LECTURE 11

1. Transformations: Who Needs Them?
2. Linear Transformations: Their Geometric and Algebraic Definition.
3. Onto and One-to-one Transformations
4. Null Space and Range Space: Their Importance
5. Examples

Linear Transformations.

The complexity of mathematics is a reflection of the complexity of causal relations that exist in the world.

One's quantitative grasp of the nature of the physical world is achieved by mathematical relations. Of these, linear relations, also known as linear transformations, are the easiest to understand, the best understood, and the most ubiquitous in a person's mathematical arsenal of concepts.

A Linear transformation expresses (= is) a relationship between vector spaces, and as such between R^m and R^n for finite-dimensional vector spaces.

The distinguishing feature of a linear transformation is that it relates <u>closed triangles</u>, more generally closed polygons, to <u>closed triangles</u>:

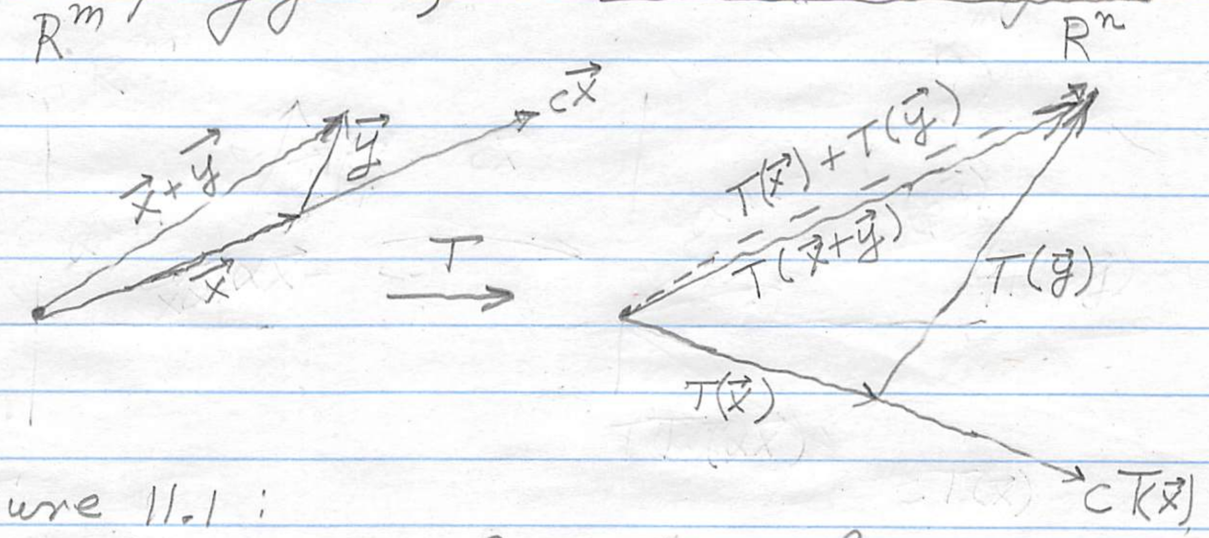


Figure 11.1: The linear tranformation of a linear combination (=superposition) of vectors into a linear superposition of the transformed vectors.

This geometrical feature is expressed algebraically by means of the following

Definition 11.1 (Linear transformation)

Let U and V be vector spaces.

