LECTURE 15

Vector spaces: Retail vs. wholesale

Isomorphim: What is it;
examples: MATLAB'S kron function.

Isomorphism via equal dimension

Coordinate Representation of a Lin. Transformation.

Extra credit 10 points: Due 5 PM toda

(solution must fit on bottom half of this sheet)

Let U and V be n-dimensional vector spaces.

Let $T: U \to V$ be a linear transformation

PROVE or DISPROVE:

a) T is 1-1 $\Rightarrow$ T is invertible.

b) T is onto $\Rightarrow$ T is invertible.

So far our analysis has focussed on the nature of individual vector spaces ("retail"), their bases, dimension, relation to their duals, etc. for each one of them.
However, vector spaces are as different as the types of elements that comprise them (space of musical notes, space of polynomials, space of matrices, space of n-tuples etc.)
Although (as we have seen in Lecture 13 and in Homework 4 and 5) there exist linear transformations between these spaces, the disparateness in the comprising types runs the danger of

shattering vector space theory into a mere juxtaposition of innumerable disconnected pieces.

Fortunately, because of the existence of certain types of linear transformations, one can claim that two vector spaces, however different in type but having equal dimension, are essentially the same. This claim is based on the definition of isomorphism (meaning the same shape)

Definition (Isomorphism)

a) A linear transformation $T: U \to V$ is called an <u>isomorphism</u> if it is one-to-one and onto.

b) Two vector spaces U and V are said to be <u>isomorphic</u> whenever there exists an isomorphism between them. In this case one writes

$$U \cong V.$$

Example 15.1
SHOW that $P_{n-1} \cong R^n$.

Solution:

1.) Let $B = \{1, x, \ldots, x^{n-1}\}$ be the standard basis for P_{n-1} and let

$$\left.\begin{aligned} &T(p(x)) = [p(x)]_B, \\ &\text{i.e.} \\ &T(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} \end{aligned}\right\} (*)$$

From Theorem 4 pages 5-4.7, 5.2 we know that

$$T(p(x)+q(x)) = [p(x) + q(x)]_B = [p(x)]_B + [q(x)]_B = T(p(x)) + T(q(x))$$

and

$$T(cp(x)) = [c\,p(x)] = c[p(x)] = c\,T(p(x))$$

Thus T is a <u>linear transformation</u> (1)

2.) Let $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \in N(T)$

Then $$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = T(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus $a_0 = \cdots = a_{n-1} = 0$ so that $p(x) = \vec{0}$.

Consequently
$$N(T) = \{\vec{0}\}, \text{ i.e}$$

Thus T is <u>one-to-one</u> (2)

3.) The dimensional consistency theorem states

$$\dim N(T) + \dim R(T) = n$$

But we know that

$$\dim \mathcal{N}(T) = 0$$

and

$$\dim R^n = n$$

Consequently,

$$\dim \mathcal{R}(T) = \dim R^n$$

Thus T is onto (3)

4) Equations (1)-(3) imply that T is an isomorphism;

$$\mathcal{P}_{n-1} \cong R^n.$$

Example 15.2

Let M_{mn} be the vector space of $m \times n$ matrices.

SHOW that M_{mn} and R^{mn} are isomorphic

Solution: Introduce the standard basis for M_{mn} as follows

Let $e_i = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{m \times 1} \leftarrow i \Big\} m;$

