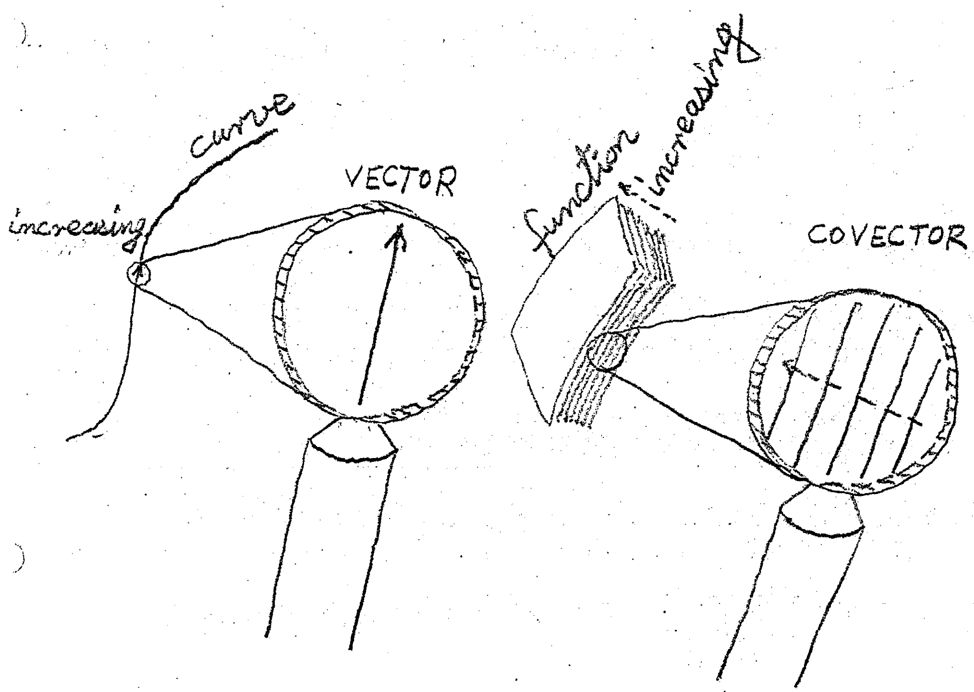
LECTURE 7 Friday

<u>Linear functions on a vectorspace V</u>

<u>The Dual Vectorspace V^*</u>

<u>Dirac's Bracket Notation</u>

<u>Basis vs Dual Basis</u>


increasing
curve
VECTOR
function
increasing
COVECTOR

The Dual of a Vector Space is
a Vector Space: the Space of Duals
of Covectors
of linear fns on V.

Consider the set of fruit inventories.

Designate a typical inventory consisting of α^a apples, α^b bananas, α^c coconuts...

by

$$\vec{x} = \alpha^a \overrightarrow{apple} + \alpha^b \overrightarrow{banana} + \alpha^c \overrightarrow{coconut} + \cdots$$

or more succintly by

$$\vec{x} = \alpha^a \vec{e}_a + \alpha^b \vec{e}_b + \alpha^c \vec{e}_c + \cdots$$

I. Fruit inventories like these form a vector space, V, closed under addition and scalar multiplication

Nota bene: In this vector space there is no distance function, no Pythagorean theorem, and no angle between any pair of inventories.

II. Next consider a particular purchase price function $\$_f$. Its values yield the cost of any fruit inventory $\vec{x}$:

$$\$_f(\vec{x}) = \$_f(\alpha^a \vec{e}_a + \alpha^b \vec{e}_b + \alpha^c \vec{e}_c + \cdots)$$

$$= \alpha^a \underbrace{\$_f(\vec{e}_a)}_{\substack{= \\ \text{purchase} \\ \text{price/apple}}} + \alpha^b \underbrace{\$_f(\vec{e}_b)}_{\substack{= \\ \text{purchase} \\ \text{price/apple}}} + \alpha^c \underbrace{\$_f(\vec{e}_c)}_{\substack{= \\ \text{purchase} \\ \text{price/apple}}} + \cdots$$

where

Next consider the cost of fruit inventory $\vec{y}$,

$$\$_f(\vec{y}) = \beta^a \$_f(\vec{e}_a) + \beta^b \$_f(\vec{e}_b) + \beta^c \$_f(\vec{e}_c) + \cdots$$

We see that

$$\$_f(\vec{x} + c\vec{y}) = \$_f(\vec{x}) + c\,\$_f(\vec{y})$$

Thus $\$_f : V \to R$
$\quad \vec{x} \rightsquigarrow \$_f(\vec{x})$

is a linear function on the vector space V of fruit inventories.

7.3

III The set purchase price fns forms a vector space.

Indeed, consider the purchase price functions $\$_f, \$_g, \$_h \cdots$ of different fruit wholesalers, and introduce combined purchase price function $\$_f + \$_g$ by the requirement that

$$(\$_f + \$_g)(\vec{x}) \equiv \$_f(\vec{x}) + \$_g(\vec{x}) \quad \forall \vec{x} \in V$$

and the c-multiple of $\$_f$ by

$$(c\,\$_f)(x) = c\,\$(x).$$

We see that the set of purchase price functions forms a vector space V^*, the space dual to V.

Algebra and Geometry of Dual Vector Spaces

I. a) The space of duals is a space of linear functions
linear functionals
covectors
duals

A dual is a derived concept, it depends on the existence and knowledge of a vector space. More precisely, we have the following

Definition 7 (Linear function)

Let V be a vector space.
Consider a scalar valued linear function f defined on V as follows:

$$f: V \longrightarrow \text{scalars}$$
$$x \rightsquigarrow f(x)$$

such that

$$\boxed{f(\alpha x+\beta y) = \alpha \cdot f(x) + \beta f(y)} \qquad x, y \in V, \quad \alpha, \beta \in \{\text{scalars}\}$$

(This is different from non-linear f'ns, obviously!)

