LECTURE 11

1. Transformations: Who Needs Them?

2. Linear Transformations: Their Geometric and Algebraic Definition.

3. Onto and One-to-one Transformations

4. Null Space and Range Space: Their Importance

5. Example.

Linear Transformations.

The complexity of mathematics is a reflection of the complexity of the relations that exist in the world.

One's quantitative grasp of the nature of the physical world is achieved by mathematical relations. Of these, linear relations, also known as linear transformations, are the easiest to understand, the best understood, and the most ubiquitous in a person's mathematical arsenal of concepts.

A Linear transformation expresses (=is) a relationship between vector spaces,

and as such between R^m and R^n for finite-dimensional vector spaces.

The distinguishing feature of a linear transformation is that it relates closed triangles, more generally closed polygons, to closed triangles:

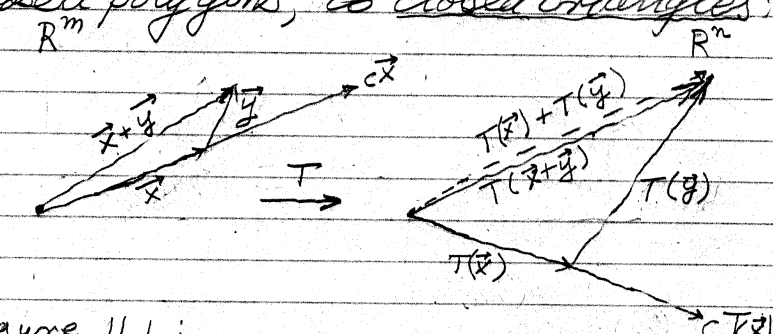

Figure 11.1: The linear tranformation of a linear combination (=superposition) of vectors into a linear superposition of the transformed vectors.

This geometrical feature is expressed algebraically by means of the following

<u>Definition 11.1</u> (<u>Linear transformation</u>)

<u>Let U and V be vector spaces.</u>

Consider the map

$$T:\ U \to V$$
$$u \rightsquigarrow T(u)$$

Then T is a linear transformation whenever

$$T(c_1 u + c_2 w) = c_1 T(u) + c_2 T(w)$$

for all $u, w \in U$ and $c_1, c_2 \in$ scalars.

<u>Comment</u>

(i) U is called the <u>domain</u> or the <u>domain space</u> of T.

(ii) V is called the <u>target</u> or the <u>target space</u> of T.

(iii) The set $\{v:\ v = T(u);\ u \in U\}$ is called the <u>range</u> or the <u>range space</u> of T.

Linear transformations come in two (not necessarily mutually exclusive) flavors:

<u>onto</u> or <u>surjective</u> and <u>one-to-one</u> or <u>injective</u> as defined by

<u>Definition 11.2a</u> (<u>onto map = surjective map</u>)

T is said to be an <u>onto</u> map if

for any $v \in V$ there exists at least one $u \in U$ such that

$$T(u) = v; \qquad (*)$$

in other words

(i) if $\exists$ a solution to Eq. $(*)$

or

(ii) if Eq. $(*)$ can be solved for u

or

(iii) if the set $\{u : T(u) = v\} \neq \phi$ (not the empty set) $\forall v \in V$.

<u>Comment</u>:

(i) $T(u)$ is called the <u>image of u</u> of (or "under") the transformation T.

(ii) The <u>set</u> $\{u : T(u) = v\} \equiv T^{-1}(v)$ is called the <u>preimage of v</u>. Note that the preimage is a <u>set</u>!

The concept of <u>one-to-oneness (or injectivity)</u> is defined by

<u>Definition 11.2b</u> (one-to-one map = injective map)

T is said to be <u>one-to-one</u> if

$$T(u_1) = T(u_2) \Rightarrow u_1 = u_2;$$

in other words, if the solution to $T(u) = v$ is <u>unique</u>.

There are <u>two sets</u> of properties central to all of linear algebra. On one hand there is the <u>spanning property</u> and the <u>linear independence</u> of a given set of elements of a vector space. On the other hand there is the <u>onto</u> and <u>one-to-one</u> property of a given transformation. These <u>two pairs</u> of properties are not mere juxtapositions. On the contrary, they are <u>logically equivalent</u>. The cause of this equivalence are the concepts of existence and uniqueness of the equations:

of solutions

existence: $\underbrace{c_1 v_1 + \cdots + c_p v_p = \vec{v}}^{\text{spanning ppty?}} \leftrightarrow T(\vec{u}) = \vec{v} \leftrightarrow T$ is onto

uniqueness: $\underbrace{c_1 v_1 + \cdots + c_p v_p = \vec{0}}_{\text{lin. independence?}} \leftrightarrow T(\vec{u}) = \vec{0} \leftrightarrow T$ is one-to-one

The criterion whether or not T is onto or/and one-to-one can be restated in terms of two fundamental subspaces that characterize T. Doing so highlights the underlying similarity that unites into an intelligible whole the plethora of concepts that make up the landscape of linear mathematics.

