LECTURE 13

Operations with Linear Transformations

I. Sum of two linear xformations

II. Composition of linear xformation

III. Inverse of an invertible lin. xformation.

Operation with Linear Transformations

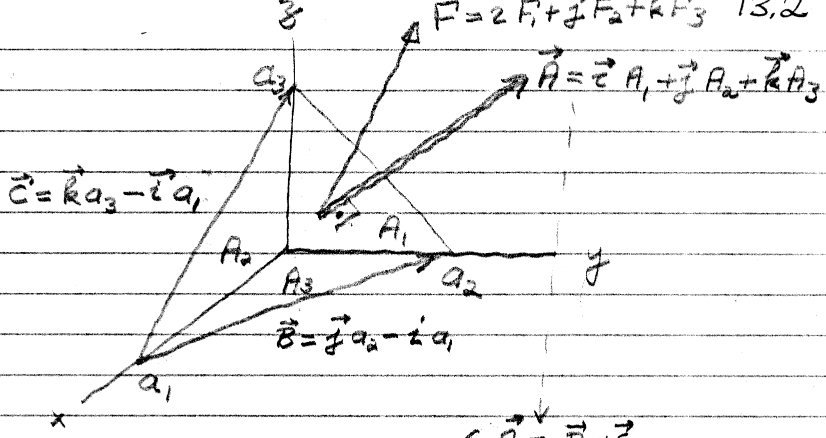
I. The Sum of Two Linear Transformations.

For a moving viscous fluid passing through a vectorial element of area $[A_i]$, (Figure 13,1 on page 13,2) the measured force vector $[F_j]$ acting on this area is expressed quantitatively in terms of the stress tensor (pressures, shear stresses) $[T_{ji}]$ which characterizes this fluid:

$$F_j = \sum_{i=1}^{3} T_{ji} A_i .$$

If there are two kinds of fluid, say type a and type b with stress tensors $[T^a_{ji}]$ and $[T^b_{ji}]$, then the force on this element of area is

$$F_j = \sum_{i=1}^{3} \left(T^a_{ji} + T^b_{ji}\right) A_i$$



$\vec{A}$: $A_1 = a_2 a_3$
$A_2 = a_3 a_1$
$A_3 = a_1 a_2$

$$\left\{ \begin{aligned} \vec{A} &= \vec{B} \times \vec{C} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a_1 & a_2 & 0 \\ -a_1 & 0 & a_3 \end{vmatrix} \\ &= \vec{i} \underbrace{a_2 a_3}_{A_1} + \vec{j} \underbrace{a_3 a_1}_{A_2} + \vec{k} \underbrace{a_1 a_2}_{A_3} \end{aligned} \right.$$

Figure 13,1:
Components of an element of area $\vec{A}$ as projections of this area onto the respective coordinate planes.

$$\boxed{(\vec{F})_j = \sum_i T_{ji} (\vec{A})_i}$$

$\uparrow$ force — $\uparrow$ area

The force – area relation

The observation of such a relation between the concept "area" and the concept "force" is mathematized by linear transformations

$$T^a : U \longrightarrow V$$
$$u_1+u_2 \mapsto T^a(u_1+u_2) = T^a(u_1) + T^a(u_2)$$

$$T^b : U \longrightarrow V$$
$$u_1+u_2 \mapsto T^b(u_1+u_2) = T^b(u_1) + T^b(u_2)$$

and the definition of their sum:

Definition 13.1 (Sum of two linear transformations)

The linear tranformation

$$T^{a+b} = T^a + T^b$$

defined by the function

$$T^{a+b} : U \longrightarrow V$$
$$u \mapsto T^{a+b}(u) = T^a(u) + T^b(u)$$

is called the sum of T^a and T^b.

It is an easy exercise to validate the following:

Theorem 13.1

If $T^a : U \to V$ and $T^b : U \to V$ are linear transformations, then so is is their sum:

$$T^{a+b} \equiv T^a + T^b : U \to V.$$

and their scalar multiple defined by

$$cT^a : U \longrightarrow V$$
$$u \mapsto (cT^a)(u) \equiv c(T(u)) \quad \forall u \in U$$

Comment:

At the risk of stating the obvious, note that the c scalar multiple of T^a depends on the validity of the c scalar multiple of a vector being another vector.

II <u>Composition of Two Transformations</u>

1. <u>Consider two rotation in R^2</u>

$$T^{\theta_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} : R^2 \to R^2$$

$$T^{\theta_2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} : R^2 \to R^2$$

<u>Then</u>

$$T^{\theta_2} T^{\theta_1} = \begin{bmatrix} \cos\theta_2 \cos\theta_2 - \sin\theta_2 \sin\theta_1 & -(\cos\theta_2 \sin\theta_1 + \sin\theta_2 \cos\theta_1) \\ \sin\theta_2 \cos\theta_1 + \cos\theta_2 \sin\theta_1 & -\sin\theta_2 \sin\theta_1 + \cos\theta_2 \cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_2+\theta_1) & -\sin(\theta_2+\theta_1) \\ \sin(\theta_2+\theta_1) & \cos(\theta_2+\theta_1) \end{bmatrix}$$

$$= T^{\theta_2+\theta_1} : R^2 \to R^2$$

Conclusion: <u>The composite of two rotations is another rotation.</u>

2. Next consider the <u>two linear transformations</u>

$T : R^2 \to P_1$ and $S = P_1 \to P_2$

$\begin{bmatrix} a \\ b \end{bmatrix} \rightsquigarrow T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)x$ and $p(x) \rightsquigarrow S(p(x)) = x\,p(x)$

<u>This example and others like it lead to the concept of the composition of two linear transformations</u> as follows

<u>Definition</u> 13.2 (<u>The composition map</u>)

If $T : U \to V$ and $S : V \to W$ <u>are linear transformations</u>, then the <u>composition of S with T</u> is the mapping defined by

$$S \circ T(u) = S(T(u))$$

where u is in U.