$f_j = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{n \times 1} \leftarrow j \Big\} n;$ $f_j^t = \underbrace{[0 \cdots 1 \cdots 0]}_{n}$ $1 \times n$

a) Using MATLAB's Kronecker product kron(X,Y) (page 11-11 in MATLAB manual version 6), generate

$$E_{ij} = kron(e_i, f_j^t) \in M_{mn}$$

$$= \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{m \times 1} \underbrace{[0 \cdots 1 \cdots 0]}_{1 \times n} = \underbrace{\begin{array}{c} \begin{array}{ccc} & j & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow i \end{array}}_{m \cdot 1 \times 1 \cdot n}$$

the standard basis

$$\{E_{ij} : i = 1 \cdots m;\ j = 1 \cdots n\}$$

of $m \times n$ matrices for M_{mn}

b) Again using MATLAB's Kronecker product $kron(X,Y)$, generate

$$kron(e_i, f_j) = \begin{bmatrix} 0 \cdot f_j \\ \vdots \\ 1 \cdot f_j \\ \vdots \\ 0 \cdot f_j \end{bmatrix} \Big\} m \cdot n \quad \in R^{mn}$$

$(i \cdot n + j)^{th}$ place $\longrightarrow$ $1 \cdot f_j$

$\underbrace{}_{m \cdot n \times 1}$ with a 1 in the $(n \cdot i + j)^{th}$ place

the standard basis.

c) The mapping T:

$$\text{span}\{kron(e_i, f_j^t)\} = M_{mn} \xrightarrow{T} \text{span}\{kron(e_i, f_j)\} = R^{mn}$$

$$A = \sum_{i}^{m} \sum_{j=1}^{n} A_{ij}\, kron(e_i, f_j^t) \quad (\in M_{mn}) \overset{T}{\rightsquigarrow}$$

$$T(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}\, kron(e_i, f_j) \quad (\in R^{mn})$$

is one-to-one and onto. Thus

$$M_{mn} \cong R^{mn}$$

The easiest way to tell whether two vector spaces are isomorphic is to check their dimension. This is achieved by the following

Theorem 15.1 (Isomorphism via equal dimension)

Let U and V be two finite-dimensional vector spaces. Then

dim U = dim V $\Leftrightarrow$ U is isomorphic to V.

Comment

This easily used theorem unites all the collection of n-dimensional vector spaces into a single type, namely R^n. It mirrors exactly their algebraic and geometric structure in its own conceptual simplicity.

Proof: $\Longrightarrow$

Let $B = \{u_1, \ldots, u_n\}$ be a basis for U and

$C = \{v_1, \ldots, v_n\}$ be a basis for V

We need to construct T and then show that it is

(i) linear,
(ii) one-to-one
(iii) onto.

Let $u \in U$. One can write

$$u = c_1 u_1 + \cdots + c_n u_n$$

where $\{c_1, \ldots, c_n\}$ are uniquely determined.

Construct $T : U \xrightarrow{T} V$
by letting
$$u \overset{T}{\longmapsto} T(u) = c_1 v_1 + \cdots + c_n v_n$$

Note T is linear:

$$T(\alpha u + \alpha' u') = T\big(\alpha(c_1 u_1 + \cdots c_n u_n) + \alpha'(c_1' u_1 + \cdots + c_n' u_n)\big)$$
$$= (\alpha c_1 + \alpha c_1')\, v_1 + \cdots + (\alpha c_n + \alpha c_n')\, v_n$$
$$= \alpha(c_1 v_1 + \cdots + c_n v_n) + \alpha'(c_1' v_1 + \cdots + c_n' v_n)$$

$$= \alpha\, T(u) + \alpha' T(u')$$
Thus T is linear

(ii)
Let $u \in \mathcal{N}(T)$

$$u = c_1 u_1 + \cdots + c_n u_n$$

Hence $0 = T(u) = c_1 v_1 + \cdots + c_n v_n$

Thus $c_1 = \cdots = c_n = 0$

Consequently
$$u = \vec{0},$$
$$\mathcal{N}(T) = \{\vec{0}\}$$

Thus T is one-to-one

(iii) The dimensional consistency theorem implies

$$\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim U \quad (1)$$
But
$$\dim \mathcal{N}(T) = 0 \quad (2)$$
$$\dim U = \dim V \quad (3)$$
(1), (2), (3) $\Rightarrow$ $\dim \mathcal{R}(T) = \dim V$

Thus $\mathcal{R}(T) = V$ i.e. T is onto

(i)-(iii) Imply that T is an isomorphism: $U \cong V$.

