LECTURE 15

Vector spaces: Retail vs. wholesale
juxtaposition vs. integration ("one in the many")

Isomorphism: What is it?
examples: MATLAB's kron function

Isomorphism via equal dimension
Coordinate Representation of a lin. trans.
So far our analysis has focused on the nature of individual vector spaces ("retail")
their bases, dimension, relation to their duals, etc., for each one of them.
However, vector spaces are as different as the types of elements that comprise
them (spaces of musical notes, spaces of polynomials, spaces of matrices, space of 2-tuples,
and others).
Although (as we have seen in Lecture 13 and in Homework 4 and 5) there exist linear transformations between these spaces, the disparities in the
comprising types run the danger of
shattering vector space theory into a mere juxtaposition of innumerable disconnected pieces.

Fortunately, because of the existence of certain types of linear transformations, one can claim that two vector spaces, however different in type but having equal dimension, are essentially the same. This claim is based on the definition of isomorphism (meaning the same shape).
Definition (Isomorphism)

1. A linear transformation $T : U \rightarrow V$ is called an isomorphism if it is one-to-one and onto.

2. Two vector spaces $U$ and $V$ are said to be isomorphic whenever there exists an isomorphism between them. In this case one writes $U \cong V$.

Example 15.1

Show that $P_{n-1} \cong \mathbb{R}^n$.

Solution:

1) Let $B = \{1, x, x^2, \ldots, x^{n-1}\}$ be the standard basis for $P_{n-1}$ and let

\[ T(p(x)) = [p(x)]_B, \]

i.e.,

\[ T(a_n + a_1 x + \cdots + a_{n-1} x^{n-1}) = \begin{bmatrix} a_n \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} \] \hspace{1cm} (*)

From Theorem 4, page 4, 7.3.2, we know that
\[ T(p(x) + q(x)) = [p(x) + q(x)]_B = [p(x)]_B + [q(x)]_B = T(p(x)) + T(q(x)) \]

and

\[ T(cp(x)) = [c \cdot p(x)] = c[p(x)] = c \cdot T(p(x)) \]

Thus \( T \) is a linear transformation \((1)\)

2) Let \( p(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \in \mathbb{P}(n) \)

Then

\[
\begin{bmatrix}
  a_0 \\
  a_1 \\
  \vdots \\
  a_{n-1}
\end{bmatrix} = T \left( a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \right) = \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}
\]

Thus \( a_0 = \ldots = a_{n-1} = 0 \) so that \( p(x) = 0 \).

Consequently

\( \mathcal{N}(T) = \{ \overrightarrow{0} \} \), i.e.

Thus \( T \) is one-to-one \((2)\)

3) The dimensional consistency theorem states

\[ \dim \mathcal{N}(T) + \dim \mathcal{R}(T) = n \]
But we know that
\[ \dim J^p(T) = 0 \]
and
\[ \dim \mathbb{R}^n = n \]

Consequently,
\[ \dim \mathbb{R}(T) = \dim \mathbb{R}^n \]

Thus \( T \) is onto \( \text{(3)} \)

4) Equations (1) - (3) imply that \( T \) is an isomorphism.

\[ \mathbb{R}^{n-1} \cong \mathbb{R}^n \]

Example 15.2

Let \( M_{nn} \) be the vector space of \( m \times n \) matrices.

Show that \( M_{nn} \) and \( \mathbb{R}^m \) are isomorphic.

Solution: Introduce the standard basis for \( M_{nn} \) as follows
Let \( \mathbf{e}_i = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -i \end{bmatrix} \) \( \in \mathbb{R}^m \).

\[
\mathbf{f}_i = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -i \end{bmatrix} \mathbf{n} \quad \mathbf{f}_i^t = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{1 \times n}
\]

\( \mathbf{n} \times 1 \)

a) Using MATLAB's Kronecker product \( \text{kron}(\mathbf{x}, \mathbf{y}) \) (page 11-11 in MATLAB manual version 6), generate \( \mathbf{E}^{ij} = \text{kron}(\mathbf{e}_i, \mathbf{f}_i^t) \) \( \in \mathbb{R}^{m \times 1 \times 1 \times n} \).