Consider the map

$$T : U \to V$$
$$u \rightsquigarrow T(u)$$

Then T is a linear transformation whenever

$$T(c_1 u + c_2 w) = c_1 T(u) + c_2 T(w)$$

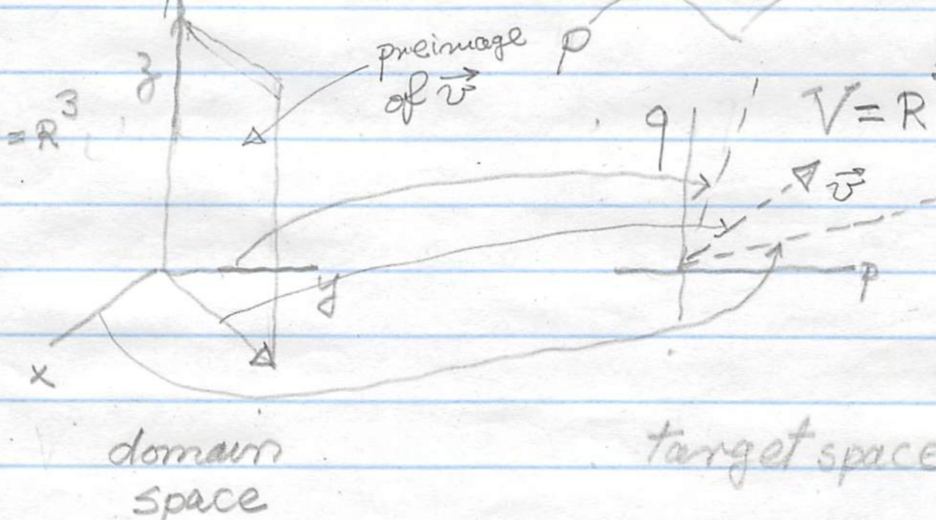
for all $u, w \in U$ and $c_1, c_2 \in$ scalars.

Ex. 1 $U = R^2$ $\qquad V = R^3$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$= x \begin{bmatrix} a \\ c \\ e \end{bmatrix} + y \begin{bmatrix} b \\ d \\ f \end{bmatrix}$$

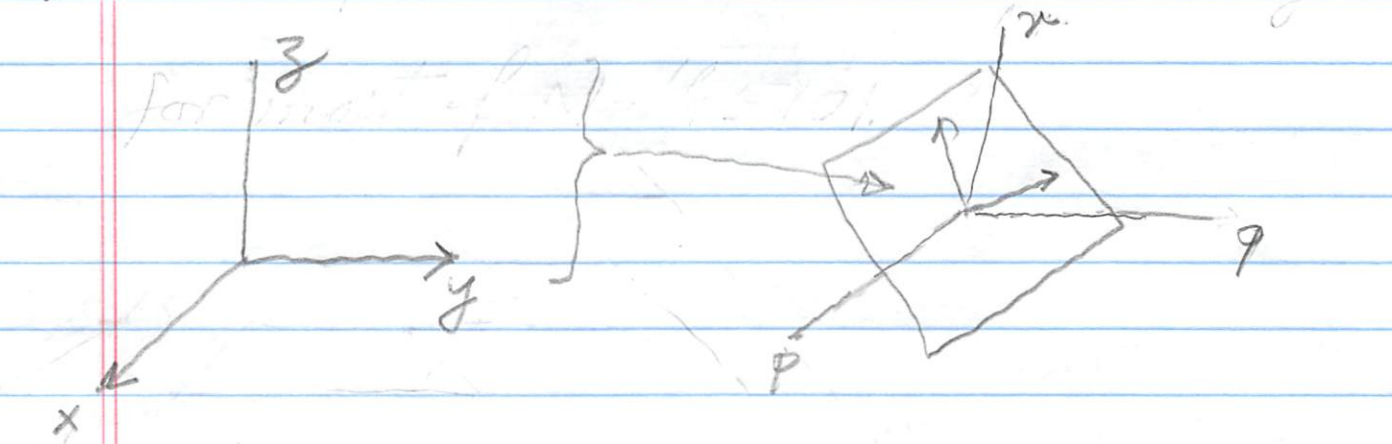
Ex. 2 $U = R^3$ $\qquad V = R^2$



$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

$$ax + by + 0z = p$$
$$cx + dy + 0z = q$$

Ex.3 $U = R^3$ $V = R^3$



$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$x\begin{bmatrix} a \\ c \\ e \end{bmatrix} + y\begin{bmatrix} b \\ d \\ f \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{matrix} p \\ q \\ r \end{matrix}$$

Comment

(i) U is called the domain or the domain space of T.
(ii) V is called the target or the target space of T.
(iii) The set $\{v: v=T(u);\ u\in U\}$ is called the range or the range space of T.

