LECTURE 10 (Revised version)

Reciprocal Vector Basis for V

Its relation to the dual basis for V^*

Geometrical relation between vectors and their linear functionals in V^*.

2.2 The Reciprocal Vector Basis

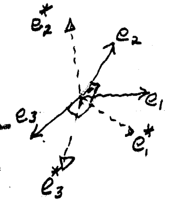
Some text books refer to x^i and x_i as the "contravariant" and "covariant" components of one and the same vector. This can actually be misleading.

Historically the "contravariant" components are *the* component relative to a choosen basis, *regardless* of what metric the vector space V is endowed with.

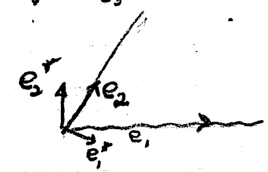
If V is endowed with a specific metric, then there exists a second well defined basis, "reciprocal" to the first one. It turns out that the components of a vector relative to this *reciprocal* basis are precisely x_j, and it is these components that have been called the "covariant" components of the vector.

The *reciprocal* basis is actually very useful because it brings the geometrical features of the dual basis into sharper focus.

The reciprocal basis and its properties arise as follows.



Definition: (reciprocal basis)
Given (i) $g = "\cdot"$, a metric on V
(ii) $\{e_i\}$, a basis for V
Then the set of vectors
$$\{e_1^*, e_2^*, \ldots, e_n^*\},$$



where
$$\boxed{e_k^* \cdot e_i = \delta_{ki}},$$
is the basis *reciprocal* to $\{e_i\}$.

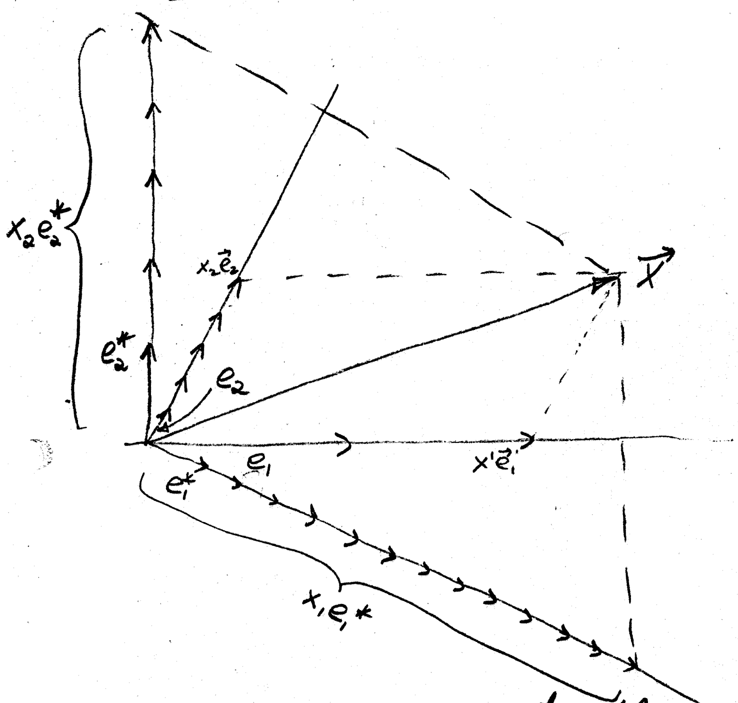
This definition says that e_k^* is a vector perpendicular to the plane spanned by $\{e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_n\}$; i.e.
$$\boxed{e_k^* \cdot e_i = 0} \quad i \neq k.$$

Furthermore, e_k^* is *scaled* such that
$$\boxed{e_k^* \cdot e_k = 1} \quad \text{NO SUM}$$

It is *also clear* that if the basis $\{e_k^*\}$ is reciprocal to $\{e_i\}$, then $\{e_i\}$ is reciprocal to $\{e_k^*\}$.

In other words, one has

$$\boxed{\vec{x} = x^i \vec{e}_i = \sum x_k e_k^*} = x^1 e_1 + x^2 e_2 = x_1 e_1^* + x_2 e_2^*$$



In terms of these components the square length of a vector has the simple form.

$$x \cdot x = (x^1 e_1 + x^2 e_2) \cdot (x_1 e_1^* + x_2 e_2^*)$$
$$= x^1 x_1 + x^2 x_2$$
$$= x^1 g_{1i} x^i + x^2 g_{2i} x^i$$
$$= g_{ji} x^j x^i$$

The fact that elements of the reciprocal basis as well as the elements of the dual basis serve as projection operators suggests that there exists a more far reaching relation between these two basis. That this is indeed the case is expressed by the following

