LECTURE 15

Vector spaces: Retail vs. wholesale

Isomorphism: What is it?
  examples: MATLAB's kron function,

Isomorphism via equal dimension

Coordinate Representation of a linear transformation.
Extra credit 10 points; Due 5 pm today.
(solution must fit on bottom half of this sheet)

Let $U$ and $V$ be $n$-dimensional vector spaces.
Let $T: U \rightarrow V$ be a linear transformation.

Prove or Disprove:

2) $T$ is 1-1 $\Rightarrow$ $T$ is invertible.

b) $T$ is onto $\Rightarrow$ $T$ is invertible.
So far our analysis has focussed on the nature of individual vector spaces ("real")
their bases, dimension, relation to their duals, etc., for each one of them.
However, vector spaces are as different as the types of elements that comprise them (space of musical notes, space of polynomials, space of matrices, space of n-tuples, etc.)

Although (as we have seen in Lecture 13 and in Homework 4 and 5) there exist linear transformations between these spaces, the interspersedness in the comprising types runs the danger of shattering vector space theory into a mere juxtaposition of innumerable disconnected pieces.

Fortunately, because of the existence of certain types of linear transformations, one can claim that two vector spaces, however different in type but having equal dimension, are essentially the same. This claim is based on the definition of isomorphism (meaning the same shape).
Definition (Isomorphism)

A linear transformation \( T: U \to V \) is called an isomorphism if it is one-to-one and onto.

Two vector spaces \( U \) and \( V \) are said to be isomorphic whenever there exists an isomorphism between them. In this case, one write \( U \cong V \).

Example 15.1

Show that \( \mathbb{R}^{n-1} \cong \mathbb{R}^n \).

Solution

1) Let \( B = \{x_1, x_2, \ldots, x_n\} \) be the standard basis for \( \mathbb{R}^n \) and let

\[
T(p) = [p(x)]_B,
\]

i.e.,

\[
T(a_0 + a_1x + \ldots + a_{n-1}x^{n-1}) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}.
\]

Thus \( a_0 = \ldots = a_{n-1} = 0 \) so that \( p(x) = 0 \).

Consequently

\( N(T) = \{0\} \), i.e.,

Thus \( T \) is one-to-one (2).

3) The dimension consistency theorem states

\[
\dim N(T) + \dim R(T) = n.
\]

From Theorem 4, page 475, we know that
But we know that
\[ \dim N(T) = 0 \]
and
\[ \dim R^n = n \]

Consequently,
\[ \dim R(T) = \dim R^n \]

Thus $T$ is onto. (3)

4) Equations (1) - (3) imply that $T$ is an isomorphism;
\[ \mathbb{R}^{n_1} \cong R^n. \]

Example 15.2

Let $M_{mn}$ be the vector space of $m \times n$ matrices.

Show that $M_{mn}$ and $R^{mn}$ are isomorphic.

Solution: Introduce the standard basis for $M_{mn}$ as follows
\[ \left\{ E_{ij} : i = 1 \ldots m, j = 1 \ldots n \right\} \]

Let $e_i = \begin{bmatrix} 0 & \ldots & 1 & \ldots & 0 \end{bmatrix}^T \in \mathbb{R}^{m \times 1}$

\[ f_i = \left| \begin{array}{c} \vdots \\ 1 \\ \vdots \end{array} \right| \quad f_i^T = \left[ \begin{array}{c} \vdots \\ 0 \\ \vdots \end{array} \right] \begin{bmatrix} n \end{bmatrix} \]

Let $\mathbf{E}_{i,j} = \text{ kron}(e_i, f_j) \in M_{mn}$

Using MATLAB's Kronecker product kron$(X,Y)$ (page 11-11 in MATLAB manual version 6), generate
\[ \mathbf{E}_{i,j} = \text{ kron}(e_i, f_j) \]
\[ = \begin{bmatrix} 0 & \ldots & 1 & \ldots & 0 \\ i & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots \\ \end{bmatrix}_{2 \times n} \]

\[ \in \mathbb{R}^{m \times 1} \quad m \cdot 1 \times n \]
15.7

1) Again using MATLAB's Kronecker product \( \text{kron}(X,Y) \), generate

\[
\text{kron}(e_i, f_j) = \begin{bmatrix} e_i \otimes f_j \end{bmatrix} \in \mathbb{R}^{mn}
\]

\((m+1)^{th}\) place

\[
\begin{bmatrix} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \end{bmatrix} \quad m \cdot n
\]

\(m \times 1\) with \(a_1\) in the \((m+1)^{th}\) place

The standard basis.

