Lecture 1

1. Three archetypical eq'ns of linear algebra
2. Vector as an aggregate of entities
3. Vector space: Definition & Examples
4. Subspace of a vector space

The two mathematical underpinnings of 20^{th} and 21^{st} century science are calculus and linear algebra.

The latter is summarized by the positing of three problems

1. $A\vec{u} = 0$ Homogeneous problem
2. $A\vec{u} = \vec{b}$ Inhomogeneous problem
3. $AG = I$ Inverse of A

These are archetypical problems:

If 1. has a non-trivial sol'n then 2. has ∞ many or none at all, depending on $\vec{b}$, and 3. has none.

More generally, in linear algebra one asks: For what value of λ do the following equations have a sol'n:

1. $(A-\lambda B)\vec{u} = 0$
2. $(A-\lambda B)\vec{u} = \vec{b}$
3. $(A-\lambda B)G = I$

These problems extend to infinite dimensions. In Math 5102 (the follow up) they will be identified as

the Sturm-Liouville Problem
the Inhomogeneous Boundary Value Problem
the Green's Function Problems

respectively.

The purpose of Math 510, is twofold

a) to develop the concepts giving rise to these equations and to identify the resulting principles and mathematical methods.

b) to connect them to reality i.e. to particular concretes.

The basic concepts are concretized by the following two examples

Example 1

(i) Consider a grocery store's old inventory of fruits consisting of

7 apples, 11½ bananas, 13 coconuts:

$$7a + 11.5b + 13c + \cdots \equiv \vec{u}$$

(ii) Consider the fruit delivery of a supply consisting of

200 apples, 100 bananas, 50 coconuts:

$$200a + 100b + 50c + \cdots \equiv \vec{v}$$

(iii) The new fruit inventory is

$$207a + 111.5b + 63c + \cdots = \vec{u} + \vec{v} \equiv \vec{w}$$

Note: The augmentation of the old inventory by the fruit delivery supply yields another fruit inventory for the grocery store.

One says that the <u>sum</u> of the old inventory and the one furnished by the delivery process obeys the <u>super position principle</u>; this is because the resulting aggregate of fruits is another inventory, namely the new one $\vec{w}\ (= \vec{u} + \vec{v})$

One also says that the space of fruit inventories is <u>closed</u>

Example 2

Consider the superposition of two audio signals

$$a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t = u(t)$$

$$b_1 \cos t + b_2 \cos 2t + b_3 \cos 3t = v(t)$$

namely

$$(a_1+b_1)\cos t + (a_2+b_2)\cos 2t + (a_3+b_3)\cos 3t = u(t) + v(t) \equiv w(t)$$

This is another audio signal of the same type, i.e. the collection of audio signals is closed under addition.

These commonality between the two examples and others like it give rise to the following

Definition: A vector space is the result of uniting a constellation of 4 ideas

1. A field F of "scalars" (e.g. reals, complex #s, rational #s, etc)
2. A set V of objects called vectors
3. An operation, called vector addition, such that $\boxed{v, w \in V \Rightarrow v+w \in V}$, i.e their sum is in V subject to
   a) $v+w = w+v$
   b) $(u+v)+w = u+(v+w)$
   c) $\exists$ a unique vector $\vec{0} \in V$ such that $v+0 = v \quad \forall v \in V$
   (In other words, V is non empty!)
   d) For each $v \in V$ $\exists$ a unique vector $-v \in V$ such that $v+(-v) = \vec{0}$

Comment: 1. These conditions say that V is "closed" under $+$.

4. An operation, called scalar multiplication such that $\boxed{c \in F \text{ and } v \in V \Rightarrow cv \in V}$, i.e the scalar multiple is in V, subject to
   a) $1 \cdot v = v \quad \forall v \in V$
   b) $(c_1 c_2)v = c_1(c_2 v)$
   c) $c(v+w) = cv + cw$
   d) $(c_1+c_2)v = c_1 v + c_2 v$

To summarize: A vector space V over the field F consists of a field, a set of vectors, and two operations.

Note that $\boxed{0\,\vec{v} = \vec{0}}$. Why? Because $0 \cdot \vec{v} = (0+0)\vec{v} = 0\vec{v} + 0\vec{v}$. Add $-0v$ to both sides and obtain

$$0 \cdot \vec{v} = 0\vec{v} + 0\vec{v} \quad \Big| -0\vec{v}$$

$$\vec{0} = 0\vec{v} + 0$$

$$\vec{0} = 0\vec{v}$$

Example 3

$C^0[a,b]$ = space of continuous functions from $[a,b]$ to the set of complex numbers $\mathbb{C}$
$\equiv V$

Explicitly one has

$$\psi \in V: \quad [a,b] \longrightarrow \mathbb{C}$$
$$\qquad x \rightsquigarrow \psi(x)$$

The process of putting into mathematical form ("mathematization of") the measured behaviour of sound, e.m. radiation, mechanical vibrations, etc, gives rise to the following laws of addition and scalar multiplication:

Let $\psi_1, \psi_2, \psi \in V$, then the sum $\psi_1+\psi_2$ and the scalar multiple $c\psi$ are defined as follows

$$\overbrace{\psi_1+\psi_2}^{\text{Function}}: \quad [a,b] \longrightarrow \mathbb{C}$$
$$(*): \quad x \rightsquigarrow \overbrace{(\psi_1+\psi_2)(x)}^{\text{Value of the function at } x} = \psi_1(x)+\psi_2(x) \quad \forall x \in [a,b]$$
$$c\psi: \quad [a,b] \longrightarrow \mathbb{C}$$
$$(**) \quad x \rightsquigarrow (c\psi)(x) = c(\psi(x)) \quad \forall x \in [a,b]$$

Conclusion: V is a vector space.

