LECTURE 14

1. Two mathematical sins

2. Invertibility
   a) is <u>unique</u>

   b) holds also for non-linear xformations

3. Invertivility for linear xformations

$\iff$ (onto + one-to-one)

Used in Math 2568
Auxiliary
Texts
J R & A 5th ed'n
INTRODUCTION TO LIN. ALGEBRA
Ch. 5
Ch. 1. matrix theory
2. Vectors in R^2 & R^3
Chap. 3 Lin. Alg. in R^n

Used in Math 568
David Poole
Lin. Algebra:
A modern introd'n
Ch. 6 Vector Spaces

There are two cardinal sins in mathematics:

no context

(i) introduce a concept whithout specifying the context from which one abstracts the concept, and

no hierarchy of antecedent data-based ideas

(ii) not specifying the hierarchy of antecedent data-based ideas, which tie the concept to reality (as perceived and quantified by our senses, augmented by instruments, if necessary)

The Definition 13.3 on page 13.9 illustrates both of these sins. The commission of these turns the defined concept "the invertibility" of a linear transformation (i.e. the existence of an "inverse" for it) into what is known as a floating abstraction, an idea disconnected and detached from reality and thereby rendering it without meaning.

The purpose of the present and next lecture is to provide

(i) the mathematical context, namely, linear transformations which are onto and one-to-one, and

(ii) the mathematical method, basis-induced matrix transformations, which provide the (hierarchical) link to ~~the~~ observable and measurable data that lie directly or indirectly at the base of all linear transformations, invertible ones included.

I. Invertible Linear Transformations. (cont'd)

1. Uniquess of T'

Recall that the defining property for the invertibility of a (linear or non-linear!) transformation

$$T : U \to V$$

is defined by the existence of at least one

$$T' : V \to U$$

with the property that

$$\boxed{T' \circ T = I_U} \text{ and } \boxed{T \circ T' = I_V} \quad (*)$$

!! → Comment: Invertibility also applies to non-linear xform'ns.

The fact that this invertibility criterion for T consists of two equation implies the uniqueness of T'

as stated by the following

Theorem 14.1 (uniqueness)

If a transformation is invertible, then its inverse is unique.

Proof:

(i) Let T' and T'' be two inverses of T:

$$T' \circ T = I_U \quad \text{and} \quad T \circ T' = I_V \quad (*)$$

$$(**) \quad T'' \circ T = I_U \quad \text{and} \quad T \circ T'' = I_V$$

Application of T'' to Eq.(*) yields

$$T'' \circ (T \circ T') = T'' \circ I_V$$

using Eq.(*)

$$(T'' \circ T) \circ T' = T''$$

$$I_U \circ T' = T''$$

$$T' = T''$$

Thus all inverses of T are the same transformation, namely,

$$T' = T'' \equiv T^{-1}.$$

Thus, an invertible transformation has only one inverse, namely T^{-1}. It satisfies

both $$T^{-1} \circ T = I_U \qquad (\star a)$$
and $$T \circ T^{-1} = I_V \qquad (\star b)$$

Comment

A) These two invertibility conditions ($\star$) are symmetric under $T \leftrightarrow T^{-1}$.

Thus if T is invertible, then T^{-1} is also invertible; in fact,
$$(T^{-1})^{-1} = T \qquad (\star\star)$$

B) Note: Relations ($\star$) $\Rightarrow$ ($\star\star$), even if T and T^{-1} are non-linear.

c) Invertibility also applies to non-linear transformation.

2. Context

The <u>invertibility</u> criterion,

$$\exists\, T^{-1} \text{ s.t } \quad T^{-1}T = I_U$$
$$\qquad\qquad\qquad TT^{-1} = I_V,$$

can only be understood within its proper context, which is given by

Thm 14.2 (Thm 16 in JRA)

A <u>linear</u> transformation

$$T: U \to V$$

is <u>invertible</u> $\Longleftrightarrow$ $\boxed{T \text{ is onto } \underline{\text{and}} \text{ one-to-one}}$

Proof in 4 Steps.

Go to next page.