Linear transformations come in two (not necessarily mutually exclusive) flavors: (I) onto or surjective and (II) one-to-one or injective as defined by

Definition 11.2a (onto map = surjective map)

T is said to be an onto map if

for any $v\in V$ there exists at least one $u\in U$ such that

$$T(u)=v; \qquad (*)$$

in other words
(i) if $\exists$ a solution to Eq. $(*)$
or
(ii) if Eq. $(*)$ can be solved for u
or

(iii) if the set $\{u : T(u) = v\} \neq \phi$ (not the empty set) $\forall v \in V$.

Comment:

(i) $T(u)$ is called the image of u of (or "under") the transformation T.

(ii) The set $\{u : T(u) = v\} \equiv T^{-1}(v)$ is called the preimage of v. Note that the preimage is a set!

The concept of one-to-oneness (or injectivity) is defined by

Definition 11.2b (one-to-one map = injective map)

T is said to be one-to-one if

$$T(u_1) = T(u_2) \Rightarrow u_1 = u_2 ,$$

in other words, if the solution to
$$T(u) = v$$
is unique.

There are two sets of properties central to all of linear algebra. On one hand there is the (a) spanning property and the (b) linear independence of a given set of elements of a vector space. On the other hand there is the (a) onto and (b) one-to-one property of a given transformation. These two pairs of properties are not mere juxtapositions. On the contrary, they are logically equivalent. The cause of this equivalence are the concepts of existence and uniqueness of ~~a~~ the equations:

of solutions

spanning ppty

existence: $c_1 v_1 + \cdots + c_p v_p = \vec{v} \leftrightarrow T(\vec{u}) = \vec{v} \leftrightarrow T$ is onto

uniqueness: $c_1 v_1 + \cdots + c_p v_p = \vec{0} \leftrightarrow T(\vec{u}) = \vec{0} \leftrightarrow T$ is one-to-one

lin. independence

The criterion whether or not T is <u>onto</u> or/and <u>one-to-one</u> can be restated in terms of two fundamental subspaces that characterize T. Doing so highlights the underlying similarity that unites into an intelligible whole the plethora of concepts that make up the landscape of linear mathematics.

GO TO PAGE 11.9

This criterion is stated in terms of the following important

Theorem 11.1 (T-induced subspaces)

Let $T: U \to V$ be a linear xformation then

(i) $\{u : T(u) = 0_V\} \equiv \mathcal{N}(T) \equiv \ker(T)$

is a subspace of U, called the nullspace or the kernel of T; $\boxed{\dim \mathcal{N}(T) = \text{nullity of } T}$

(ii) $\{v : v = T(u) \text{ for some } u \in U\} \equiv \mathcal{R}(T)$ is a

is a subspace of V, called the range of T. $\boxed{\dim \mathcal{R}(T) = \text{rank of } T}$

(iii) T is onto $\iff V = \mathcal{R}(T)$ see page 11.12

(iv) T is one-to-one $\iff \mathcal{N}(T) = \{0_U\}$. see page 11.13

Example 0
Example 1
Example 2
Example 3
Example 4 skip,
Example 5

Example #0

$$T: R^3 \to R^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightsquigarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

T is onto but not 1-1.

$\mathcal{R}(T) = R^2$

$\mathcal{N}(T) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Example #1

$$T: P_2 \longrightarrow R^1$$

$$p(x) = a_0 + a_1 x + a_2 x^2 \rightsquigarrow p(t_1) = a_0 + a_1 t_1 + a_2 t_1^2$$

T is onto but not 1-1

$\mathcal{R}(T) =$ entire real line

$\mathcal{N}(T) = \{ a(x - t_1)(x - b) : a, b \in \text{reals} \}$

Example #2

Let $B = \{u_1, \ldots, u_p\}$ be a basis for U.

$$T: U \longrightarrow R^p$$

$$u \rightsquigarrow T(u) = [u]_B = p\text{-tuple of coord. components rel've to } B.$$

T is onto and 1-1

$\mathcal{R}(T) = R^p$

$\mathcal{N}(T) = \{0_v\}$

Only mention this one

11.11

Example #3 ($y' + ay = g$)