Proposition: The image of e_k^* under the metric is $\underset{\sim}{\omega}^k$; in other words, the metric establishes a one-to-one relationship between the elements of the reciprocal basis and those of the dual basis:

$$e_k^* \rightsquigarrow \underset{\sim}{\omega}^k$$

proof:

we have:

$$g(e_k^*, \vec{x}) = e_k^* \cdot \vec{x} = e_k^* \cdot \vec{e}_i x^i = x^k$$

also: $\langle \omega^k | \vec{x} \rangle = \quad = \langle \omega^k | e_i \rangle x^i = x^k$

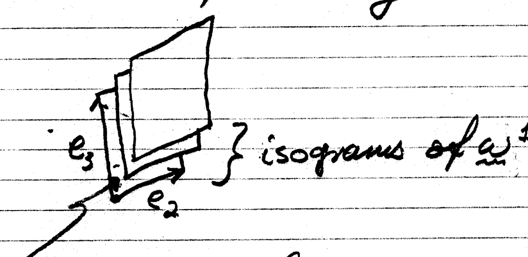
This holds for all vectors $\vec{x}$. Consequently,

$$g(e_k^*, \ ) = \text{"}e_k^* \cdot\text{"} = \underset{\sim}{\omega}^k,$$

which is the same as

$$e_k^* \rightsquigarrow g(e_k^*, \ ) = \underset{\sim}{\omega}^k$$

The geometrical interpretation of this result is that the vector e_k^* is perpendicular to the level surfaces of $\underset{\sim}{\omega}^k$. This is because all basis vectors e_i, where $i \neq k$, lie in the isogram $\underset{\sim}{\omega}^k = 0$ of the linear functional $\underset{\sim}{\omega}^k$. In other words, $\underset{\sim}{\omega}^k = 0$ is the subspace in V spanned by these e_i's



Locus of points where $\underset{\sim}{\omega}^1 = 0$

On the other hand, e_k^* is perpendicular to these e_i's. Thus e_k^* is perpendicular to the level surfaces of $\underset{\sim}{\omega}^k$ indeed.

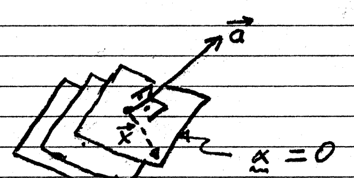
One can extend this orthogonality property to arbitrary linear functionals.

Indeed, let $\underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k$ be an arbitrarily chosen linear functional on V.

Question: Is there a unique vector $\perp$ to the level surfaces of $\underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k$?

Answer: Yes. This vector is $\vec{a} =$

$$\vec{a} = \sum_{k=1}^{n} \alpha_k e^*_k .$$



Discussion: Let $\vec{x}$ be any vector. Then

$\underset{\sim}{\alpha}(\vec{x}) = \langle \alpha_k \omega^k | \vec{x} \rangle = \alpha_k \langle \omega^k | \vec{x} \rangle = \alpha_k e^*_k \cdot \vec{x} = \vec{a} \cdot \vec{x}$ (*)

Suppose that $\vec{x}$ is any vector lying in the plane $\underset{\sim}{\alpha} = 0$, i.e. $\{\vec{x} : \underset{\sim}{\alpha}(\vec{x}) = 0\}$.

We now have the following

II Proposition

Given the liner function (covector)

$$\underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k$$

Conclusion:

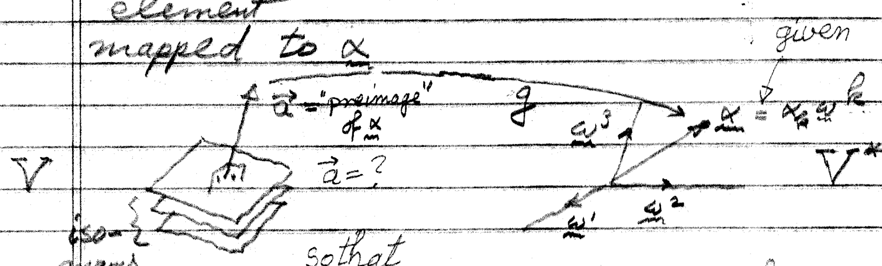
(a) The preimage under the metric g is the unique vector

$$\vec{a} = \sum_k \alpha_k e^*_k$$

(b) This vector is perpendicular to the isograms of $\underset{\sim}{\alpha}$.

Discussion:

(a) "Preimage" of $\underset{\sim}{\alpha}$ is the set of vectors that get mapped to $\underset{\sim}{\alpha}$. Here the preimage consists of a single element mapped to $\underset{\sim}{\alpha}$



so that

$$\vec{a} \rightsquigarrow g(\vec{a}) = \underset{\sim}{\alpha} = \underbrace{\alpha_k \omega^k}_{\text{given}}$$

a) (cont'd)