Validation:
One must verify that 3 a-d and 4 a-d are satisfied. For example

$$3a: \quad (\psi_1+\psi_2)(x) = \psi_1(x)+\psi_2(x) = \psi_2(x)+\psi_1(x)$$
$$= (\psi_2+\psi_1)(x) \quad \forall x \in [a,b]$$

Hence $\boxed{\psi_1+\psi_2 + \psi_2+\psi_1}$

$$3b: \quad \psi_1+[\psi_2+\psi_3] = [\psi_1+\psi_2]+\psi_3$$

$$3c: \quad \vec{0}: \quad x \in [a,b] \rightsquigarrow 0 \in \mathbb{C} \quad \forall x \in [a,b]$$

is zero function

3d: For each $\psi \in V$ define $-\psi$ by

$$(-\psi)(x) \equiv -(\psi(x)) \qquad \forall x \in [a,b]$$

Consequently,

$$[\psi + (-\psi)](x) = \psi(x) + (-\psi)(x) = \psi(x) - (\psi(x)) = 0 \quad \forall x$$

$\therefore\ \psi + (-\psi) = \vec{0}$ ("zero function)

One verifies 4 a–d using Eq. (**)

For example

4a) $(1\cdot\psi)(x) \underset{\text{Eq}(**)\text{ on P 1.10}}{=} 1(\psi(x)) = \psi(x) \qquad \forall x$

Hence $1\cdot\psi = \psi$

4c) For the sake of clarity set $\psi_1 + \psi_2 = \phi \in V$

Then

$$[c(\psi_1+\psi_2)](x) = [c\,\phi](x) \underset{\text{Eq}(**)\text{ on P 1.10}}{=} c[\phi(x)] = c\phi(x) =$$

$$= c(\psi_1(x) + \psi_2(x)) = c\psi_1(x) + c\psi_2(x)$$

$$= [c\psi_1](x) + [c\psi_2](x) \qquad \forall x$$

Thus $c(\psi_1+\psi_2) = c_1\psi_1 + c\psi_2$

Example 4

$\mathcal{P}_2$ = set of real polynomial of degree two or less

$$= \{ p(x) = a_2x^2 + a_1x + a_0 :\ a_2, a_1, a_0 \in R^1 \}$$

is a vector space

Definition 2 (Subspace)

Let V, W both be vector spaces

$W \subset V$ (i.e. W is a subset)

$W \neq \phi$ (W is not empty) this implies that

then W is called a **subspace of V**.

Comment: Subspaces are of practical importance in problems involving approximations, optimization, diff'l eq'ns

1. They are used to answer questions like

a) "How can we find good approximations to complicated functions?"

b) "How can we generate good approximate solutions to d. eq'ns?"

Subspaces are easy to recognize because

Theorem 1 Let a) $W \subset V$ ← vector space
b) W nonempty

(i.e W is a nonempty subset of V)

$\boxed{W \text{ is a subspace}} \iff$

$$\boxed{\left.\begin{array}{ll}(1) & u+v \in W \\ (2) & cu \in W\end{array}\right\} \begin{array}{l}\text{whenever } u, v \in W \\ \text{" } \quad c \text{ any scalar}\end{array}}$$

Comment: (1)&(2) $\iff$ $cu+v \in W$ whenever $u, v \in W$, c any scalar

$0v = (0+0)v = 0v$
$= 0v + 0v$

Proof: $\Rightarrow$ obvious because W is a vector space

$\Leftarrow$ (1) commutativity & assoc & distributivity are inherited from V.

These are properties a-d

ZERO: (2) W contains the zero vector! Why?

because W contains $0 \cdot \vec{v}$ and $0 \cdot \vec{v} = \vec{0}$ (SHOWN in class)

property (3d)

(3) W INVERSE: For any $v \in W$, $(-1)v \in W$.
$(-1)v = -v$ (Homework probl. 2)

(5) W inherits ppties 4 a-d from V.

Comment: Theorem 1 simplifies the task of determining whether or not W is a vector space, all one has to do is verify that

$\forall\, v, w \in W \Rightarrow cv + w \in W$!!!

In fact: $cv + w \in W \iff \begin{array}{l} v+w \in W \\ cv \in W \end{array}$
whenever $v, w \in W$

Example. For $n < m$, P_n is a subspace of P_m

We already know that P_n & P_m are vector spaces

$P_n \subset P_m$. Thus P_n is a subspace.