the average paper is read by the author, the referee, and perhaps one other person. Classic papers are read by thousands. We are concerned with research that will matter in the long run and become more than a footnote in history.

If you are to do important work then you must work on the right problem at the right time in the right way. Without any one of the three, you may do good work but you will almost certainly miss real greatness.

Greatness is a matter of style. For example, after learning the elements of painting, you study under a master. While studying you pay attention what the master says in discussing your work, but you know if you are to achieve greatness then you must find your own style. Furthermore, a successful style in one age is not necessarily appropriate for another age. Cubism would not have gone over big during the realism period.

Similarly, there is no simple formula for doing great science or engineering, I can only talk around the topic. The topic is important because, so far as we have any solid evidence, you have but one life to live. Under these circumstances it seems better to live a life in which you do important things

(important in your eyes, of course) than to merely live out your life. No sense in frittering away your life on things that will not even appear in the footnotes.

Choosing the problem

I begin with the choice of problem. Most scientists spend their time working on problems that even they admit are neither great or are likely to lead to great work; hence, almost surely, they will not do important work. Note that importance of the results of a solution does not make the problem important. In all the 30 years I spent at Bell Telephone Laboratories (before it was broken up) no one to my knowledge worked on time travel, teleportation, or antigravity. Why? Because they had no attack on the problem. Thus an important aspect of any problem is that you have a good attack, a good starting place, some reasonable idea on how to begin.

To illustrate, consider my experience at BTL (Bell Telephone Laboratories). For the first few years I ate lunch with the mathematicians. I soon found that they were more interested in fun and games than in serious work, so I shifted to eating with the physics table. There I stayed for a number of years until the Nobel Prize, promotions, and offers from other companies removed most of the interesting people. So I shifted to the corresponding chemistry table, where I had a friend.

At first I asked what were the important problems in chemistry, then what important problems they were working on, problems that might lead to important results. One day I asked, "If what they were working on was not important, and was not likely to lead to important things, then why were they working on them?" After that I had to eat with the engineers!

About four months later my friend stopped me in the hall and remarked that my question had bothered him. He had spent the summer thinking about the important problems in his area, and while he had not changed his research he thought it was well worth the effort. I thanked him and kept walking. A few weeks later I noticed that he was made head of the department. Many years later he became a member of the National Academy of Engineering. The one person who could hear the question went on to do important things, and all the others – so far as I know– did not do anything worth public attention.

There are many right problems, but very few people search carefully for

them. Rather they simply drift along doing what comes to them, following the easiest path to tomorrow. Great scientists spend a lot of time and effort examining the important problems in their field. Many have a list of 10 to 20 problems that might be important if they had a decent attack. As a result, when they notice something new that they had not known but seems to be relevant, then they are prepared to turn to the corresponding problem, work on it and get there first.

Some people work with their doors open in clear view of those who pass by, while others protect themselves from interruptions. those with the door open get less work done each day, but those with their door closed tend not to know what to work on, nor are they apt to hear the clues to the missing piece to one of their "list" problems. I cannot prove that the open door produces the open mind, or the other way around. I can only observe the correlation. I suspect that each reinforces the other, that an open door will more likely lead you to important problems than will a closed door.

Hard work is a trait which most scientists have. Edison said that genius was 99 % perspiration and 1 % inspiration. Newton said that if others worked as hard as he did then they would get similar results. Hard work is necessary but it is not sufficient. Most people do not work as hard as they easily could. However, many who do work hard – work on the wrong problem, at the wrong time, in the wrong way, and have very little to show for it.

You are all aware that frequently more than one person starts working on the same problem at about the same time. In biology, both Darwin and Wallace had the idea of evolution at about the same time. In the area of special relativity, many people besides Einstein were working on it, including Poincaré. However, Einstein worked on it in the right way.

The first person to produce definite results generally gets all the credit. Those who come in second are soon forgotten. Thus working on the problem at the right time is essential. Einstein tried to find a unified theory, spent most of his later life working on it, and died in a hospital still working on it with no significant results. Apparently he attacked the problem too early, or perhaps it was the wrong problem.

There are a pair of errors that are often made when working on what you think is the right problem at the right time. One is to give up too soon, the other is to persist and never get any results. The second is quite common. Obviously. if you start on the wrong problem and refuse to give up, you are automatically condemned to waste the rest of your life (see Einstein above). Knowing when to persist is not easy—if you are wrong then you are stubborn;

but, you turn out to be right, then you are strong willed.