The validation of the fact that under g the linear function $\underset{\approx}{\alpha} = \alpha_k \underset{\approx}{\omega}^k$ has the unique vector $\vec{a} = \sum_k \alpha_k e^*_k$ as its preimage is a achieved by the observation that

$$g(\vec{a}, \vec{x}) = \sum_k \alpha_k\, g(e^*_k, \vec{x})$$
$$= \sum_k \alpha_k\, e^*_k \cdot x$$
$$= \sum_k \alpha_k\, e^*_k \cdot e_i x^i$$
$$= \sum \alpha_k x^k \qquad e^*_k \cdot e_i = \delta_{ki}$$
$$= \sum_k \alpha_k \omega^k(\vec{x})$$
$$\overset{\text{given}}{=} \underset{\approx}{\alpha}(\vec{x}) \qquad \forall \vec{x}$$

On the other hand

$$g(\vec{a}, x) = g_{ij}\, \underset{\approx}{\omega}^i(\vec{a})\, \omega^j(x) \quad \forall \vec{x}$$

Consequently

$$\alpha_k \underset{\approx}{\omega}^k \overset{\text{given}}{=} \underset{\approx}{\alpha} = g_{ij}\, \underset{\approx}{\omega}^i(\vec{a}) \otimes \underset{\approx}{\omega}^j$$

i.e. $\vec{a} = \alpha_k \vec{e}^*_k \in V$ is a preimage of $\underset{\approx}{\alpha}$ under

$$g = g_{ij}\, \underset{\approx}{\omega}^i \otimes \omega^j$$

The linear independence of $\{\underset{\approx}{\omega}^k\}_{k=1}^n$ implies

$$\alpha_k = g_{ik}\, \underset{\approx}{\omega}^i(\vec{a})$$
$$= g_{ik}\, a^i \qquad k = 1, \ldots, n$$

The uniqueness of $\{a^i\}_{i=1}^n$, and hence of $a^i e_i = \vec{a} (= \alpha_k e^*_k)$, follows from the fact that the matrix $[g_{ik}]$ is assumed to be non-singular.

Thus, any $\underset{\approx}{\alpha} = \alpha_k \underset{\approx}{\omega}^k$ in V^* has a unique g-induced preimage $\vec{a} = \alpha_k e^*_k$ in V.

Moreover if expansion of $\vec{a}$ relative to the basis $\{e_i\}$ is

$$\vec{a} = a^i e_i = \omega^i(\vec{a})\, e_i,$$

then these expansion coefficients yield

$$\alpha_k = g_{kj}\, a^j$$

In summary one has

$$\vec{a} = a^i e_i = \sum_k \alpha_k e^*_k \quad \in V$$

$$\updownarrow$$

$$\underset{\sim}{\alpha} = \alpha_k \omega^k \quad \in V^*$$

where

$$\alpha_k = g_{ki} a^i.$$

(b) To validate that $\vec{a}$ with the property that

$$g(\vec{a}, \;) = \underset{\sim}{\alpha} \quad (= \alpha_k \underset{\sim}{\omega}^k \leftarrow \text{given})$$

is a vector which is $\perp$ to the isograms of $\underset{\sim}{\alpha}$, let $\vec{x}$ be any vector in V. Then

$$\underset{\sim}{\alpha}(\vec{x}) = \langle \alpha_k \omega^k | \vec{x} \rangle$$
$$= \alpha_k \langle \omega^k | \vec{x} \rangle$$
$$= \alpha_k x^k$$
$$= \sum_k \alpha_k e^*_k \cdot \vec{x} \qquad \forall \vec{x}$$

Next let $\vec{x}$ be any vector lying in the isogram where $\underset{\sim}{\alpha}$ has the value zero, i.e. $\underset{\sim}{\alpha}(\vec{x}) = 0$

Consequently,

$\vec{x} \in \{\vec{x} : \underset{\sim}{\alpha}(\vec{x}) = 0\} =$ $\qquad \underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k$

$= \{\vec{x} : \vec{a} \cdot \vec{x} = 0\}$ $\qquad \vec{a} = \sum \alpha_k e^*_k$

$\vec{x}$ lies in the zero isogram of $\underset{\sim}{\alpha}$

$\vec{a}$ is $\perp$ to the zero isogram of $\underset{\sim}{\alpha}$

The fact that $\vec{a}$ is perpendicular to all vectors $\vec{x}$ lying in planar isogram $\underset{\sim}{\alpha} = 0$ is what is meant by the statement that $\vec{a}$ is the vector perpendicular to all the level planar isograms of $\underset{\sim}{\alpha}$. Why? Because they are parallel planes.

Conclusion:

$$\vec{a} = \sum_k \alpha_k e^*_k = \sum_i a^i \vec{e}_i$$

is the (unique) vector which is perpendicular to the isograms of

$$\underset{\sim}{\alpha} = \sum_k \alpha_k \underset{\sim}{\omega}^k$$

where (i) $\alpha_k = g_{ki} a^i$

(ii) $g_{ki} = e_k \cdot e_i$; $e^*_k \cdot e_i = \delta_{ki}$

and (iii) $\langle \omega^j | e_i \rangle = \delta^j_i$;

ANOTHER APPLICATION

Bragg Diffraction expressed in terms of Laue's Eq'n

Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be basis ("primitive translation" vectors for a crystal lattice V.