Observe that S∘T maps U to W:

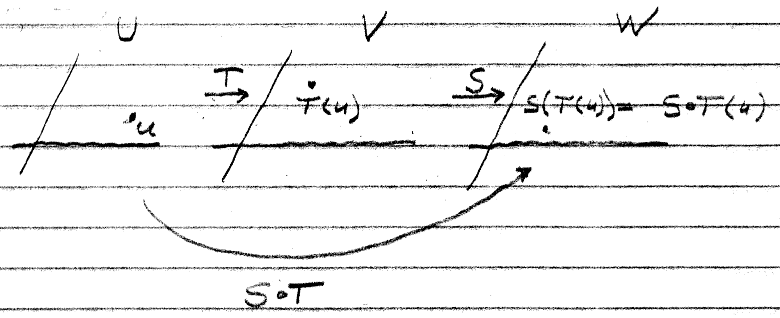


Figure 13.2
Composite of two linear transformations

Returning to the example at the top of page 13.5,

FIND $S\circ T\left(\begin{bmatrix}5\\-4\end{bmatrix}\right)$ and $S\circ T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$.

SOLUTION

Computing, one obtain

$$S\circ T\left(\begin{bmatrix}5\\-4\end{bmatrix}\right) = S\left(T\begin{bmatrix}5\\-4\end{bmatrix}\right) = S\left(5+(5-4)x\right) = S\left(5+x\right)$$

$$= x(5+x) \quad = 5x+x^2$$

and

$$S\circ T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = S\left(T\begin{bmatrix}a\\b\end{bmatrix}\right) = S\left(a+(a+b)x\right) = ax+(a+b)x^2$$

As expected, one has the following

Theorem 13.2 (Composition preserves Linearity)

Given:
$T: U \to V$ is linear
$S: V \to W$ is linear
Conclusion:
$S\circ T$ is linear,

Proof:
The validation of this conclusion hinges on the validity of the equation

$$S\circ T(c_1 u_1 + c_2 u_2) = c_1\, S\circ T(u_1) + c_2\, S\circ T(u_2),$$

$\forall\, u_1, u_2 \in U$ and $\forall$ scalars c_1 and c_2.

It is an easy exercise to verify this using the Definition 13.2 and 11.1 on pages 13.6 and 11.3 respectively.

III. Invertible Linear Transformations.

There is a third way of obtaining a new transformation from a given linear transformation. This method consists of *inverting* a given linear transformation. The result of this process is the *inverse* of that transformation.

However, it is only *invertible* transformations which can be inverted. Such transformations are identified by means of the following

Definition 13.3 (Inverse of a transformation)

A linear transformation

$$T: U \rightarrow V$$

is said to be *invertible* is there exists a transformation

$$T': V \rightarrow U$$

with the property that

$$T' \circ T = I_U \quad \text{and} \quad T \circ T' = I_V$$

where I_U and I_V are the identity transformations on U and V respectively. If there exists such a T', one says that T' is an *inverse* for T.

Comment

1. The domain space U and the target space V do not have to be the same, as they do in the case of invertible matrix transformations.

2. If T' is an inverse for T, then the definition implies that T is an inverse for T'.

3. We shall see that T is invertible whenever T is onto and one-to-one.

A. There are two ways of determining whether T is onto:

(i) Determine whether $\forall\, v \in V$ the equation

$$T(u) = v$$

can be solved for $u \in U$.

(ii) Use the dimension argument to determine whether $R(T) = V$ by the following 4-step argument:

1. Suppose V is finite dimensional, say

$$\dim V = p. \qquad (*)$$

and

$$\dim R(T) = \dim V. \qquad (**)$$

2. The fact that

$$R(T) \subseteq V \qquad (\star)$$

implies that $R(T)$ is a subspace, which according to Eqs. $(**)$ and $(*)$ is p-dimensional and hence has a basis

$$\{T(u_1), \ldots, T(u_p)\} \equiv Q,$$

which is a linearly independent set.

3. Theorem 7 on page 6.9 implies that Q is a spanning set for V:

$$\begin{aligned} V &\subseteq Sp\, Q \\ &= Sp\left(\{T(u_1), \dots, T(u_p)\}\right) \\ &= R(T) \qquad (\star\star) \end{aligned}$$

4. Combining Eqs. ($\star$) and ($\star\star$) one finds

$$R(T) = V$$

i.e. T is <u>onto</u>.

<u>Comment</u>:

Example 6 in Ch. 4 of J.R.&A illustrates this line of reasoning.