2) The mapping \( T \):

\[
\text{span}\{\text{kron}(e_i, f_j)\} = M_{mn} \rightarrow \text{span}\{\text{kron}(e_i, f_j)\} = \mathbb{R}^{mn}
\]

\( T(A) = \sum_{i,j=1}^{m \times n} A_{ij} \odot \text{kron}(e_i, f_j) \in \mathbb{R}^{mn} \)

is one-to-one and onto. Thus

\[
M_{mn} \cong \mathbb{R}^{mn}
\]

15.8

The easiest way to tell whether two vector spaces are isomorphic is to check their dimension. This is achieved by the following

**Theorem 15.1** (Isomorphism via equal dimension)

Let \( U \) and \( V \) be two finite-dimensional vector spaces. Then

\[
\dim U = \dim V \iff U \text{ is isomorphic to } V.
\]

**Comment**

This easily used theorem unites all \( n \)-dimensional vector spaces into a single type, namely \( \mathbb{R}^n \). It mirrors exactly their algebraic and geometric structure in its own conceptual simplicity.
Proof: \[ \Rightarrow \]

Let \( B = \{ u_1, \ldots, u_m \} \) be a basis for \( U \) and \( C = \{ v_1, \ldots, v_n \} \) be a basis for \( V \).

We need to construct \( T \) and then show that it is:
(i) linear,
(ii) one-to-one,
(iii) onto.

Let \( u \in U \). One can write
\[ u = c_1 u_1 + \cdots + c_m u_m \]
where \( c_1, \ldots, c_m \) are uniquely determined.

Construct \( T: U \to V \) by letting
\[ u \mapsto T(u) = c_1 v_1 + \cdots + c_m v_m \]

Note \( T \) is linear.

\[ T(au + a'u) = T((c_1 u_1 + \cdots + c_m u_m) + (c_1' u_1 + \cdots + c_m' u_m)) = (c_1 + c_1') v_1 + \cdots + (c_m + c_m') v_m = a(c_1 v_1 + \cdots + c_m v_m) + a'(c_1' v_1 + \cdots + c_m' v_m) \]

Thus \( T \) is linear.

Hence \( \text{dim } N(T) = c_1 + \cdots + c_m = 0 \)

Consequently
\[ u = 0 \]
Thus \( \text{dim } N(T) = 0 \)

Thus \( T \) is one-to-one.

(ii) The dimension consistency theorem implies
\[ \text{dim } N(T) + \text{dim } R(T) = \text{dim } U \tag{1} \]
But
\[ \text{dim } N(T) = 0 \]
\[ \text{dim } U = \text{dim } V \tag{2} \]
\[ \text{dim } \text{dim } R(T) = \text{dim } V \tag{3} \]

(1), (2), (3) \( \Rightarrow \) \( \text{dim } R(T) = \text{dim } V \)
Thus \( R(T) = V \), i.e., \( T \) is onto.

(2) - (iii) imply that \( T \) is an isomorphism: \( U \cong V \).
Proof: ≤
U is isomorphic to V means that 
\[ T : U \rightarrow V \]
such that T is one-to-one and onto, consequently, the dimensional consistency relation
\[ \dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim U \]
becomes
\[ 0 + \dim \mathcal{R}(T) = \dim U. \]
That T is onto implies \( \mathcal{R}(T) = V \).

Thus \( \dim V = \dim U \) Q.E.D.
So far we have characterized those properties of a linear transformation which do not depend on any choice of bases for $U$ and $V$.

