LECTURE 11

1. Transformations: Who Needs Them?

2. Linear Transformations: Their Geometric and Algebraic Definition.

3. Onto and One-to-one Transformations

4. Null Space and Range Space: Their Importance

5. Example
Linear Transformations

The complexity of mathematics is a reflection of the complexity of the relations that exist in the world.

One's quantitative grasp of the nature of the physical world is achieved by mathematical relations. Of these, linear relations, also known as linear transformations, are the easiest to understand, the best understood and the most ubiquitous in a person's mathematical arsenal of concepts.

A linear transformation expresses (= is) a relationship between vector spaces.

and as such, between $\mathbb{R}^m$ and $\mathbb{R}^n$ for finite-dimensional vector spaces.

The distinguishing feature of a linear transformation is that it relates closed triangles, more generally closed polygons, to closed triangles:

Figure 11.1

The linear transformation of a linear combination (superposition) of vectors into a linear superposition of the transformed vectors.

This geometrical feature is expressed algebraically by means of the.
following

Definition 11.1 (Linear Transformation)

Let $U$ and $V$ be vector spaces.

Consider the map

$$T: U \rightarrow V$$

$$u \mapsto Tu$$

Then $T$ is a linear transformation whenever

$$T(c_1u + c_2w) = c_1Tu + c_2Tw$$

for all $u, w \in U$ and $c_1, c_2 \in \text{scalars}$. 

Comment

(i) $U$ is called the domain or the domain space of $T$.

(ii) $V$ is called the target or the target space of $T$.

(iii) The set $\{v \in V : v = Tu \} : u \in U \}$ is called the range or the range space of $T$.

Linear transformations come in two (not necessarily mutually exclusive) flavors:

onto or surjective and one-to-one or injective as defined by

Definition 11.2a (onto map = surjective map)

$T$ is said to be an onto map if

for any $v \in V$ there exists at least one $u \in U$ such that

$$T(u) = v$$

in other words

(i) if $T$ has a solution to Eq. (★)

or

(ii) if Eq. (★) can be solved for $u$
(b) If the set \( \{ u : T(u) = v \} \neq \emptyset \), \( \forall u \in V \).

Comment:
(i) \( T(u) \) is called the image of \( u \) of \( u \) under \( T \).

(ii) The set \( \{ u : T(u) = v \} \neq \emptyset \) is called  
the preimage of \( v \). Note that the  
preimage is a set!

The concept of one-to-oneness (or injectivity)  
is defined by

Definition 11.26 (one-to-one map = injective map)

\[ T(u_1) = T(u_2) \Rightarrow u_1 = u_2 \]

In other words, if the solution to 
\[ T(u) = v \]
is unique,

There are two sets of properties central to all of linear algebra. On one hand  
there is the spanning property and the  
linear independence of a given set of 
elements of a vector space. On the other hand  
there is the onto and one-to-one  
property of a given transformation.

These two pairs of properties are not mere  
juxtapositions. On the contrary, they  
are logically equivalent. The cause  
of this equivalence are the concepts  
of existence and uniqueness of  
the equations:

- **Existence**: \( c_1 v_1 + \ldots + c_m v_m = 0 \Leftrightarrow T(c_1 v_1 + \ldots + c_m v_m) = 0 \Leftrightarrow T\) is onto
- **Uniqueness**: \( c_1 v_1 + \ldots + c_m v_m = 0 \Rightarrow T(c_1 v_1 + \ldots + c_m v_m) = 0 \Leftrightarrow T\) is one-to-one
The criterion whether or not $T$ is onto or one-to-one can be restated in terms of two fundamental subspaces that characterize $T$. Doing so highlights the underlying similarity that unites into an intelligible whole the plethora of concepts that make up the landscape of linear mathematics.

This criterion is stated in terms of the following important theorem 11.1.

\[ T: U \rightarrow V \text{ be a linear transformation, then} \]

(i) \( \exists \{ \mathbf{u} \in T(\mathbf{u}) = \mathbf{0} \} \Rightarrow \text{nullspace of } T \]

is a subspace of $V$, called the nullspace or the kernel of $T$.

(ii) \( \{ \mathbf{v} : \mathbf{v} = T(\mathbf{u}) \text{ for some } \mathbf{u} \in U \} = R(T) \)

is a subspace of $V$, called the range of $T$.

(iii) $T$ is onto $\iff V = R(T)$

(iv) $T$ is one-to-one $\iff N(T) = \{ \mathbf{0} \}$.

Example 0
Example 1
Example 2
Example 3
Example 4  skip
Example 5
Example #1

\[ T : \mathbb{R}^2 \to \mathbb{R}^2 \]
\[
(x, y) \mapsto (u, v) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}
\]

\( T \) is onto but not 1-1

\( \mathcal{R}(T) = \mathbb{R}^2 \)

\( \mathcal{N}(T) = \text{Span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\} \)

Example #2

\[ T : \mathbb{R} \to \mathbb{R} \]
\[
p(x) = a_0 + a_1 x + a_2 x^2 \mapsto p'(x) = a_0 + a_1 x + a_2 x^2
\]

\( T \) is onto but not 1-1

\( \mathcal{R}(T) = \text{entire real line} \)

\( \mathcal{N}(T) = \{a(x-b)(x-c): a, b, c \in \text{real}\} \)