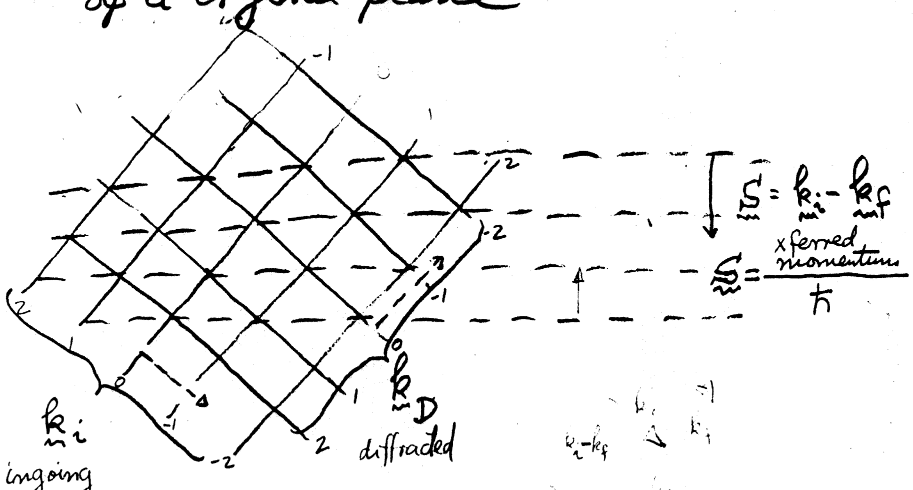
Let $\{\underset{\sim}{\omega}^1, \omega^2, \omega^3\}$ be the basis of duals: $\langle \omega^i, e_j \rangle = \delta^i_j$

Let $\underset{\sim}{k}_i$ = <u>incident propagation</u> covector for the wave $\exp(i\langle \underset{\sim}{k}_i, \vec{x}\rangle) = \exp[i(k_1 x^1 + k_2 x^2 + k_3 x^3)]$

Let $\underset{\sim}{k}_D$ = <u>diffracted propagation</u> covector

Let $\underset{\sim}{S} = \underset{\sim}{k}_i - \underset{\sim}{k}_D$ [note: $(\underset{\sim}{k}_i - \underset{\sim}{k}_D)\hbar$ = momentum transferred to the crystal]

Let h, k, and ℓ be integers, the "Miller indeces" of a crystal plane



Then an incident wave $e^{i\langle \underset{\sim}{k}_i, \vec{x}\rangle}$ will get diffracted by the crystal lattice into a diffracted wave $e^{i\langle \underset{\sim}{k}_D, \vec{x}\rangle}$ (i.e. Bragg diffraction occurs) provided

$$\langle \underset{\sim}{S}, \vec{e}_1 \rangle = 2\pi h = \text{phase along } \vec{e}_1$$
$$\langle \underset{\sim}{S}, \vec{e}_2 \rangle = 2\pi k = \text{phase along } \vec{e}_2$$
$$\langle \underset{\sim}{S}, \vec{e}_3 \rangle = 2\pi \ell = \text{phase along } \vec{e}_3$$

These are Laue's eq'ns. They can be restated

$$\boxed{\begin{gathered}\frac{1}{2\pi}\underset{\sim}{S} = h\underset{\sim}{\omega}^1 + k\underset{\sim}{\omega}^2 + \ell\underset{\sim}{\omega}^2 \\ h, k, \ell = \text{integers}\end{gathered}}$$

Laue's eq'n for Bragg diffraction

This eq'n says that Bragg diffraction $\Longleftrightarrow$ level surfaces of $\underset{\sim}{S}$ must coincide with crystal plane (h, k, ℓ)

For details see Charles Kittel: "Introduction to Solid State Physics."

$\frac{1}{2\pi}\vec{S} = h\overset{*}{e}_1 + k\overset{*}{e}_2 + \ell\overset{*}{e}_3 \in$ Reciprocal Lattice

Lattice of crystal planes,

used in X-Ray crystallography & quantum theory of solids.

LECTURE 10

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Historically the "contravariant" components are *the* component relative to a choosen basis, *regardless* of what metric the vector space V is endowed with.

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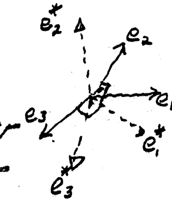
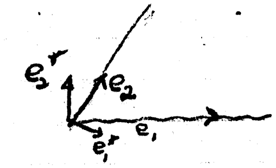
The *reciprocal* basis is actually very useful because it brings the geometrical features of the dual basis into sharper focus.

The reciprocal basis and its properties arise as follows.



