Simple Harmonic Oscillator Driven by Periodic Forces

GIVEN:

$$\underbrace{\left[\frac{d^2}{dt^2}+\omega^2\right]}_{T}(\psi) = F_1\cos(\omega_1 t+\delta_1)+F_2\cos(\omega_2 t+\delta_2)+F_3\cos(\omega_3 t+\delta_3)$$

$$T : U \longrightarrow V = Sp\left(\{\cos\omega_i t, \sin\omega_i t\}_{i=1}^{3}\right)$$

U ← Space of sol'ns to $T(\psi)=\sum_{i=1}^{3} F_i\cos(\omega_i t+\delta_i)$

$$\psi \rightsquigarrow T(\psi)$$

$$\mathcal{N}(T) = Sp(\{\underbrace{\cos\omega t}_{\vec{x}_1}, \underbrace{\sin\omega t}_{\vec{x}_2}\})$$

$$\mathcal{R}(T) = Sp(\{\underbrace{\cos\omega_1 t}_{\vec{v}_1}, \underbrace{\cos\omega_2 t}_{\vec{v}_2}, \underbrace{\cos\omega_3 t}_{\vec{v}_3}, \underbrace{\sin\omega_1 t}_{\vec{v}_{3+1}}, \underbrace{\sin\omega_2 t}_{\vec{v}_{3+2}}, \underbrace{\sin\omega_3 t}_{\vec{v}_{3+3}}\})$$

SOLUTION:

$$\frac{w_1}{-\omega_1^2} \;\mapsto\; \text{etc}$$

$$\psi = c_1\cos\omega t + c_2\sin\omega t + d_1\cos\omega_1 t + d_2\cos\omega_2 t + d_3\cos\omega_3 t$$

$$+ d_{3+1}\underbrace{\sin\omega_1 t}_{\frac{w_4}{-\omega_1^2}} + d_{3+2}\sin\omega_2 t + d_{3+3}\sin\omega_3 t \quad \ldots \text{etc}$$

c_1, c_2: determined by initial condxns

d_i terms: a particular solution to $T(\psi) = \sum F_i\cos(\omega_i t+\delta_i)$

Basis for $U = \{\cos\omega t, \cos\omega t, \cos\omega_1 t, \cos\omega_2 t, \cos\omega_3 t, \sin\omega_1 t, \sin\omega_2 t, \sin\omega_3 t\}$

$$\therefore \boxed{\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim V}$$