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This criterion is stated in terms of the following important

Theorem 11.1

Let $T : U \to V$ be a linear xformation then

(i) $\{u : T(u) = 0_V\} \equiv \mathcal{N}(T) \equiv \mathrm{Ker}(T)$ is a subspace of U, called the nullspace or the kernel of T.

(ii) $\{v : v = T(u) \text{ for some } u \in U\} \equiv \mathcal{R}(T)$ is a is a subspace of V, called the range of T.

(iii) T is onto $\Leftrightarrow V = \mathcal{R}(T)$

(iv) T is one-to-one $\Leftrightarrow \mathcal{N}(T) = \{0_U\}$.

~~Example 0~~
~~Example 1~~
Example 2
Example 3
Example 4 skip,
Example 5

Example #0

$T: R^3 \to R^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightsquigarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

T is onto but not 1-1

$\mathcal{R}(T) = R^2$

$\mathcal{N}(T) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Example #1

$T: \mathcal{P}_2 \longrightarrow R^1$

$p(x) = a_0 + a_1 x + a_2 x^2 \rightsquigarrow p(t_1) = a_0 + a_1 t_1 + a_2 t_1^2$

T is onto but not 1-1

$\mathcal{R}(T)$ = entire real line

$\mathcal{N}(T) = \{ a(x-t_1)(x-b) : a, b \in \text{reals} \}$

Example #2

Let $B = \{u_1, \dots, u_p\}$ be a basis for U.

$T: U \longrightarrow R^p$

$u \rightsquigarrow T(u) = [u]_B$ = p-tuple of coord. components rel've to B.

T is onto and 1-1

$\mathcal{R}(T) = R^p$

$\mathcal{N}(T) = \{0_v\}$

Example #3

Let $C^1(0,1)$ = set of all fns with cont. 1st derivatives in $(0,1)$

Let $a(x) \in C(0,1)$ = set of cont. fns on $(0,1)$

$T: \quad C^1(0,1) \to C(0,1)$

$f \rightsquigarrow T(f) = \frac{df}{dx} + a(x) f$

T is onto, but not 1-1

$\mathcal{R}(T) = C(0,1)$ because for any $g(x) \in C(0,1)$, the diff'l equation

$y' + a(x) y = g(x)$

has a solution,

i.e. $\exists\, y \in C^1(0,1) \ni T(y) = g(x)$

i.e. any $g \in C(0,1)$ has a preimage

$$T^{-1}(g) = \left\{ y(x) = e^{-\int^x a\,dx'} \cdot \int^x e^{\int^{x'} a\,dx''} g(x')\,dx' \right\}$$

Note: $y' = -a y + e^{-\int^x a\,dx'} e^{\int^x a\,dx''} g(x)$

$\mathcal{N}(T)$ = set of sol'ns to $y' + a(x) y = 0$,

$= \{ f = c_1 \exp(-\int^x a\,dx') \}$

Example# 4

$T: \mathcal{P}_k \to \mathcal{P}_{k-1}'$

$p(x) \rightsquigarrow p'(x)$

$\mathcal{R}(T) = \mathcal{P}_{k-1}$ because $p'(x) = b_0 + b_1 x + \cdots + b_{k-1} x^{k-1}$, can always be solved for p

$\mathcal{N}(T) = \{ p(x) = a_0 \mid a_0 \text{ any real } \# \}$

Example#5

Let $C[R]$ = set of cont. fns defined on R

Let $f \in C[R] : R \to R$.

$T: C[R] \to C^1[R]$

$f \rightsquigarrow T(f)(x) = \int_0^x f(t)\,dt \quad \forall x \in R$

T is 1-1 but not onto.

$\mathcal{N}(T) = \{0\}$ because $0 = T(f)(x) = \int_0^x f(t)\,dt \ \forall x$

$\Rightarrow f(x) = \text{zero} \ \forall x$

$\mathcal{R}(T) = \{ g(x) : g(x) \in C^1 \text{ with } g(0) = 0 \} \subset C[R]$

Validation of Theorem 11.1 on page 11.8

(i) $\mathcal{N}(T)$ is a subspace

proof: 1) $\mathcal{N}(T) \subset U$

2) $\mathcal{N}(T)$ is closed under lin. comb'ns. QED.