Definition 9: (reciprocal basis)
Given (i) $g = \text{"}\cdot\text{"}$, a metric on V
(ii) $\{e_i\}$, a basis for V
Then the set of vectors
$$\{e_1^*, e_2^*, \ldots, e_n^*\},$$
where
$$\boxed{e_k^* \cdot e_i = \delta_{ki}},$$
is *the basis reciprocal* to $\{e_i\}$.

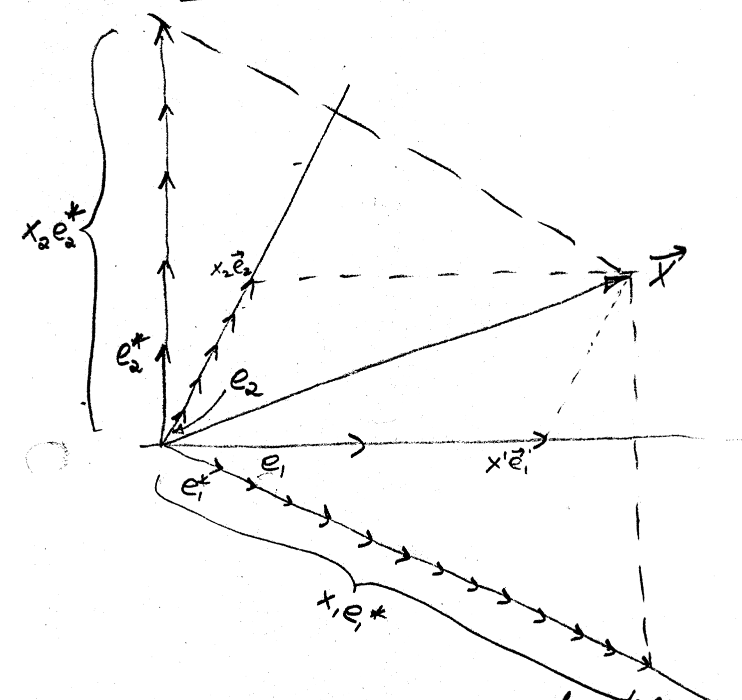


This definition says that e_k^* is a vector perpendicular to the plane spanned by $\{e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_n\}$; i.e.
$$\boxed{e_k^* \cdot e_i = 0} \qquad i \neq k.$$
Furthermore, e_k^* is scaled such that
$$\boxed{e_k^* \cdot e_k = 1} \qquad \text{NO SUM}$$

It is also clear that if the basis $\{e_k^*\}$ is reciprocal to $\{e_i\}$, then $\{e_i\}$ is reciprocal to $\{e_k^*\}$.

In other words, one has

$$\boxed{\vec{x} = x^i \vec{e}_i = \sum x_k e^*_k} = x^1 e_1 + x^2 e_2 = x_1 e^*_1 + x_2 e^*_2$$



In terms of these components the square length of a vector has the simple form.

$$x \cdot x = (x^1 e_1 + x^2 e_2) \cdot (x_1 e^*_1 + x_2 e^*_2)$$
$$= x^1 x_1 + x^2 x_2$$
$$= x^1 g_{1i} x^i + x^2 g_{2i} x^i$$
$$= g_{ji}\, x^j x^i$$

We know that the elements of the dual basis $\{\omega^j\}$ serve as operators which project out ~~the~~ a vector's components relative to the given basis. The elements of the reciprocal basis serve the same function. These facts are summarized by the following

proof: ① $e^*_k \xrightarrow{g} g(e^*_k, \ ) = "e^*_k \cdot" \equiv \underset{\sim}{\omega}^k$

② Evaluate $\underset{\sim}{\omega}^k$ for any basis vector e_i:

$\underset{\sim}{\omega}^k(e_i) = g(e^*_k, e_i) = \delta_{ki}$

③ These are the properties of $\underset{\sim}{\omega}^k$. Hence

$$e^*_k \xrightarrow{g} \omega^k (\equiv "e_k \cdot")$$

Both the reciprocal basis vectors and the dual 1-forms are useful because they serve as projection operators. In fact we have

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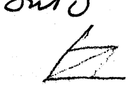
Proposition: (Projection operators)

The dual basis as well as the reciprocal basis elements serve as projection operators that yield the coordinates of a vector:

$$\langle \underset{\sim}{\omega}^k | \vec{x} \rangle = \langle \underset{\sim}{\omega}^k | x^i \vec{e}_i \rangle = x^i \delta^k_i = x^k$$

$$\vec{e}^*_k \cdot \vec{x} = x^i \vec{e}^*_k \cdot e_i = x^i \delta_{ki} = x^k$$

Here $x^k =$ is the parallel projection of $\vec{x}$ onto the coordinate axis of e_k



$=$ contravariant component of $\vec{x}$.

