# LECTURE 12

1. T-induced Basis Properties

2. Dimensional Consistency.

3. Direct Sum of two Vector spaces

# II. T-induced Linear Independence and Spanning Properties.



The remarkable aspect of Theorem 11.1 (about T-induced subspaces) on page 11.9 concerning the linear transformation

$$T : U \to V,$$

namely

| | |
|---|---|
| T is onto $\Leftrightarrow$ $V = R(T)$ | 2 particular kinds of T-induced subspaces. |
| T is 1-1 $\Leftrightarrow$ $N(T) = \{0_U\}$, | |

is the fact that it is universal: It holds regardless of one's choice of basis; the boxed result is a <u>coordinate independent result</u>; it is a very abstract and hence all encompassing result in that it is very powerful; it is a pair of statements that applies to all of finite dimensional linear mathematics regardless of which particular bases one happens to be working with.

However, to prevent these statements from being mere _floating_ abstractions i.e. abstractions disconnected and detached from reality, they must be logically tied to it by means of some chosen bases.

This is achieved by identifying the T-induced relations between sets of vectors in regard to their linear independence and their spanning properties, and hence in regard to one's chosen basis.

One of the benefits of this identification is that one can turn this induction process around and obtain a (or the) transformation from the relation between the set of vectors in $U$ and $V$.

The two $T$-induced relations between the spanning sets and linearly independent sets are captured by means of the following

**Theorem 12.1** (T-induced Basis Properties)

Given: a) $T: U \rightarrow V$ a linear transformation

b) $B = \{u_1, u_2, \cdots, u_p\}$ a basis for $U$.

**Conclusion**

(1) $\text{Span}\{T(u_1), \cdots, T(u_p)\} = R(T)$

(2) $T$ is 1-1 $\Leftrightarrow \{T(u_1), \cdots, T(u_p)\}$ is lin. independent in $V$.

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The validation of (1) is straight forward.

a) Let $v \in R(T)$ (*)

Then $\exists\, u$ s.t. $T(u) = v$

Expand $u$ in terms of $\mathcal{B}$: $\exists \{c_1, \ldots, c_p\}$ s.t.

$$u = c_1 u_1 + \cdots + c_p u_p$$

Thus $v = T(c_1 u_1 + \cdots + c_p u_p) = c_1 T(u_1) + \cdots + c_p T(u_p)$

i.e. $v \in Sp\{T(u_1), \ldots, T(u_p)\}$. (**)

Eqs. (*) and (**) $\Longrightarrow$ $\boxed{R(T) \subseteq Sp\{T(u_i)\}}$ ☆

b) Let $v \in Sp\{T(u_i)\}_{i=1}^{p}$

Hence $\exists \{c_i\}$ s.t.

$$v = c_1 T(u_1) + \cdots + c_p T(u_p)$$

$$= T(c_1 u_1 + \cdots + c_p u_p) \in R(T)$$

Thus $\boxed{Sp\{T(u_i)\}_i \subseteq R(T)}$ (☆☆)

Eqs. (☆) and (☆☆) $\Longrightarrow$ $\boxed{Sp\{T(u_i)\} = R(T)}$

The validation of (2) is as follows:

$\Rightarrow$ :

Consider

$$c_1 T(u_1) + \cdots + c_p T(u_p) = \vec{0}_V \qquad (1)$$

Thus $T(c_1 u_1 + \cdots + c_p u_p) = \vec{0}_V$

$T$ is one-to-one $\Rightarrow \mathcal{N}(T) = \{\vec{0}_U\}$

$$\Longrightarrow c_1 u_1 + \cdots + c_p u_p = \vec{0}_U.$$

The set $\{u_i\}$ is a basis, hence is a lin. indep. set

Consequently

$$c_1 = \cdots = c_p = 0 \qquad (2)$$

Thus $\{T(u_i)\}$ is lin. independent indeed.