Example 1: Let $V = R^n = \{(x_1, x_2, \dots, x_n)^T\}$
Then $f: R^n \longrightarrow R$

Then $(x_1, x_2, \dots, x_n)^T \rightsquigarrow f(x_1, \dots, x_n) = \xi_1 x_1 + \dots + \xi_n x_n$
is a linear function on R^n. Note that f is determined totally by $\xi_1, \dots, \xi_n$.

R^3  $\xrightarrow{f} R$

parallel planes = loci of pts where f = constant

Example 2

Consider $V = C[a,b]$, the vector space of functions ψ continuous on $[a,b]$:

$$V = C[a,b] = \{\psi : \psi(s) \text{ is continuous on } [a,b]\}$$

We shall now consider three different linear scalar-valued functions on V:

(i) For any point $s_1 \in [a,b]$ the "s_1-evaluation map" (i.e. the "s_1-sampling function") f

$$V = C(a,b) \xrightarrow{f} R \qquad \text{(reals)}$$
$$\psi \rightsquigarrow f(\psi) = \psi(s_1)$$

f is a linear function because

$$f(c_1 \psi + c_2 \phi) = c_1 \psi(s_1) + c_2 \phi(s_1)$$
$$= c_1 f(\psi) + c_2 f(\phi)$$

(ii) Let $\{s_1, s_2, \ldots, s_n\} \subset [a,b]$ be a specified collection of points in $[a,b]$, and let $\{k_j\}_1^n$ be a set of scalars. Then

the function g defined by

$$V \xrightarrow{g} R$$

$$\psi \rightsquigarrow g(\psi) = \sum_{j=1}^{n} k_j \, \psi(s_j),$$

is a linear function on V.

(iii) Similarly, the map h defined

defined by

$$h(\psi) = \int_a^b \psi(s)\, ds$$

is also a linear function on $V = C[a,b]$

Example 3 (skip this one in class)

Consider the vector space

$$V = C^\infty(a,b) = \{\psi : \psi \text{ is } C^\infty \text{ on } (a,b)\}$$

of functions infinitely differentiable on (a,b). Furthermore, let

$$d^j\psi(s) = \left.\frac{d^j\psi}{dx^j}\right|_{x=s}$$

be the j^{th} derivative of ψ at $x=s$.

Then for any fixed $s \in (a,b)$, the map

h defined by

$$V = C^\infty(a,b) \xrightarrow{h} R$$

$$\psi \rightsquigarrow h(\psi) = \sum_{j=1}^{n} a_j \, d^j\psi(s)$$

is a linear function on

$V = C^\infty(a,b)$

II. The Vector Space V^* dual to V.

Given a vector space V, the consideration of all possible linear functions defined on V gives rise to

V^* = set of all linear functions on V.

These linear functions form a vector space in its own right, the dual space of V. Indeed, one has the following

Theorem 8

The set V^* is a vector space

Comment. a) The elements of the vector space V^* of duals are called covectors.

b) That V^* does indeed form a vector space is verified by observing that the collection of linear functions satisfies the 10 properties of a vector space defined on page 1.7

on page 1.7

If f, g, h are linear functions (i.e. elements of V^*), and $\alpha, \beta \in R$, then

1. $f+g$ is also a linear function defined by

$$(f+g)(\vec{x}) = f(\vec{x}) + g(\vec{x}) \qquad \forall \vec{x} \in V$$

such that

a) $f+g = g+f$
b) $f+(g+h) = (f+g)+h$
c) zero element is the constant zero function
d) the additive inverse of f is $-f$

2. αf is a linear function defined by

$$(\alpha f)(\vec{x}) = \alpha f(\vec{x}) \qquad \forall \vec{x} \in V$$

such that 7.10

a) $\alpha(\beta f) = (\alpha\beta) f$

b) $1 \cdot f = f$

c) $\alpha(f+g) = \alpha f + \alpha g$

d) $(\alpha+\beta) f = \alpha f + \beta f$

III Dirac's Bracket Notation

To emphasize the duality between the two vector spaces, one introduces the *bra-cket notation*, which is familiar from quantum mechanics.

If f is a linear function on V and $f(x)$ its value at $x \in V$, then one also writes

$$f(x) = \langle f | x \rangle = \langle \underset{\sim}{f} | \vec{x} \rangle. \qquad (\star)$$

Thus the twiddle under f is a reminder that $\underset{\sim}{f} \in V^*$, while x, or better $\vec{x}$, is an element of V.

We say that f *operates* on the vector x and produces

$$\langle f | x \rangle \qquad (\in R)$$

To emphasize that f is a linear "machine", we write

$$f = \langle \underset{\sim}{f} | \qquad (\in V^*)$$

for the covector, and

$$\vec{x} = | \vec{x} \rangle \qquad (\in V)$$

for the vector. They combine to form

$$\langle \underset{\sim}{f} | \vec{x} \rangle \quad \in R$$

IV Construction of a Linear Function

The existence and uniqueness of a vector's scalar coefficients, relative to a chosen basis is what makes the concept of duality so important geometrically and computationally.

This assessment arises from the following:

Duality Principle:

For each chosen basis of a finite dimensional vector space V, there exists a corresponding basis for V^*, and vice versa. This is done in the next lecture.