I now turn to the major excuse given for not working on important problems. People are always claiming that success is a matter of luck, but as Pasteur pointed out, "Luck favors the prepared mind."

A great deal of direct experience, vicarious experience through the questioning of others, and reading extensively, convinces me of the truth of this statement. Outstanding successes are too often done by the same people for it to be a matter of random chance.

For example, when I met Feynman at Los Alamos during the WWII, I believed that he would get a Nobel Prize. His energy, his style, his abilities, all indicated that he would do many things, and at least one would be important. Einstein, around the age of 12 or 14, asked himself what a light wave would look like when he went at the speed of light. He knew that Maxwell's theory did not support a local, stationary maximum, but was what he ought to see if the current theory was correct. So it is not surprising that he later developed the special theory of relativity - he had prepared his mind long before.

Many times a discussion with a person who has done something important will produce a description of how they were led, almost step by step, to the result. It is usually based on things they had done, or intensely thought about years ago. You succeed because you have prepared yourself with the necessary background long ago, without, of course, knowing then that it would prove to be a necessary step to success.

Personal traits

These traits are not all essential, but tend to be present in most doers of great things in science. First, successful people tend to exhibit more activity, energy, than most people do. They look more places, they work harder, they think longer than less successful people. Knowledge is much like compound interest - the more you do the more you can do, and the more opportunities are open for you. Thus, among other things, it was Feynman's energy and his constantly trying new things that made one think he would succeed.

This trait must be coupled with emotional commitment. Perhaps the ablest mathematician I have watched up close seldom, if ever, seemed to care deeply about the problem he was working on. He has done a great deal of first class work, but not of the highest quality. Deep emotional commitment seems to be necessary for success. The reason is obvious. The emotional

commitment keeps you thinking about the problem morning, noon, and night, and that tends to beat out mere ability.

While I was at Los Alamos after the war, I got to thinking about the famous Buffon needle problem where you calculate the probability of a needle tossed at random of crossing one of a series of equally spaced parallel lines. I asked myself if was essential that the needle be a straight line segment (if I counted multiple crossings)? No. Need they be equally spaced or is it only the average density of the lines on the plane? Is it surprising that some years later at Bell Telephone Laboratories when I was asked by some metallurgists how to measure the amount of grain boundary on some microphotographs I simply said, "Count the crossings of a random line of fixed length on the picture?" I was led to it by the previous, careful thought about an interesting, and I thought important, result in probability. The result is not great, but illustrates the mechanism of preparation and emotional involvement.

The above story also illustrates what I call the "extra mile." I did more than the minimum, I looked deeper into the nature of the problem. This constant effort to understand more than the surface features of a situation obviously prepares you to see new and slightly different applications of your knowledge. You cannot do many problems such as the above needle problem before you stumble on an important application.

Courage is an attribute of those who do great things. Shannon is a good example. For some time he would come to work at about 10:00 a.m., play chess until about 2:00 p.m. and go home.

The important point is how he played chess. When attacked he seldom, if ever, defended his position, rather he attacked back. Such a method of playing soon produces a very interrelated board. He would then pause a bit, think, and advance his queen saying, "I ain't ascaired of nothin'." It took me a while to realize that of course that is why he was able to prove the existence of good coding methods. Who but Shannon would think to average over all random codes and expect to find that the average was close to the ideal? I learned from him to say the same to myself when stuck, and on some occasions his approach enabled me to get significant results.

Without courage you are unlikely to attack important problems with any persistence, hence not likely to do important things. Courage brings selfconfidence, an essential feature for doing difficult things. However, it can border on overconfidence at times, which is more of a hindrance than a help.

There is another trait that took me many years to notice, and that is the ability to tolerate ambiguity. Most people want to believe what they

learn is the truth; there are a few people who doubt every thing. If you believe too much then you are not likely to find the essentially new view that transforms a field, and if you doubt too much you will not be able to do much at all. It is a fine balance between believing what you learn and at the same time doubting thing things. Great steps forward usually involve a change of viewpoint to outside the standard ones in the field.

While you are learning things you need to think about them and examine them from many sides. By connecting them in many ways with what you already know$\cdots\cdots$you can later retrieve them in unusual situations. It took me a long time to realize that each time I learned something I should put "hooks" on it. This is another face of the extra effort, the studying more deeply, the going the extra mile, that seems to be characteristic of great scientists.

The evidence is overwhelming that steps that transform a field often come from outsiders. In archeology, carbon dating came from physics. The first airplane was built by the Wright brothers who were bicycle experts.