$\Leftarrow$ :

Consider $v \in Sp.\{T(u_i)\}$:

$$c_1 T(u_1) + \cdots + c_p T(u_p) = v \in \mathcal{R}(T)$$

Let $\vec{s}$ and $\vec{t}$ be two pre-image vects of $v$:

$$T(\vec{s}) = v = T(\vec{t})$$

Expanding in terms of basis elements of $B$, one has

$$T(\overbrace{s_1 u_1 + \cdots + s_p u_p}^{\vec{s}}) = v = T(\overbrace{t_1 u_1 + \cdots + t_p u_p}^{\vec{t}})$$

Consequently,

$$T\left((s_1 - t_1)u_1 + \cdots + (s_p - t_p)u_p\right) = \vec{0}_V$$

$$(s_1 - t_1)\,T(u_1) + \cdots + (s_p - t_p)\,T(u_p) = \vec{0}_V$$

The set $\{T(u_i)\}$ is linearly independent.
Thus

$$s_1 - t_1 = \cdots = s_p - t_p = 0$$

or

$$s_1 = t_1 ; \; \cdots ; \; s_p = t_p$$

Consequently

$$\vec{s} = \vec{t}$$

Putting everything together, one has

$$T(\vec{s}) = v = T(\vec{t}) \Rightarrow \vec{s} = \vec{t},$$

i.e. T is one-to-one indeed.

Dimensional Consistency

Every linear transformation,

$$T: U \to V,$$

is chacterized by its two fundamental subspaces; its null space (or kernel) $\boxed{\mathcal{N}(T) \subseteq U}$ and its range space $\boxed{\mathcal{R}(T) \subseteq V}$.

They are subspaces of $T$'s domain space $U$ and target space $V$ respectively.

In finite dimensional linear mathematics each of these four vector spaces has its own $T$-determined dimension.

Question: Is there a constraint on their dimensions?

The following two examples suggest

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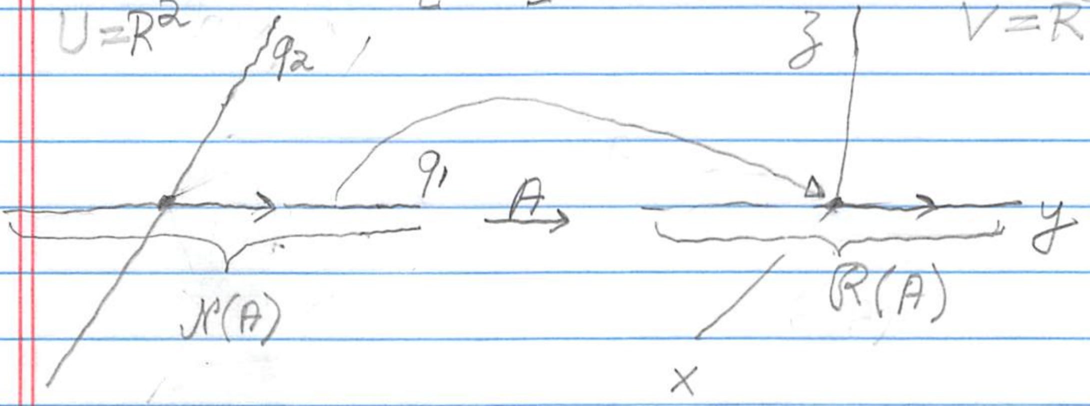
an answer to this question

## Example 1.

Let $T$: $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ : $U = R^2 \longrightarrow V = R^3$