This differs from

$e_j \cdot \vec{x} = e_j \cdot x^i \vec{e}_i = x^i g_{ji} = x_j =$ "covariant" component of $\vec{x}$

The fact that elements of the reciprocal basis as well as the elements of the dual basis serve as projection operators suggests that there exists a more far reaching relation between these two basis. That this is indeed the case is expressed by the following

Proposition: The image of e^*_k under the metric is $\underset{\sim}{\omega}^k$; in other words, the metric establishes a one-to-one relationship between the elements of the reciprocal basis and those of the dual basis:

$$e^*_k \overset{g}{\rightsquigarrow} \underset{\sim}{\omega}^k$$

proof:

we have:

$$g(e^*_k, \vec{x}) = e^*_k \cdot \vec{x} = e^*_k \cdot \vec{e}_i x^i = x^k$$

also: $\langle \omega^k | \vec{x} \rangle = \quad = \langle \omega^k | e_i \rangle x^i = x^k$

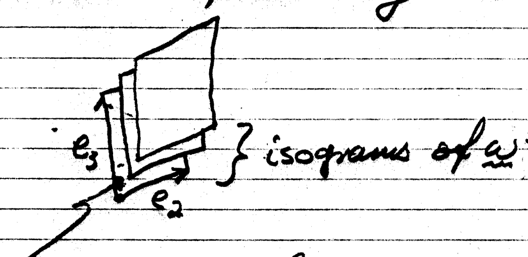
This holds for all vectors $\vec{x}$. Consequently

$$g(e^*_k, \;) = \text{"} e^*_k \cdot \text{"} = \underset{\sim}{\omega}^k,$$

which is the same as

$$e^*_k \rightsquigarrow g(e^*_k, \;) = \underset{\sim}{\omega}^k$$

The geometrical interpretation of this result is that the vector e^*_k is perpendicular to the level surfaces of $\underset{\sim}{\omega}^k$. This is because all basis vectors e_i, where $i \neq k$, lie in the isogram $\underset{\sim}{\omega}^k = 0$ of the linear function $\underset{\sim}{\omega}^k$. In other words, $\underset{\sim}{\omega}^k = 0$ is the subspace in V spanned by these e_i's



Locus of points where $\underset{\sim}{\omega}^1 = 0$

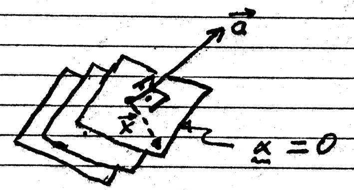
On the other hand, e^*_k is perpendicular to these e_i's. Thus e^*_k is perpendicular to the level surfaces of $\underset{\sim}{\omega}^k$ indeed.

One can extend this orthogonality property to arbitrary linear functionals. Indeed, let $\underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k$ be an arbitrarily chosen linear functional on V.

Question: Is there a unique vector $\perp$ to the level surfaces of $\underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k$?

Answer: Yes. This vector is $\vec{a} =$

$$\vec{a} = \sum_{k=1}^{n} \alpha_k e_k^*.$$



Discussion: Let $\vec{x}$ be any vector. Then

$$\underset{\sim}{\alpha}(\vec{x}) = \langle \alpha_k \omega^k | \vec{x} \rangle = \alpha_k \langle \omega^k | \vec{x} \rangle = \alpha_k e_k^* \cdot \vec{x} = \vec{a} \cdot \vec{x} \quad (*)$$

Suppose that $\vec{x}$ is any vector lying in the plane $\underset{\sim}{\alpha} = 0$, i.e. $\{\vec{x} : \underset{\sim}{\alpha}(\vec{x}) = 0\}$.

We now have the following

II Proposition

Given the liner function (covector)

$$\underset{\sim}{a} = \alpha_k \underset{\sim}{\omega}^k$$

Conclusion:

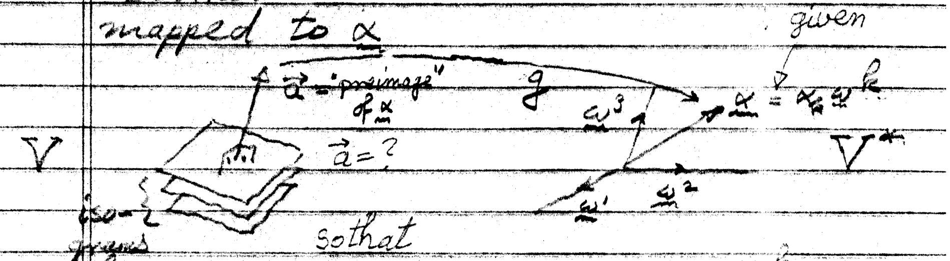
(a) The preimage under the metric g is the unique vector

$$\vec{a} = \sum_k \alpha_k e_k^*$$

(b) This vector is perpendicular to the isograms of $\underset{\sim}{\alpha}$.