Thus, as an expert in your field you face a difficult problem. There is apparently, an ocean of kooks with their crazy ideas. However, if there is a great step forward it probably will be made by one of them! If you listen too much to them you will not get any of your own work done, but if you ignore them then you may miss your great chance. I have no simple answer except do not dismiss the outsider too abruptly as is generally done by the insiders.

"Brains" are nice to have, but often the top graduate students do not contribute as much as some lower rated ones. Brains come in all kinds of flavors. Experimental physicists do not think in the same way that theoreticians do. Some experimentalists think with their hands, i.e. playing with equipment lets them think more clearly. It took me few years to realize that people who did not know a lot of mathematics still could contribute. Just because they could not solve a quadratic equation immediately in their head does not mean that I should ignore them. When someone's flavor of brains does not match your may be more reason for paying attention to them.

Vision

You need a vision of who you are and where your field is going. A suitable parable is that of the drunken sailor. He staggers one way and then the other with independent random steps. In n steps he will be, on the average,

about $\sqrt{n}$ steps away from where he started. But if there is a pretty girl in one direction he will get a distance proportional to n. The difference, over a life time of choices, between $\sqrt{n}$ and n is very large and represents the difference between having no vision and having a vision. The particular vision you have is less important than just having one –there are many paths to success. Therefore it is wise to have a vision of what you may become, of where you want to go, as well as how to get there. No vision, not much chance of doing great work; with a vision you have good chance.

Another topic I must discuss is that of age. Historically, the greatest contributions of mathematicians, theoretical physicists, and astrophysicists are done when they are young. On the other hand, apparently in music composition, politics, and literature, the later works are most valued by society. Other area seem to fall in between these two extremes, and you need to realize that in some areas you had better get going promptly.

People often complain about the working conditions they have to put up with, but it is easily observed that some of the greatest work was done under unfavorable conditions. What most people believe is the best working conditions for them, is seldom, if ever, true. In my opinion the Institute for Advanced Study in Princeton has ruined more good people than it has helped. You only have to judge their work before they were appointed and afterwards to come to this conclusion. There are exceptions, to be sure, but on the average the ideal working conditions seem to sterilize people.

Another obvious trait of great people is that they do their work in such a fashion that others can build on top of it. Newton said, "If I have seen farther than others it is because I stood on the shoulders of giants." Too many people seem not to want others to build on top of their work but they rather want to hoard it to themselves. Don't do things in such a fashion that next time it must be repeated by you, or by others, but rather in a way that represents a significant step forward.

Selling

I must now take up the unpleasant topic of selling your ideas. Too many scientists think that this is beneath them, that the world is waiting for their results. In truth, the other researchers are busy with their own work. You must present your results so that they will stop their own work and listen to you. Presentation comes in three forms: published papers, prepared talks,

and impromptu situations. You must master all three forms.

Lots of good work has been lost because of poor presentation only to be rediscovered later by others. There is a real danger that you will not get credit for what you have done. I know of all too many times when the discoverer could not be bothered to present things clearly, and hence his or her work was of no importance to society.

Finally, I must address the question of whether greatness is worth the large effort it requires. Those who have done really great things generally report, privately, that it is better than wine, the opposite sex, and song put together. The realization that you have done it is overwhelming.

Of course, I have consulted only those who did great things, and have not dared to ask those who did not. Perhaps they would reply differently. But as is often said, it is the in struggle and not the success that real gain appears. In striving to do great things, you change yourself into a better person, so they claim. The actual success is of less importance, so they say. And I tend to believe this theory.

No one ever told me the kind of things I have just related to you; I had to find them out for myself. Since I now have told you how to succeed, you have no excuse for not trying and doing great work in your chosen field.

About the author

Dr. Richard W. Hamming is best known for the Hamming code, Hamming distance and the Hamming spectral window along with numerical methods. In 1986, the IEEE [Institute for Electrical and Electronic Engineers] established an award in his name for exceptional contributions to information sciences and systems. The award includes a gold medal and $10,000.

From 1945 to 1946 he worked on computing atomic bomb designs via simulations for the Manhattan Project at Los Alamos, New Mexico. In 1946, Dr. Hamming joined Bell Telephone Laboratory where he ran analog as well as digital computer centers in the Math and Computing Research Department until July 1976. Hamming then then joined the faculty of the Naval Postgraduate School, Monterey (CA), where he teaches in the Department of Computer Science. For an indepth view of Hamming's outlook on life and the little events that created the long lasting ones read "Interview: Richard Hamming Interviewed by David Gilbert," IEEE COMPUTING FUTURES, SPRING 1991, pp.10-17.

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