$$V \xrightarrow{T} W$$

induces

$$V^* \xleftarrow{T^t} W^*$$

A transformation between vector spaces
($V \xrightarrow{T} W$)
induces a transformation between their

I dual spaces $V^* \xleftarrow{T^t} W^*$

$\left[\mathbb{R} \xleftarrow{\vec{v}} \right]$ V^* W^*

$\left[\vec{v}(g) \equiv g(\vec{v}) \leftarrow \right] g$ f

$$V \xrightarrow{T} W \xrightarrow{f} \mathbb{R}$$

$$\vec{v} \rightsquigarrow \vec{w} = T(\vec{v}) \overset{f}{\rightsquigarrow} f(\vec{w}) = f(T(\vec{v})) = g(\vec{v})$$

$\forall \vec{v}$

$$f \circ T(\vec{v}) = g(\vec{v})$$

i.e

$$g = f \circ T$$

Thus for any $f \in W^*$ $\exists$ a unique g, namely

$$g = f \circ T$$

Thus we have

$$\boxed{\begin{array}{l} W^* \xrightarrow{T^t} V^* \\ f \rightsquigarrow T^t(f) = f \circ T \end{array}}$$

II a) Let $B = \{e_i\}_1^n$ for V, $C = \{e'_{k'}\}^m$ for W be bases

$$e_i \rightsquigarrow T(e_i) = \sum^m e'_{k'} A^{k'}{}_{i} \qquad i = 1 \cdots n$$

$$\vec{v} \rightsquigarrow T(e_i v^i) \overset{!}{=} e'_{k'} A^{k'}{}_i v^i$$

$$\{v^i\} \rightsquigarrow \{A^{k'}{}_i v^i\} = [A]_{CB} [v]_B$$

b) Let $B^* = \{\omega^j\}_1^n$ for V^* $\quad C^* = \{\omega'^{\ell'}\}_1^m$ be the bases for W^*

Consequently, $\omega^j(e_i) = \delta^j_i$ $\quad \omega'^{\ell'}(e'_{k'}) = \delta^{\ell'}_{k'}$

$n \times n$ $\qquad m \times m$

III

a) Consider $T(\vec{v}) = \sum_k \vec{e}'_{k'} A^{k'}{}_i v^i$

and hence

$$w^{\ell} = \omega^{\ell'}(\vec{w}) = \omega^{\ell'}(T(\vec{v})) = \omega'^{\ell'}\left(\sum_{k'} \vec{e}'_{k'} A^{k'}{}_i v^i\right)$$

$$= \sum_{k'} \delta^{\ell'}_{k'} A^{k'}{}_i v^i$$

$$\boxed{w^{\ell'} = A^{\ell'}{}_i v^i}$$

Thus we have

$$\{v^i\} \rightsquigarrow \{w^{\ell}\} = \{A^{\ell}{}_i v^i\}$$

or in terms of their matrix representatives

$$[v]_B \rightsquigarrow [w_C] = [A]_{CB}[v]_B$$

$[A]_{CB}$ is $m \times n$

b) Consider an arbitrary $f \in W^*$. We have

$f = f_{k'} \omega'^{k'}$

Q: What are the components $\{g_i\}$ of

the corresponding $g = f \circ T \equiv g_i \underset{\sim}{\omega}^i$

A: Evaluate

$$g_j = g(\vec{e}_j) = \underset{\sim}{f} \circ T(e_j) = \underset{\sim}{f}(\vec{e}_{\ell'} A^{\ell'}{}_j)$$

$$= f_{k'} \underbrace{\underset{\sim}{\omega}^{k'}(\vec{e}_{\ell'})}_{\delta^{k'}{}_{\ell'}} A^{\ell'}{}_j$$

$$\boxed{g_j = f_{k'} A^{k'}{}_j \quad \Longleftrightarrow \quad [g_1, \cdots, g_n] = [f_1, \cdots, f_m][A]_{CB}}$$

Thus

$$\boxed{\underset{\sim}{g} = f_{k'} A^{k'}{}_j \underset{\sim}{\omega}^j} = T^t(\underset{\sim}{f}) \; [= f \circ T] \in W^*$$

which is to be compared with

$$\boxed{\vec{w} = \vec{e}_{k'} A^{k'}{}_i v^i} = T(\vec{v}) \in W$$

IV. Summary

a)

$$\begin{array}{ccc} g = & & f \\ \omega^i & & \omega^{k'} \\ V^* & \xleftarrow{T^t} & W^* \end{array}$$

$\{g_j = f_{k'} A^{k'}{}_j\}_1^n$ $\longleftarrow$ $\{f_{k'}\}_1^m$

or

$$[g_1, \cdots, g_n] = [f_1, \cdots, f_m] \underset{\substack{\uparrow \\ m \times n \\ \text{matrix}}}{[A]_{CB}} \longleftarrow [f_1, \cdots, f_m]$$

b) $\vec{v} \rightsquigarrow T(\vec{v}) = e_{k'} w^{k'} = T(e_j) v^j = e_{k'} A^{k'}{}_j v^j$

$$e_j \rightsquigarrow T(e_j) = e_{k'} A^{k'}{}_j$$

$$V \xrightarrow{T} W$$

$$\{v^j\} \rightsquigarrow \{w^{k'} = A^{k'}{}_j v^j\}$$

$$\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \rightsquigarrow \begin{bmatrix} w^1 \\ \vdots \\ w^m \end{bmatrix} = [A]_{CB} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

$m \times n$
matrix

which is to be compared with $\underline{IV}$ a)

$$[A]^t [f_1, \dots, f_m]^t = [g_1, \dots g_n]^t$$

i.e.

$$A^t \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$