Discussion:

(a) "Preimage" of $\underset{\sim}{\alpha}$ is the set of vectors that get mapped to $\underset{\sim}{\alpha}$. Here the preimage consists of a single element mapped to $\underset{\sim}{\alpha}$



so that

$$\vec{a} \rightsquigarrow g(\vec{a}) \equiv \underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k$$

given

a) (cont'd)

To validate the claim that g maps

$\vec{a} = \sum \alpha_k e^*_k$ to $\underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k$

one observes that

$$g(\vec{a}, \vec{x}) = \sum_k \alpha_k \, g(e^*_k, \vec{x})$$
$$= \sum_k \alpha_k \, e^*_k \cdot x$$
$$= \sum_k \alpha_k \, e^*_k \cdot e_i x^i \qquad e^*_k \cdot e_i = \delta_{ki}$$
$$= \sum \alpha_k x^k$$
$$= \sum_k \alpha_k \omega^k(\vec{x})$$

given $\overset{\text{given}}{=} \underset{\sim}{\alpha}(\vec{x}) \qquad \forall \vec{x}$

On the other hand

$$g(\vec{a}, x) = g_{ij} \, \underset{\sim}{\omega}^i(\vec{a}) \, \underset{\sim}{\omega}^j(x) \quad \forall \vec{x}$$

Consequently,

$$g_{ij} \, \underset{\sim}{\omega}^i(\vec{a}) \otimes \underset{\sim}{\omega}^j \overset{\text{!}}{=} \underset{\sim}{\alpha} \qquad (*)$$

i.e. $\vec{a} = \alpha_k \vec{e}^*_k \in V$ is a preimage of $\underset{\sim}{\alpha}$ under

$$g = g_{ij} \, \underset{\sim}{\omega}^i \otimes \underset{\sim}{\omega}^j$$

Having identified

$$\vec{a} = \sum_k \alpha_k e^*_k$$

as the preimage of $\underset{\sim}{\alpha} = \sum \alpha_k \underset{\sim}{\omega}^k$ under g, one needs to determine the expansion coefficients a^i of $\vec{a}$ relative to the given basis $\{\vec{e}_i\}$; in other words, one needs to solve

$$\vec{a} = \sum_i a^i e_i$$

for a^i. This is done with the help of Eq. (*) at the bottom of p 10.10, namely

$$g_{ij} \, \omega^i(\vec{a}) \, \underset{\sim}{\omega}^j = \alpha_j \underset{\sim}{\omega}^j$$

These expansion cofficients, we recall, are the coordinate components of $\vec{a}$, namely,

$$a^i = \omega^i(a)$$

Thus one ~~concludes~~ obtains

$$g_{ij} a^i = \alpha_j \qquad j = 1, \ldots, n$$

which can be solved for a^i whenever $[g_{ij}]$ is a non-singular matrix.

Summary. The covector $\underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k \in V^*$ has under g the unique preimage $\vec{a} = \sum \alpha_k e^*_k \in V$.

Moreover, if one expands $\vec{a}$ relative to the basis $\{e_i\}$

$$\vec{a} = a^i e_i = \omega^i(\vec{a}) e_i,$$

then these expansion coefficients yield

$$\alpha_k = g_{kj} a^j$$

In summary one has

$$\vec{a} = a^i e_i = \sum_k \alpha_k e^*_k \quad \in V$$
$$\updownarrow$$
$$\underset{\sim}{\alpha} = \alpha_k \omega^k \quad \in V^*$$

where
$$\alpha_k = g_{ki} a^i.$$

(b) To validate that $\vec{a}$ with the property that
$$g(\vec{a}, \;) = \underset{\sim}{\alpha} \quad (= \alpha_k \underset{\sim}{\omega}^k \text{ given})$$
is a vector which is $\perp$ to the isograms of $\underset{\sim}{\alpha}$, let $\vec{x}$ be any vector in V. Then
$$\underset{\sim}{\alpha}(\vec{x}) = \langle \alpha_k \omega^k | \vec{x} \rangle$$
$$= \alpha_k \langle \omega^k | \vec{x} \rangle$$
$$= \alpha_k x^k$$
$$= \sum_k \alpha_k e^*_k \cdot \vec{x} \qquad \forall \vec{x}$$

Next let $\vec{x}$ be any vector lying in the isogram where $\underset{\sim}{\alpha}$ has the value zero, i.e.
$$\underset{\sim}{\alpha}(\vec{x}) = 0$$

Consequently,

$\vec{X} \in \{\vec{X} : \underset{\sim}{\alpha}(\vec{X}) = 0\} = \qquad \underset{\sim}{\alpha} = \alpha_k \underset{\sim}{\omega}^k$

$= \{\vec{X} : \vec{a}\cdot\vec{X} = 0\} \qquad \vec{a} = \sum \alpha_k e^*_k$

$\boxed{\vec{X} \text{ lies in the zero isogram of } \underset{\sim}{\alpha}}$ $\boxed{\vec{a} \text{ is } \perp \text{ to the zero isogram of } \underset{\sim}{\alpha}}$

The fact that $\vec{a}$ is perpendicular to all vectors $\vec{x}$ lying in planar isogram $\underset{\sim}{\alpha} = 0$ is what is meant by the statement that $\vec{a}$ is the vector perpendicular to <u>all</u> the level planar isograms of $\underset{\sim}{\alpha}$. Why? Becaus they are parallel planes.

