~~Augmentations for P 14.3 and 14.6 of Lecture 14~~

LECTURE 14

1. Two mathematical sins

2. Invertibility
   a) is unique

   b) holds also for non-linear xformations

3. Invertivility for linear xformations

$\iff$ (onto + one-to-one)

Used in Math 2568
Auxiliary
Texts
J R & A 5th ed'n
INTRODUCTION TO LIN. ALGEBRA
Ch. 5
Ch. 1. matrix theory
2. Vectors in R^2 & R^3
Chap. 3 Lin. Alg. in R^n

Used in Math 568
David Poole
Lin. Algebra:
A modern introd'n
Ch. 6 Vector spaces

Good axilliary texts.

There are two cardinal sins in mathematics:

(i) introduce a concept whichout specifying the context from which one abstracts the concept, and

(ii) not specifying the hierarchy of antecedent data-based ideas, which tie the concept to reality (as perceived and quantified by our senses, augmented by instruments, if necessary)

Friday's Definition 13.3 on page 13.9 illustrates both of these sins. The commission of these turns the defined concept "the invertibility" of a linear transformation (i.e. the existence

of an "inverse" for it) into what is known as a floating abstraction, an idea disconnected and detached from reality and thereby rendering it without meaning.

The purpose of the present and next lecture is to provide

(i) the mathematical context, namely, linear transformations which are onto and one-to-one, and

(ii) the mathematical method, basis-induced matrix transformations, which provide the hierarchical link to observable and measurable data that lie directly or indirectly at the base of all linear transformations, invertible ones included.

I. Invertible Linear Transformations (cont'd)

1. Uniquess

Recall that the defining property for the invertibility of a (linear or non-linear!) transformation

$$T : U \to V$$

is defined by the existence of at least one

$$T' : V \to U$$

with the property that

$$T' \circ T = I_U \quad \text{and} \quad T \circ T' = I_V \qquad (*)$$

!! → Comment: Invertibility also applies to non-linear xform'ns.

The fact that this invertibility criterion for T consists of two equation implies the uniqueness of T'

as stated by the following

Theorem 14.1 (uniqueness)

An invertible transformation is unique

Proof:

(i) Let T' and T'' be two invertible transformations:

$$T' \circ T = I_U \quad \text{and} \quad T \circ T' = I_V$$
$$T'' \circ T = I_U \quad \text{and} \quad T \circ T'' = I_V$$

Application of T'' yields

$$T'' \circ (T \circ T') = T'' \circ I_V$$
$$(T'' \circ T) \circ T' = T''$$
$$I_U \circ T' = T''$$
$$T' = T''$$

Thus all inverses of T are the same transformation, namely,

$$T' = T'' \equiv T^{-1}.$$

Thus, an invertible transformation has only one inverse, namely T^{-1}. It satisfies

both $\quad T^{-1}\circ T = I_U \qquad (\star a)$

and $\quad T\circ T^{-1} = I_V \qquad (\star b)$

Comment

A) These two invertibility conditions ($\star$) are symmetric under $T \leftrightarrow T^{-1}$.

Thus if T is invertible, then T^{-1} is also invertible; in fact,

$$(T^{-1})^{-1} = T \qquad (\star\star)$$

B) relations $(\star) \Rightarrow (\star\star)$, even if T and T^{-1} are non-linear.

c) Invertibility also applies to non-linear transformation.

2. Context:

The invertibility criterion, Eq. ($\star$), can only be understood within its proper context, which is given by the following

Theorem 14.2

A linear transformation $T: U \to V$ is invertible if and only if ($\Leftrightarrow$)

$\boxed{T \text{ is onto and one-to-one.}}$

Proof in Four Steps:

Step 1: $\boxed{(\star) \Rightarrow T \text{ is onto}}$

Let $v \in V$.

Invertibility implies $\exists\, T^{-1}$ s.t. $T\circ T^{-1} = I_V$

Let $u = T^{-1}(v)$

so that $T(u) = T(T^{-1}(v)) = T\circ T^{-1}(v) = v$

Consequently, $T(u) = v$

has a solution, namely

$$u = T^{-1}(v).$$

$\boxed{\text{Thus } T \text{ is onto}}$

Step 2: $\boxed{(\star) + T \text{ is linear} \Rightarrow T^{-1} \text{ is linear}}$

Let $T(u_1) = v_1$ and $T(u_2) = v_2$

Invertibility implies $\exists\, T^{-1}$ s.t. $T^{-1}T = I_u$

Thus $u_1 = T^{-1}v_1$ and $u_2 = T^{-1}v_2$ $(\star)$

Linearity of T implies $T(u_1) + T(u_2) = T(u_1+u_2)$

$$v_1 + v_2 = T(u_1+u_2)$$

$$T^{-1}(v_1+v_2) = \underbrace{T^{-1}T}_{I_u}(u_1+u_2)$$

Invertibility $\Rightarrow$

$$= u_1 + u_2$$

$(\star) \Rightarrow$ $\boxed{T^{-1}(v_1+v_2) = T^{-1}v_1 + T^{-1}v_2}$

One proves $\boxed{T^{-1}(cv) = c\,T^{-1}(v)}$

in an analogous way. $\boxed{\text{Thus } T^{-1} \text{ is linear}}$

Step 3: $(\star) + T^{-1}$ is linear $\Rightarrow$ T is 1-1

Let $u \in \mathcal{N}(T)$. Thus $T(u) = \vec{0}_V$

T^{-1} is linear $\Rightarrow$ $\underbrace{T^{-1}T}_{=\, I_U}(u) = T^{-1}(\vec{0}_V) = \vec{0}_U$

$\star$

$$u = \vec{0}_V$$

$\boxed{T \text{ is 1-1}}$

Step 4a: $\boxed{\text{Show that } \left\{\begin{matrix} T \text{ onto} \\ T \text{ 1-1} \end{matrix}\right\} \Rightarrow \exists\, T^{-1} \text{ s.t. } TT^{-1} = I_V}$

Let v be any element in V

$\left.\begin{matrix} T \text{ onto} \\ T \text{ 1-1} \end{matrix}\right\} \Rightarrow \exists$ unique u s.t. $T(u) = v$

Define T^{-1}: $v \mapsto T^{-1}(v) = u$ where $T(u) = v$

unique!

Apply T: $TT^{-1}(v) = T(u) = v \quad \forall v$

$\therefore$ $\boxed{TT^{-1} = I_V}$

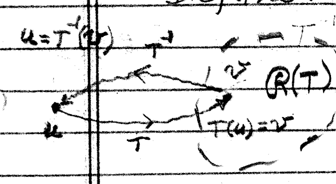
Step 4b: Show that $\left\{\begin{matrix} T \text{ onto} \\ T \text{ 1-1} \end{matrix}\right\} \Rightarrow \exists\, T^{-1}$ s.t. $T^{-1}T = I_U$

Let $u \in U$ any element in U

$T(u) = v \in \mathcal{R}(T)$

Define $T^{-1}: \mathcal{R}(T) \longrightarrow U$

$v \mapsto T^{-1}(v) = u$ where $T(u) = v$



$T(u) \mapsto T^{-1}T(u) = u \quad \forall u$

$\therefore$ $\boxed{T^{-1}T = I_U}$