Let $C^1(0,1)$ = set of all fns with cont. 1st derivatives in $(0,1)$

Let $a(x) \in C(0,1)$ = set of cont. fns on $(0,1)$

$$T: \quad C'(0,1) \to C(0,1)$$
$$f \mapsto T(f) = \frac{df}{dx} + a(x) f$$

T is onto, but not 1-1

$\boxed{R(T) = V = C(0,1)}$ because for any $g(x) \in C(0,1)$, the diff'l equation

$\boxed{\dim R(T) = \infty}$ $\quad$ $\boxed{y' + a(x) y = g(x) \text{ has a sol'n}}$

i.e. $\exists\ y \in C^1(0,1) \ni T(y) = g(x)$

i.e. any $g \in C(0,1)$ has a preimage

$$T^{-1}(g) = \left\{ y(x) = e^{-\int^x a\,dx'} \cdot \int^x e^{\int^{x'} a\,dx''} g(x')\,dx' \right\}$$

Note: $y' = -ay + e^{-\int^x a\,dx'} e^{\int^x a\,dx''} g(x)$

$N(T)$ = set of sol'ns to $y' + a(x) y = 0$,
$= \{ f = c_1 \exp(-\int^x a\,dx') \}$

$\dim N(T) = 1$

Read 4.7 (5.7) Thm 14, 15

only mention this one

11.12

Example 3b ($y'' + Py' + Qy = g$)

$T: \; U = C^2(0,1) \longrightarrow V = C(0,1)$

$$y \mapsto T(y) = \left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\right](y)$$

T is onto:

$\forall g \in C(0,1) \; \exists$ a y s.t. $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Qy = g(x)$

i.e. $\forall g \in V \; \exists \; y \in U$ s.t. $T(y) = g$

i.e. one can always solve the d.e. $T(u) = g$

i.e.

$\boxed{R(T) = C(0,1)}$ $\dim R(T) = \infty$

T is not 1-1

$N(T) = \{ y : y = c_1 u_1 + c_2 u_2$ where $\left[\frac{d^2}{dx^2} + P\frac{d}{dx} + Q\right] u_i = 0,\; i = 1, 2 \}$

= set of sol'ns to the homogeneous eq'n $T(y) = 0$

$\dim N(T) = 2$

Example# 4

$$T : \mathcal{P}_k \to \mathcal{P}_{k-1}'$$
$$p(x) \rightsquigarrow p'(x)$$

$\mathcal{R}(T) = \mathcal{P}_{k-1}$ because $p'(x) = b_0 + b_1 x + \cdots + b_{k-1} x^{k-1}$ can always be solved for p

$\mathcal{N}(T) = \{ p(x) = a_0 \mid a_0 \text{ any real } \# \}$

Example#5

Let $C[R]$ = set of cont. fns defined on R

Let $f \in C[R] : R \to R$

$$T : C[R] \to C^1[R]$$
$$f \rightsquigarrow T(f)(x) = \int_0^x f(t)\,dt \quad \forall x \in R$$

T is 1-1 but not onto.

$\mathcal{N}(T) = \{0\}$ because $\theta = T(f)(x) = \int_0^x f(t)\,dt \;\; \forall x$

$\Longrightarrow f(x) = \text{zero} \;\; \forall x$

$\mathcal{R}(T) = \{ g(x) : g(x) \in C^1 \text{ with } g(0) = 0 \} \subset C[R]$

Validation of Theorem 11.1 on page 11.8

(i) $\mathcal{N}(T)$ is a subspace

proof: 1) $\mathcal{N}(T) \subset U$ & $\mathcal{N}(T)$ is non-empty

2) $\mathcal{N}(T)$ is closed under lin. comb'ns. QED

$\boxed{\text{dim } \mathcal{N}(T) \text{ is called the nullity of } T}$

(ii) $\mathcal{R}(T) = \{v : v = T(u) \text{ for some } u \in U\}$ is a subspace $\subset V$

proof: 1) Let $\boxed{v \in \mathcal{R}(T)}$. This means

$T: U \to V$

$u \rightsquigarrow T(u) = \boxed{v \in V} \Rightarrow \mathcal{R}(T) \subset V$:

$\mathcal{R}(T)$ is a non-empty subset of V.