$\boxed{\text{dim } \mathcal{N}(T) \text{ is called the nullity of } T}$

(ii) $\mathcal{R}(T) = \{v : v = T(u) \text{ for some } u \in U\}$ is subspace

proof: 1) $v = \mathcal{R}(T) \Rightarrow$

$T: U \to V$

$u \rightsquigarrow T(u) = v \in V \Rightarrow \mathcal{R}(T) \subset V$

2) Let $v_1, v_2 \in \mathcal{R}(T)$

We must show that $c_1 v_1 + c_2 v_2 \in \mathcal{R}(T)$

a) $\exists\, u_1 \,\&\, u_2 \ni v_1 = T(u_1) \,\&\, v_2 = T(u_2)$

b) $c_1 v_1 + c_2 v_2 = c_1 T(u_1) + c_2 T(u_2)$

$= T(c_1 u_1 + c_2 v_2)$

c) $\therefore c_1 v_1 + c_2 v_2 \in \mathcal{R}(T)$

$\mathcal{R}(T)$ is closed, hence a subspace.

$\boxed{\text{dim } \mathcal{R}(T) \text{ is called the rank of } T}$

(iii) T is onto $\Leftrightarrow V = \mathcal{R}(T)$

proof: $\Rightarrow$: $\mathcal{R}(T) = \{v : v = T(u) \text{ for some } u\} \subseteq V \ (*)$

T is onto: Let $v \in V$. Then $v = T(u)$ for some u.

i.e. $v \in \mathcal{R}(T)$. $\therefore V \subseteq \mathcal{R}(T) \ (**)$

Thus $V = \mathcal{R}(T)$.

$\Leftarrow$: $V = \mathcal{R}(T) \Rightarrow v \in V \Rightarrow v \in \mathcal{R}(T)$, i.e. $v = T(u)$ for some u.

i.e. T is onto.

(iv) T is 1-1 $\Leftrightarrow \mathcal{N}(T) = \{\vec{0}_U\}$

$\Rightarrow$: Let $u \in \mathcal{N}(T)$ i.e. $\left.\begin{array}{l} T(u) = 0 \\ T(0) = 0 \end{array}\right\} \Rightarrow T(u) = T(0)$

T is 1-1 $\Rightarrow u = \vec{0}$ $\therefore \mathcal{N}(T) = \{\vec{0}\}$.

$\Leftarrow$: Consider u_1 and $u_2 \ni T(u_1) = T(u_2)$

then $T(u_1 - u_2) = 0$

But $\mathcal{N}(T) = \{0\}$ $\therefore u_1 - u_2 = 0$

Thus $u_1 = u_2$ $\therefore$

i.e. T is one-to-one.

Corollary to Theorem 11.1

Given: $T: U \to V$ be a 1-1 lin. xformation

$\{u_1, \dots, u_m\}$ lin. indep set in U.

$\Leftrightarrow \{T(u_1), \dots, T(u_m)\}$ lin. indep. set in V

i.e. 1-1 xformations preserve lin. indep. sets.

proof $\Rightarrow$: consider

$c_1 T(u_1) + \cdots + c_m T(u_m) = 0$

$T(c_1 u_1 + \cdots + c_m u_m) = 0$

T is 1-1 $\Rightarrow c_1 u_1 + \cdots + c_m u_m = 0$

$\{u_1 \cdots u_m\}$ lin indep. $c_1 = \cdots = c_m = 0$ is the only sol'n.

$\therefore \{T(u_1), \dots, T(u_m)\}$ is lin indep.

proof $\Leftarrow$: Consider $c_1 u_1 + \cdots c_m u_m = \vec{0}_U$

$c_1 T(u_1) + \cdots + c_m T(u_m) = \vec{0}$

$c_1 = c_2 = \cdots = c_m = 0$ is the only solution $\therefore u_1 \cdots u_m$ ol. i.

Note: We did not use the fact that T is 1-1.

11.14

Given a linear transformation

$$T: U \to V,\quad u \rightsquigarrow v = T(u)$$

Recall that

$\{v : v = T(u) \text{ for some } u \in U\} = R(T) \subset V =$ "target space" of T

is called the range (space) of T.

Let

$$B = \{e_1, \dots, e_p\}$$

be a basis for U, the domain (space) of T.

$$\boxed{\text{Proposition: } R(T) = \text{span}\{T(e_1), \dots, T(e_p)\}}$$ Thm 15.1 in J&R.

proof: 1.) Let $v \in R(T) = \{v : v = T(u) \text{ for some } u \in U\}$. def'n

$$v = T(u) = T(c_1 e_1 + \dots + c_p e_p)$$
$$= c_1 T(e_1) + \dots + c_p T(e_p) \Rightarrow v \in sp\{T(e_i)\}_{i=1}^{p}$$
$$\therefore R(T) \subseteq sp\{T(e_i)\}_1^p$$

2. Let $v \in sp\{T(e_i)\}_1^p \Rightarrow \exists\, c_1, \dots, c_p$ such that

$$v = c_1 T(e_1) + \dots + c_p T(e_p)$$
$$= T(c_1 e_1 + \dots + c_p e_p) \Rightarrow v \in R(T).$$
$$\therefore sp\{T(e_i)\}_1^p \subseteq R(T)$$

3.) $sp\{T(e_i)\} = R(T)$.