Proof: $\Leftarrow$

U is isomorphic to V means that $\exists$ 15,11

$T: U \to V$

such that T is one-to-one and onto. Consequently, the dimensional consistency relation

$$\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim U$$

becomes

$$0 + \dim \mathcal{R}(T) = \dim U.$$

That T is onto implies $\mathcal{R}(T) = V$,

Thus $\dim V = \dim U$ Q.E.D.

SKIP Section 4.9 in J.R.&A.

So far be have characterized those properties of a linear transformation $\boxed{T: U \to V}$ which do not depend on any choice of basis for U and V.

~~Rev~~ ~~new~~ problem: Make a list of these properties and be able to illustrate them with specific examples.

Knowing these basis independent properties for any given transformation is necessary but not enough: We must be able to do actual computations with a given linear transformation.

In other words, suppose we have chosen (or are given) a basis for U: $B = \{u_1, \dots, u_n\}$ and a basis for V: $C = \{v_1, \dots, v_m\}$

Then any vector $u \in U$ can be represented by $[u]_B \in R^n$
" " $v \in V$ " " represented by $[v]_C \in R^m$

For example, if $u = c_1 u_1 + \dots + c_n u_n$, then $[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in R^n$.

Consider a linear transformation

$$T: U \longrightarrow V$$

T has numerous properties which are independent of a choice of basis for U or for V.

- Linearity
- $\mathcal{N}(T)$
- $\mathcal{R}(T)$
- 1-1 or not 1-1; onto or not onto.

We now consider properties of T which do depend on a choice of basis. This will, among others, lead to the concept of a matrix (or coordinate) representation for T.

Let $B = \{u_1, \dots, u_n\}$ be a basis for U

$C = \{v_1, \dots, v_m\}$ be a basis for V

Then $u = u_1 c_1 + \dots + u_n c_n \equiv u_i c_i$ for any $u \in V$
(Note the order!)

Note that we have

$$T(u) = v$$

$$T(u_i c_i) = v_k d_k$$

We have

$$U \xrightarrow{T} V$$

$$u \rightsquigarrow, T(u) = v$$

$$[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \rightsquigarrow [T(u)]_C = [v]_C = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} \quad (*)$$

$$R^n \xrightarrow{Q} R^m$$

Question: $Q = ?$

Answer: Eq'n$(*)$ yields

$$[T(u)]_C = [T(u_i c_i)]_C = [T(u_i) c_i]_C$$

$$= [T(u_i)]_C \, c_i = [T(u_1)]_C \, c_1 +$$

$$= [T(u_1)]_C c_1 + [T(u_2)]_C c_2 + \cdots + [T(u_n)]_C c_n$$

$$= \begin{bmatrix} q_{11} \\ \vdots \\ q_{m1} \end{bmatrix} c_1 + \begin{bmatrix} q_{12} \\ \vdots \\ q_{m2} \end{bmatrix} c_2 + \begin{bmatrix} q_{1n} \\ \vdots \\ q_{mn} \end{bmatrix} c_n$$

coord. rep'tve of $T(u_1)$

$$= \underbrace{\begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{m1} & \cdots & q_{mn} \end{bmatrix}}_{Q} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} \in R^m$$

Thus the representation of T relative to B & C is

$$\boxed{Q = \left[[T(u_1)]_C \; [T(u_2)]_C \; \cdots \; [T(u_n)] \right]}$$

i.e. Q obtained as follows:

1) start with $\{u_1, u_2, \cdots, u_n\} = B$ basis for U

2) Compute their transforms

$$\{T(u_1), T(u_2), \cdots, T(u_n)\}$$

3) Take their C-coordinate rep'ves as columns and form matrix

$$Q : R^n \to R^m$$

the B-C representative of $T: U \to V$.