New problem: Make a list of these properties and be able to illustrate them with specific examples.

Knowing these basis independent properties for any given transformation is necessary but not enough: We must be able to do actual computations with a given linear transformation. In other words, suppose we have chosen a basis for $U$: $B = \{u_1, \ldots, u_n\}$ and a basis for $V$: $C = \{v_1, \ldots, v_m\}$.

Then any vector $u \in U$ can be represented by $[u]_B \in \mathbb{R}^n$ and $u \in U$ is represented by $[u]_B \in \mathbb{R}^n$.

For example, if $u = c_1 u_1 + \cdots + c_n u_n$, then $[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$.

Consider a linear transformation $T: U \to V$.

It has numerous properties which are independent of a choice of basis for $U$ or for $V$.

- Linearity
  - $T(u + v) = T(u) + T(v)$
  - $T(au) = aT(u)$

We now consider properties of $T$ which do depend on a choice of basis. This will, among others, lead to the concept of a matrix (coordinate) representation for $T$.

Let $B = \{u_1, \ldots, u_n\}$ be a basis for $U$ and $C = \{v_1, \ldots, v_m\}$ be a basis for $V$.

Then $u = u_1 c_1 + \cdots + u_n c_n \Rightarrow u_i c_i$, for any $u \in V$ (Note the order $S$)

Note that we have

$T(u) = v$

$T(u_i c_i) = v_k d_k$
We have \( U \xrightarrow{T} V \)

\[ u \xrightarrow{\text{ } T \text{ } } v \]

\[
\begin{bmatrix}
\mathbf{u}\end{bmatrix}_B =
\begin{bmatrix}
\mathbf{v}\end{bmatrix}_C =
\begin{bmatrix}
d_1 \\
\vdots \\
d_m
\end{bmatrix}
\]

\( \mathbb{R}^n \xrightarrow{Q} \mathbb{R}^m \)

Question: \( Q = ? \)

Answer: Eqn(1) yields

\[
\begin{bmatrix}
T(v)
\end{bmatrix}_C =
\begin{bmatrix}
T(u_1, \ldots, u_n)
\end{bmatrix}_C =
\begin{bmatrix}
\ldots
\end{bmatrix}_C
\]

\[ = \begin{bmatrix}
\ldots \\
T(u) \\
\ldots
\end{bmatrix}_C C_i = \begin{bmatrix}
T(u) \\
\ldots
\end{bmatrix}_C C_i
\]

\[ = \begin{bmatrix}
T(u_1) \\
\ldots
\end{bmatrix}_C C_1 + \begin{bmatrix}
T(u_2) \\
\ldots
\end{bmatrix}_C C_2 + \ldots + \begin{bmatrix}
T(u_n) \\
\ldots
\end{bmatrix}_C C_n
\]

\[
\text{coord. rep}\text{. of } T(v)
= \begin{bmatrix}
q_{11} & \ldots & q_{1n} \\
q_{21} & \ldots & q_{2n} \\
\vdots & \ddots & \vdots \\
q_{m1} & \ldots & q_{mn}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
\ldots \\
C_n
\end{bmatrix}
\]

\[ = \begin{bmatrix}
d_1 \\
\vdots \\
d_m
\end{bmatrix} \in \mathbb{R}^m
\]

Thus the representation of \( \text{T relative to } B \& C \) is

\[
Q = \begin{bmatrix}
T(u_1) \\
\ldots \\
T(u_n)
\end{bmatrix}_C
\]

i.e. \( Q \) obtained as follows:

1) Start with \( \{ u_1, u_2, \ldots, u_n \} = B \) basis for \( U \)

2) Compute their transforms

\[ \{ T(u_1), T(u_2), \ldots, T(u_n) \} \]

3) Take their \( C \)-coordinate rep's as columns and form matrix

\[ Q : \mathbb{R}^n \to \mathbb{R}^m \]

the \( B-C \) representative of \( T : U \to V. \)