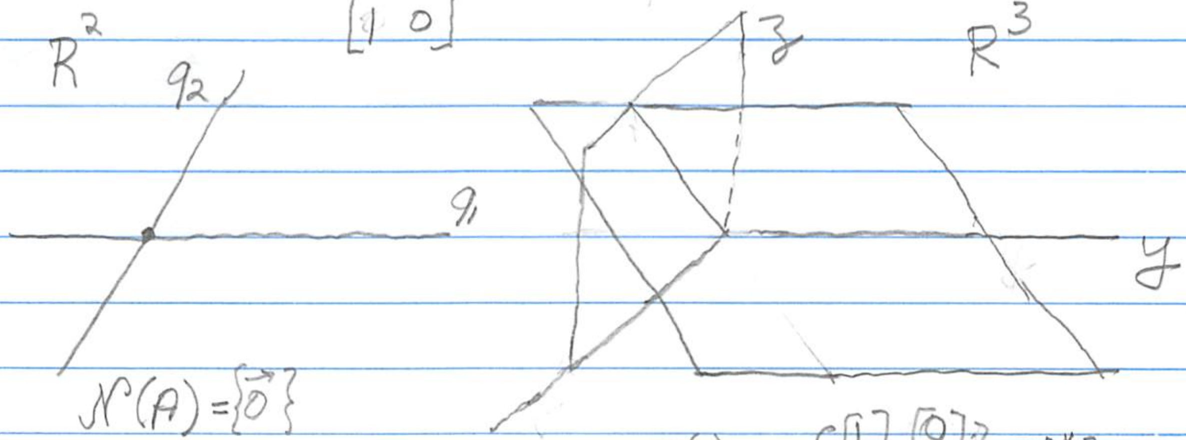



$\mathcal{N}(A) = sp\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$  $\mathcal{R}(A) = sp\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

Note: $\dim \mathcal{N}(A) + \dim \mathcal{R}(A) = \dim U$ ; $1+1=2$

## Example 2.

Let $T$: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ : $U = R^2 \longrightarrow V = R^3$




$\mathcal{N}(A) = \{\vec{0}\}$

$\mathcal{R}(A) = sp\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - z = 0\right\}$

Note:

$\underbrace{\dim \mathcal{N}(A)}_{\text{"Nullity (A)"}} + \underbrace{\dim \mathcal{R}(A)}_{\text{"Rank(A)"}} = 2 \ (= \dim U)$ for $\begin{cases} 1. & 1+1=2 \\ 2. & 0+2=2 \end{cases}$

## Example 3 (Differential Transformation)

Let $T = \frac{d^2}{dx^2}$ ; $U = \mathbb{P}_4 \longrightarrow V = \mathbb{P}_2$

$p \mapsto T(p) = p''$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \mapsto T(p)(x) = 2a_2 + 6a_3 x + 12 a_4 x^2$

PROBLEM:

What is a basis for $\mathbb{R}(T)$, $\mathcal{N}(T)$? Is $T$ onto, one-to-one?

SOLUTION:

1.) Start with a basis for the domain space $\mathbb{P}_4$,

$\{1, x, x^2, x^3, x^4\}$

2.) Consider its images under $T = \frac{d^2}{dx^2}$,

$T(1) = 0,\ T(x) = 0,\ T(x^2) = 2,\ T(x^3) = 6x,\ T(x^4) = 12x^2$

In accordance with (the Conclusion) Eq. (1), at the bottom of Page 12.3, these images form a spanning set for $R(T)$.

3.) a) Identify a linearly independent subset

$\{T(x^2) = 2,\ T(x^3) = 6x,\ T(x^4) = 12x^2\}$

as a basis for $R(T)$

b) $T$ is an onto map because

$R(T) = Sp\{T(x^n)\}_{n=2}^{4} = \mathbb{P}_2$

4. a) Identify $\{1, x\}$ as a basis for

$$\mathcal{N}(T) = sp\{1, x\}$$

b) $T$ is <u>not</u> <u>one-to-one</u>

5.) However,

$$\dim \mathcal{N}(T) + \dim R(T) =$$

$$= 2 + 3 = 5 = \dim P_4 \; (= \text{domain}(T))$$

These two examples illustrate the

Theorem 12.2 (Dimensional Consistency)

For a linear transformation

$$T: U \to V$$

$$\boxed{\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim U}.$$

In words one has: the dimension of the nullspace of T plus the dimension of its range space equals the dimension of it domain space.

[Before proving this theorem, consider]

Consider a system of $r$ linear inhomogeneous diff'l eq'ns

$$\boxed{\underset{\substack{\uparrow\\ \text{diff'l operator}}}{T}(\vec{x}) = \vec{v}_j \qquad j = 1, \ldots, r}$$

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12.12.

Proof of Theorem 12.2.