2) Let $v_1, v_2 \in \mathcal{R}(T)$

We must show that $c_1 v_1 + c_2 v_2 \in \mathcal{R}(T)$

a) $\exists\, u_1 \,\&\, u_2 \ni v_1 = T(u_1) \,\&\, v_2 = T(u_2)$

b) $c_1 v_1 + c_2 v_2 = c_1 T(u_1) + c_2 T(u_2)$

$= T(c_1 u_1 + c_2 v_2)$

c) $\therefore c_1 v_1 + c_2 v_2 \in \mathcal{R}(T)$

i.e. $\mathcal{R}(T)$ is closed, hence, by the subspace thm,

$\boxed{\mathcal{R}(T) \text{ is a subspace.}}$

$\boxed{\text{dim } \mathcal{R}(T) \text{ is called the rank of } T}$

We know $\mathcal{R}(T) \subseteq V$. Q: But is it true that $\mathcal{R}(T) = V$?

(iii) $\boxed{T \text{ is onto} \Leftrightarrow V = \mathcal{R}(T)}$ $\boxed{\text{defining ppty of } T}$

proof: $\Rightarrow$: $\boxed{\mathcal{R}(T) = \{v : v = T(u) \text{ for some } u\} \subseteq V}$ (*)

T is onto: Let $v \in V$. Then $v = T(u)$ for some u.

i.e. $v \in \mathcal{R}(T)$. $\therefore$ $\boxed{V \subseteq \mathcal{R}(T)}$ (**)

Thus $V = \mathcal{R}(T)$.

$\Leftarrow$: $V = \mathcal{R}(T) \Rightarrow v \in V \Rightarrow v \in \mathcal{R}(T)$, i.e. $v = T(u)$ for some u,

i.e. T is onto.

(iv) $\boxed{T \text{ is } 1\text{-}1 \iff \mathcal{N}(T) = \{\vec{0}_U\}}$

$\Rightarrow$: Let $u \in \mathcal{N}(T)$ i.e. $\left.\begin{array}{l} T(u) = 0 \\ T(0) = 0 \end{array}\right\} \Rightarrow T(u) = T(0)$

T is 1-1 $\Rightarrow u = \vec{0}$. $\therefore \mathcal{N}(T) = \{\vec{0}\}$.

$\Leftarrow$: Consider u_1 and $u_2 \ni T(u_1) = T(u_2)$
then $T(u_1 - u_2) = 0$
But $\mathcal{N}(T) = \{0\}$ $\therefore u_1 - u_2 = 0$
Thus $u_1 = u_2$ $\therefore$
i.e. T is one-to-one.

SKIP THIS BECAUSE IT'S DONE IN Lecture 12 as Thm 12.1 on page 12.3

Corollary to Theorem 11.1
Given: $T: U \to V$ be a 1-1 lin. xformation

$\{u_1, \dots, u_m\}$ lin. indep set in U.

$\iff \{T(u_1), \dots, T(u_m)\}$ lin. indep. set in V

i.e. 1-1 xformations preserve lin. indep. sets.
i.e. if $\{u_1, \dots u_m\}$ into lin. indep. directions, then so will $\{T(u_1), \dots, T(u_m)\}$

proof $\Rightarrow$: consider

$$c_1 T(u_1) + \cdots + c_m T(u_m) = 0$$
$$T(c_1 u_1 + \cdots + c_m u_m) = 0$$

T is 1-1 $\Rightarrow$ $c_1 u_1 + \cdots + c_m u_m = 0$
$\{u_1, \dots u_m\}$ lin indep. $c_1 = \cdots = c_m = 0$ is the only sol'n.
$\therefore \{T(u_1), \dots, T(u_m)\}$ is lin. indep.

proof $\Leftarrow$: Consider $c_1 u_1 + \cdots c_m u_m = \vec{0}_U$

$$c_1 T(u_1) + \cdots + c_m T(u_m) = \vec{0}_V$$

$c_1 = c_2 = \cdots = c_m = 0$ is the only solution $\therefore u_1, \cdots u_m$ l. i.
Note: We did not use the fact that T is 1-1.