<u>Conclusion</u>:

$$\vec{a} = \sum_k \alpha_k e^*_k = \sum_i a^i \vec{e}_i$$

is <u>the</u> (unique) vector which is perpendicular to the isograms of

$$\underset{\sim}{\alpha} = \sum_k \alpha_k \underset{\sim}{\omega}^k$$

where (i) $\alpha_k = g_{ki}\, a^i$

(ii) $g_{ki} = e_k \cdot e_i$; $e^*_k \cdot e_i = \delta_{ki}$

and (iii) $\langle \omega^j | e_i \rangle = \delta^j_i$;

ANOTHER APPLICATION

Bragg Diffraction expressed in terms of Laue's Eq'n

Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be basis ("primitive translation" vectors for a crystal lattice V.

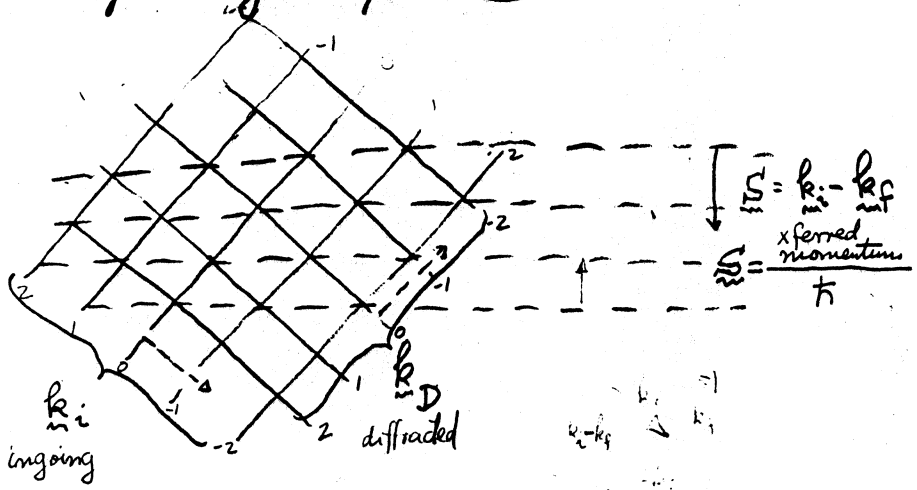
Let $\{\underset{\sim}{\omega}^1, \omega^2, \omega^3\}$ be the basis of duals: $\langle \omega^i, e_j \rangle = \delta^i_j$

Let $\underset{\sim}{k}_i$ = incident propagation covector for the wave $\exp(i\langle \underset{\sim}{k}_i, \vec{x}\rangle) = \exp[i(k_1x^1 + k_2x^2 + k_3x^3)]$

Let $\underset{\sim}{k}_D$ = diffracted propagation covector

Let $\underset{\sim}{S} = \underset{\sim}{k}_i - \underset{\sim}{k}_D$ [note: $(\underset{\sim}{k}_i - \underset{\sim}{k}_D)\hbar$ = momentum transferred to the crystal]

Let h, k, and l be integers, the "Miller indeces" of a crystal plane



Then an incident wave $e^{i\langle \underset{\sim}{k}_i, \vec{x}\rangle}$ will get diffracted by the crystal lattice into a diffracted wave $e^{i\langle \underset{\sim}{k}_D, \vec{x}\rangle}$ (i.e. Bragg diffraction occurs) provided

$$\langle \underset{\sim}{S}, \vec{e}_1 \rangle = 2\pi h = \text{phase along } \vec{e}_1$$
$$\langle \underset{\sim}{S}, \vec{e}_2 \rangle = 2\pi k = \text{phase along } \vec{e}_2$$
$$\langle \underset{\sim}{S}, \vec{e}_3 \rangle = 2\pi l = \text{phase along } \vec{e}_3$$

These are Laue's eq'ns. They can be restated

$$\boxed{\frac{1}{2\pi}\underset{\sim}{S} = h\underset{\sim}{\omega}^1 + k\underset{\sim}{\omega}^2 + l\underset{\sim}{\omega}^2 \qquad h, k, l = \text{integers}}$$

Laue's eq'n for Bragg diffraction

This eq'n says that Bragg diffraction $\Longleftrightarrow$ level surfaces of $\underset{\sim}{S}$ must coincide with crystal plane (h, k, l)

For details see Charles Kittel: "Introduction to Solid State Physics."

$\frac{1}{2\pi}\vec{S} = h\overset{*}{e}_1 + k\overset{*}{e}_2 + l\overset{*}{e}_3 \in$ Reciprocal Lattice

Lattice of crystal planes.

used in X-Ray crystallography & quantum theory of solids.