$\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim(\text{Domain}(T))$ $(*)$

The idea of the proof is to construct a linearly independent spanning of $U$ from a basis of $\mathcal{N}(T)$ and a basis of $\mathcal{R}(T)$ directly and thereby arrive at Eq. $(*)$ as follows:

[sol'n to hom. eq'ns]

Let $\{x_1, \ldots, x_k : T(x_i) = 0,\ i = 1, \ldots, k\} \subseteq U$ be a basis for $\mathcal{N}(T)$

Let $\{v_1, \ldots, v_r\}$ be a basis for $\mathcal{R}(T)$

Let $\{w_1, \ldots, w_r : T(w_1) = v_1, \ldots, T(w_r) = v_r\} \subseteq U$ basis

be its preimage. (particular sol'n to the system.

We shall now show that

$\{x_1, \ldots, x_k, w_1, \ldots, w_r\} \equiv B$ is a basis for $U$

Domain

$T\{c_1 w_1 + \cdots c_r w_r = 0\}$

$c_1 T(w_1) + \cdots c_n T(w_r) = 0$

a) To show that

B is a <u>lin. indep. set</u>:

(i) consider

$$c_1 x_1 + \cdots + c_k x_k + d_1 w_1 + \cdots + d_r w_r = \vec{0}_U$$

(ii) apply T:

$$T( \text{ " } + \cdots + \text{ " } \quad + \quad \text{ " } + \cdots + \text{ " } ) = \vec{0}_V$$

$$\vec{0} + d_1 \underbrace{T(w_1)}_{v_1} + \cdots + d_r \underbrace{T(w_r)}_{v_r} = \vec{0}_V$$

(iii) note that

$\{v_1, \ldots, v_r\}$ is lin. indep $\Rightarrow d_1 = \cdots = d_n = 0$ (☆)

(iv) From step (i) note that

$$c_1 x_1 + \cdots + c_k x_k = \vec{0}_V,$$

which in light of

"$\{x_1, \ldots, x_k\}$ is lin. indep" implies that

$c_1 = \cdots = c_k = 0$ (☆☆)

(v) Eqs (☆) and (☆☆) imply that B is a

lin. indep. set

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b) That B is a <u>spanning set for U</u> is proved as follows

(i.e. any sol'n is some lin. comb'n of the elements of B)

(i) Let $u \in U$

(ii) Thus $T(u) \in R(T)$

Expand $T(u)$ in terms of $(v_i)$:

$$T(u) = d_1 v_1 + \cdots + d_r v_r.$$

(iii) Introduce the preimages of $v_1, \dots, v_r$. Thus one has

$$T(u) - [d_1 T(w_1) + \dots + d_r T(w_r)] = 0_V$$

or

$$T(u - d_1 w_1 - \dots - d_r w_r) = 0_V$$

(iv) This eq'n implies that the argument of $T$ belongs to the null space $\mathcal{N}(T)$:

$$u - d_1 w_1 - \dots - d_r w_r \in \mathcal{N}(T)$$

(v) Using the basis $\{x_1, \dots, x_k\}$ of $\mathcal{N}(T)$ one obtains

$$u - d_1 w_1 - \dots - d_r w_r = c_1 x_1 + \dots + c_k x_k$$

or

$$u = c_1 x_1 + \dots + c_k x_k + d_1 w_1 + \dots + d_r w_r \quad (*)$$

i.e.

B is a spanning set, and hence a basis for $U$. The partial sums in Eq. (*) refer to $\mathcal{N}(T)$ and the preimages of $\mathcal{R}(T)$. One has

$$\boxed{\begin{array}{c} k + r = \dim U \\ \text{or} \quad \dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim U \end{array}}$$

QED.

<u>Comment:</u>

The proof that

$$B = \{x_1, \ldots, x_k, w_1, \ldots, w_r\} \text{ is a basis for } U$$

implies that

$$\mathcal{N}(T) = \underbrace{Sp\{x_1, \ldots, x_k\}}_{\equiv X} \cap \underbrace{Sp\{w_1, \ldots, w_r\}}_{\equiv W} = \vec{0}_U$$

Moreover, for any $u \in V$, on has

$$u = x + w$$

where $x$ is a unique vector in $X$
and
$w$ is a unique vector in $W$.

This leads to the following

<u>Definition</u> (Direct Sum)

Let $X$ and $W$ be two subspace of $U$ such that

$$X + W = U$$

Then $U = X \oplus W$, the <u>direct sum</u> of $X$ and $W$ whenever

$$X \cap W = \{\vec{0}_U\}.$$