Example #3

Let \( \mathcal{C}(\mathbb{R}) = \text{set of all functions with continuous 1st derivatives in} \ \mathbb{R} \)

Let \( \mathcal{C}(\mathbb{R}) = \text{set of continuous functions on} \ \mathbb{R} \)

\[ T : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R}) \]
\[
f \mapsto T(f) = \frac{df}{dx} + a f
\]

\( T \) is onto, but not 1-1

\( \mathcal{R}(T) = \mathcal{C}(\mathbb{R}) \) because for any \( g \in \mathcal{C}(\mathbb{R}) \), the differential equation
\[
y' + a x y = g(x)
\]

has a solution.

i.e. \( \exists y \in \mathcal{C}(\mathbb{R}) \in T(y) = g(x) \)

i.e. any \( g \in \mathcal{C}(\mathbb{R}) \) has a preimage

\[ T(g) = \left\{ y(x) = e^{-\frac{a}{a-1}x} \int e^{\frac{a}{a-1}x} g(x) \ dx \right\} \]

Note: \( y' = -a y + e^{-\frac{a}{a-1}x} \int g(x) \ dx \)

\( \mathcal{N}(T) = \text{set of solutions to} \ y' + a(x) y = 0 \)
\[
= \left\{ f \in \text{exp}(\int a(x) \ dx) \right\}
\]
Example #4

\[ T : P_k \to P_{k-1} \]
\[ \psi \mapsto \psi'(0) \]

\( \mathcal{R}(T) = P_{k-1} \) because \( p(x) = b_0 + b_1 x + \ldots + b_k x^{k-1} \) can always be solved for \( p \)

\( \mathcal{N}(T) = \{ p(x) = a_0 \mid a_0 \text{ any real } \} \)

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Example #5

Let \( C(R) = \) set of cont. func. defined on \( R \)

Let \( f \in C(R) : R \to R \)

\[ T : C(R) \to C^1(R) \]
\[ f \mapsto T(f)(x) = \int_0^x f(t) dt \quad \forall x \in R \]

\( T \) is 1-1 but not onto.

\( \mathcal{N}(T) = \{ 0 \} \) because \( 0 = T(f)(x) = \int_0^x f(t) dt \quad \forall x \)

\[ \Rightarrow f(x) = 0 \quad \forall x \]

\( \mathcal{R}(T) = \{ g(x) : g \in C^1 \text{ with } g(0) = 0 \} \subset C(R) \)
Validation of Theorem 11.1 on page 11.8

(1) \( N(T) \) is a subspace

- Proof: 1. \( N(T) \subseteq V \)

- 2. If \( v_1, v_2 \in N(T) \), then \( T(v_1 + v_2) = 0 \)

\( \dim N(T) \) is called the nullity of \( T \)

(2) \( R(T) = \{ \mathbf{v} \mid T(\mathbf{v}) = \mathbf{0} \} \) for some \( u \in U \) is subspace

- Proof: \( u + v \in R(T) \)

\( T(u + v) = T(u) + T(v) = 0 + 0 = 0 \) \( \Rightarrow \) \( u + v \in R(T) \)

\( R(T) \) is called the range of \( T \)

Let \( U, V \in R(T) \)

We must show that \( c_1 u_1 + c_2 u_2 \in R(T) \)

\( u_1, u_2 \) \( \in \) \( T(u_1) = T(u_2) \)

\( c_1 u_1 + c_2 u_2 = c_1 T(u_1) + c_2 T(u_2) = T(c_1 u_1 + c_2 u_2) \)

(3) \( R(T) \) is closed hence a subspace.

\( \dim R(T) \) is called the rank of \( T \)

\( \therefore \) \( R(T) \) is a subspace.

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Given a linear transformation

\[ T : U \rightarrow V \]

\[ u \mapsto \mathbf{v} = T(u) \]

Recall that \[ \mathcal{R}(T) = \{ \mathbf{v} \in V : \mathbf{v} = T(u) \text{ for some } u \in U \} \]

is called the range (space) of \( T \).

Let \( \mathcal{B} = \{ e_1, \ldots, e_p \} \)

be a basis for \( U \), the domain (space) of \( T \).

**Proposition**: \( \mathcal{R}(T) = \text{span} \{ T(e_1), \ldots, T(e_p) \} \)

**Proof**: \( \forall \mathbf{v} \in \mathcal{R}(T) = \{ \mathbf{v} \in V : \mathbf{v} = T(u) \text{ for some } u \in U \} \), define

\[ \mathbf{v} = T(u) = T(c_1 e_1 + \cdots + c_p e_p) \]

\[ = c_1 T(e_1) + \cdots + c_p T(e_p) \Rightarrow \mathbf{v} \in \text{span} \{ T(e_i) \} \]

\[ \therefore \mathcal{R}(T) \subseteq \text{span} \{ T(e_i) \} \]

2. Let \( \mathbf{v} \in \text{span} \{ T(e_i) \} \Rightarrow \exists c_1, \ldots, c_p \) such that

\[ \mathbf{v} = c_1 T(e_1) + \cdots + c_p T(e_p) \]

\[ = T(e_1 c_1 + \cdots + e_p c_p) \Rightarrow \mathbf{v} \in \mathcal{R}(T) \]

\[ \therefore \text{span} \{ T(e_i) \} \subseteq \mathcal{R}(T) \]

3. \( \text{span} \{ T(e_i) \} = \mathcal{R}(T) \).