Step I

We need to show that

$$\boxed{\left.\begin{array}{l} T^{-1}\circ T = I_U \\ T\circ T^{-1} = I \end{array}\right\} + T \text{ is linear} \Rightarrow T^{-1} \text{ is linear}}$$

Let $T(u_1) = v_1$ and $T(u_2) = v_2$

Invertibility implies $\exists\, T^{-1}$ s.t. $T^{-1}T = I_u$

Thus $u_1 = T^{-1}v_1$ and $u_2 = T^{-1}v_2$ (*)

Linearity of T implies $T(u_1) + T(u_2) = T(u_1+u_2)$

$$v_1 + v_2 = T(u_1+u_2)$$

$$T^{-1}(v_1+v_2) = \underbrace{T^{-1}T}_{I_u}(u_1+u_2)$$

Invertibility implies this

$$= u_1+u_2$$

(*) $\Rightarrow$ $\boxed{T^{-1}(v_1+v_2) = T^{-1}v_1 + T^{-1}v_2}$

One proves $\boxed{T^{-1}(cv) = cT^{-1}(v)}$

in an analogous way. $\boxed{\text{Thus } T^{-1} \text{ is linear}}$

We need to show that

Step II $\boxed{\exists T^{-1} \text{ s.t. } T^{-1}T = I_U,\ TT^{-1} = I_V \Rightarrow T \text{ is onto.}}$

Let $v \in V$. Question: Can we solve $T(u) = v$ for u?

Answer: a) Invertibility $\Rightarrow \exists T^{-1}$ s.t. $TT^{-1} = I_V$

b) apply this to v:

$$T\,T^{-1}(v) = I_V(v) = v$$

$$T\left(T^{-1}(v)\right) = v$$

c) The sol'n to $T(u) = v$ is $u = T^{-1}(v)$.

Hence $\boxed{T \text{ is onto}}$ indeed.

Step III A) First we show that

$$\boxed{T^{-1}T = I_U \ \& \ TT^{-1} = I_V \quad \text{plus "}T^{-1}\text{ is linear"} \Rightarrow T \text{ is 1-1}}$$

Indeed, let $u \in \mathcal{N}(T)$. Then $T(u) = \vec{0}_V$;

$$T^{-1}T(u) = T^{-1}(\vec{0}_V) = \vec{0}_U \quad \text{because "}T^{-1}\text{ is linear"}$$

Applying $T^{-1}T = I_U$ to the l.h.s., one finds

that $u = \vec{0}_U$, which shows that $\mathcal{N}(T) = \{0_U\}$ and that hence $\boxed{T \text{ is 1-1}}$

B) Next we combine Steps II and III. The result is

$$\boxed{\exists\, T^{-1} \text{ s.t. } TT^{-1} = I_V \Rightarrow \left\{\begin{matrix} T \text{ is onto} \\ T \text{ is 1-1} \end{matrix}\right\} \text{ whenever } T \text{ is linear}}$$

Step IV To show that $\left\{\begin{matrix} T \text{ onto} \\ T \text{ 1-1} \end{matrix}\right\} \Rightarrow T$ is invertible

we must show that

a) $\exists\, T_R'$ such that $TT_R' = I_V$

b) $\exists\, T_L'$ such that $T_L'T = I_U$

IV a) : ("Right inverse of T)

Let v be any element in V. Then

$\left.\begin{matrix} T \text{ onto} \\ T\ 1\text{-}1 \end{matrix}\right\} \Rightarrow \exists$ a unique element u s.t. $T(u) = v$

(i) The fact that this observation holds for every $v \in V$ is conceptualized into the definition T_R' (as the right inverse of T) as follows:

$T_R' : V \to U$

$v \mapsto T_R'(v) = u$ where u is the unique solution to $T(u) = v$

(ii) Now apply T:

$$T\,T_R'(v) = T(u) = v,$$

which holds $\forall v$. This $\boxed{T\,T_R' = I_v}$

IV b) ("Left inverse of T)

(i) Pick any element in U, say u; then

$$T(u) = v \in \mathcal{R}(T)$$

The fact that T is 1-1 implies that this equation is satisfied only by u and no other element in U. This observation is conceptualized into the definition of T_L^{-1} (as the left inverse of T) as follows:

$$T_L^{-1} : \quad \mathcal{R}(T) \to U$$
$$v \rightsquigarrow T_L^{-1}(v) = u$$

where u is the unique sol'n to $T(u) = v$.

(ii) Thus for all $u \in U$ we have

$$u = T_L^{-1}(v) = T_L^{-1} T(u)$$

$$\therefore \quad \boxed{T_L^{-1} T = I_U}$$

Summary: $\boxed{T \text{ is invertible}}$