LECTURE 13

Operations with Linear Transformations

I. Sum of two linear transformations

II. Composition of linear transformation

III. Inverse of an invertible linear transformation
Operation with Linear Transformations

1. The Sum of Two Linear Transformations:

For a moving viscous fluid passing through a vectorial element of area \([A_1]\) the measured force vector \([F_1]\) acting on this area is expressed quantitatively in terms of the stress tensor (pressures, shear stresses) \([T_1]\) which characterizes this fluid:

\[
F_i = \sum_{j=1}^{3} T_{ij} A_j.
\]

If there are two kinds of fluid say type a and type b with stress tensors \([T_{ij}^a]\) and \([T_{ij}^b]\), then the force on this element of area is

\[
F_i = \sum_{j=1}^{3} \left( T_{ij}^a + T_{ij}^b \right) A_j.
\]

\[\hat{F} = F_1 + F_2 + F_3\]

\[\hat{A} = A_1 + A_2 + A_3\]

\[\hat{E} = E_1 - E_2\]

Figure 13.1:
Components of an element of area \([A]\) as projections of this area onto the respective coordinate planes.

\[\sum \overrightarrow{T}_{ij} \cdot \overrightarrow{A}_i = \overrightarrow{F}_j\]

The force - area relation.
The observation of such a relation between the concept "area" and the concept "force" is mathematized by linear transformations:

\[ T^a : U \rightarrow V \quad u \rightarrow T^a(u) = T^a_1(u_1) + T^a_2(u_2) \]
\[ T^b : U \rightarrow V \quad u \rightarrow T^b(u) = T^b_1(u_1) + T^b_2(u_2) \]

and the definition of their sum:

**Definition 13.1 (Sum of two linear transformations)**

The linear transformation defined by the function

\[ T^{a+b} : U \rightarrow V \quad u \rightarrow T^{a+b}(u) = T^a(u) + T^b(u) \]

is called the sum of \( T^a \) and \( T^b \).

It's an easy exercise to validate the following:

**Theorem 13.1**

If \( T^a : U \rightarrow V \) and \( T^b : U \rightarrow V \)
are linear transformations, then so is their sum:

\[ T^{a+b} = T^a + T^b : U \rightarrow V. \]

and their scalar multiple defined by

\[ cT^a : U \rightarrow V \quad u \mapsto cT^a(u) = c(T^a(u)) \quad \forall u \in U \]

Comment:

At the risk of stating the obvious, note that the \( c \) scalar multiple of \( T^a \) depends on the validity of the \( c \) scalar multiple of a vector being another vector.
II Composition of Two Transformations

1. Consider two rotation in $\mathbb{R}^2$

\[ T^\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2 \]

\[ T^{\theta_2} = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2 \]

Then

\[ T^{\theta_2}T^\theta = \begin{bmatrix} \cos \theta \cos \theta_2 + \sin \theta \sin \theta_2 & -\sin \theta \cos \theta_2 + \cos \theta \sin \theta_2 \\ \sin \theta \cos \theta_2 + \cos \theta \sin \theta_2 & \cos \theta \cos \theta_2 + \sin \theta \sin \theta_2 \end{bmatrix} \]

\[ = \begin{bmatrix} \cos(\theta + \theta_2) & -\sin(\theta + \theta_2) \\ \sin(\theta + \theta_2) & \cos(\theta + \theta_2) \end{bmatrix} \]

\[ = T^{\theta + \theta_2} : \mathbb{R}^2 \to \mathbb{R}^2 \]

Conclusion: The composite of two rotations is another rotation.

2. Next consider the linear transformations

\[ T : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{and} \quad S = \mathbb{R}^2 \to \mathbb{R}^2 \]

\[ [T] \mapsto T([v]) = a + (a+b)x \quad \text{and} \quad p(z) \mapsto S(p(z)) = k(p(z)) \]

This example and others like it lead to the concept of the composition of two linear transformations as follows.

Definition 13b (The composition map)

If $T : U \to V$ and $S : V \to W$

are linear transformations, then the composition of $S$ with $T$ is the mapping

\[ S \circ T(u) = S(T(u)) \]

where $u$ is in $U$. 
Observe that $S \circ T$ maps $U$ to $W$:

$T: U \rightarrow V$ is linear

$S: V \rightarrow W$ is linear

$S \circ T$ is linear.

As expected, we have the following

**Theorem 13.2 (Composition preserves linearity)**

Given:

$T: U \rightarrow V$ is linear

$S: V \rightarrow W$ is linear

Conclusion:

$S \circ T$ is linear.

Proof:

The validation of this conclusion hinges on the validity of the equation

$S \circ T(c \cdot u + c_2 \cdot u_2) = c \cdot S \circ T(u) + c_2 \cdot S \circ T(u_2)$

for all $u, u_2 \in U$ and $c, c_2$ scalars.

It is an easy exercise to verify this using the definitions 13.2 and 11.1 on pages 13.6 and 11.3 respectively.
III. Invertible Linear Transformations

There is a third way of obtaining a new transformation from a given linear transformation. This method consists of inverting a given linear transformation. The result of this process is the inverse of that transformation.

However, it is only invertible transformations which can be inverted. Such transformations are identified by means of the following definition.

Definition 13.3 (Inverse of a transformation)

A linear transformation

\[ T: U \rightarrow V \]

is said to be invertible if there exists a transformation

\[ T': V \rightarrow U \]

with the property that

\[ T' \circ T = I_u \quad \text{and} \quad T \circ T' = I_v \]

where \( I_u \) and \( I_v \) are the identity transformations on \( U \) and \( V \) respectively.

If there exists such a \( T' \), one says that \( T' \) is an inverse for \( T \).

Comment

The domain space \( U \) and the target space \( V \) do not have to be the same, as they do in the case of invertible matrices.
2. If $T'$ is an inverse for $T$, then the definition implies that $T$ is an inverse for $T'$.

3. We shall see that $T'$ is invertible whenever $T$ is onto and one-to-one.

There are two ways of determining whether $T$ is onto:

(i) Determine whether $V \in V$ the equation

$$T(u) = v$$

can be solved for $u \in U$.

(ii) Use the dimension argument to determine whether $R(T) = V$ by the following 4-step argument:

1. Suppose $V$ is finite dimensional, say $\dim V = p$.
2. Then, $\dim R(T) = \dim V = p$.
3. The fact that $R(T) \subseteq V$ implies that $R(T)$ is a subspace, which according to Eqs. (***) and (*) is $p$-dimensional and hence has a basis $\{T(u), \ldots, T(up)\}$ which is a linearly independent set.
3. Theorem 7 on page 6.9 implies that \( \mathbf{a} \) is a spanning set for \( V \):

\[ V \subseteq \text{span} \{ \mathbf{a} \} \]

\[ = \text{span} \{ T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p) \} \]

\[ = \mathbf{R}(T) \quad (\star \star) \]

4. Combining Eqs. (\star) and (\star \star) one finds

\[ \mathbf{R}(T) = V \]

i.e. \( T \) is onto.

Comment:

Example 6 in Ch. 4 of J.R. & A

illustrates this line of reasoning.