\[
\begin{align*}
\mathbf{E}^{ij} & = \text{kron}(\mathbf{e}_i, \mathbf{f}_i^t) \\
& = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -i \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
& \in \mathbb{R}^{m \times 1 \times 1 \times n}
\end{align*}
\]

the standard basis

\[ \left\{ \mathbf{E}^{ij} : i = 1, \ldots, m; j = 1, \ldots, n \right\} \]

of \( \mathbb{R}^{m \times n} \) matrices for \( \mathbb{R}^{m \times n \times m \times n} \).
b) Again using MATLAB's Kronecker product \( \text{kron}(x,y) \), generate an \( \text{m} \times \text{n} \) dimensional vector, each is \( n \)-dimensional column.

\[
\text{kron}(e_i, f_j) = \begin{bmatrix} 0 \cdot f_j \\ \ddots \\ 1 \cdot f_j \\ 0 \cdot f_j \end{bmatrix} \in \mathbb{R}^{m \times n}
\]

\( (in+j)^{th} \) place

\( m \times n \times 1 \) with a 1 in the \( (n \cdot i+j)^{th} \) place

the standard basis.

c) The mapping \( T: \)

\[
\text{span}\{\text{kron}(e_i, f_j)\} = M_{mn} \rightarrow \text{span}\{\text{kron}(e_i, f_j)\} = \mathbb{R}^{mn}
\]

\[
A = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} \text{kron}(e_i, f_j^T) (\in M_{mn}) \quad \text{and}
\]

\[
T(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} \text{kron}(e_i, f_j) (\in \mathbb{R}^{mn})
\]

**Claim:** \( T \) is one-to-one and onto.

Thus \( M_{mn} \cong \mathbb{R}^{mn} \)
The easiest way to tell whether two vector spaces are isomorphic is to check their dimension. This is achieved by the following.

**Theorem 15.1**  (Isomorphism via equal dimension)

Let \( U \) and \( V \) be two finite-dimensional vector spaces. Then

\[
\text{dim } U = \text{dim } V \iff U \text{ is isomorphic to } V.
\]

**Comment**

This easily used theorem unites all \( n \)-dimensional vector spaces into a single type, namely \( \mathbb{R}^n \). It mirrors exactly their algebraic and geometric structure in its own conceptual simplicity.
proof $1 \iff$ (very easy; see P15.11)

Proof $1 \implies$

Let $B = \{u_1, \ldots, u_m\}$ be a basis for $U$ and $C = \{v_1, \ldots, v_n\}$ be a basis for $V$

We need to construct $T$ and then show that it is

(i) linear,

(ii) one-to-one

(iii) onto.

Let $u \in U$. One can write

$$u = c_1 u_1 + \cdots + c_m u_m$$

where $c_1, \ldots, c_m$ are uniquely determined.

Construct: $T : U \rightarrow V$

by letting

$$u \mapsto T(u) = c_1 v_1 + \cdots + c_m v_m$$

Note $T$ is linear:

$$T(\alpha u + \alpha' u') = T\left(\alpha (c_1 u_1 + \cdots + c_m u_m) + \alpha' (c'_1 u'_1 + \cdots + c'_m u'_m)\right)$$

$$= \alpha (c_1 v_1 + \cdots + c_m v_m) + \alpha' (c'_1 v'_1 + \cdots + c'_m v'_m)$$
\[ = \alpha T(u) + \alpha' T(u') \]

Thus \( T \) is linear

\[(2')\]

Let \( u \in N(T) \)

\[ u = c_1 u_1 + \cdots + c_n u_n \]

Hence \( o^T(u) = c_1 u_1 + \cdots + c_n u_n \)

Thus \( c_1, \ldots, c_n = 0 \)

Consequently

\[ u = 0 \]

\[ N(T) = \{0\} \]

Thus \( T \) is one-to-one

\[(iii')\] The dimensional consistency theorem implies

\[ \dim N(T) + \dim R(T) = \dim U \quad (1) \]

But

\[ \dim N(T) = 0 \quad (2) \]

\[ \dim U = \dim V \quad (3) \]

\[ (1), (2), (3) \Rightarrow \dim R(T) = \dim V \quad (4) \]

\[ R(T) = V \quad (5) \]

\[ (4), (5) \Rightarrow R(T) = V \text{ i.e. } T \text{ is onto} \]

\[(i)-(iii)\] Imply that \( T \) is an isomorphism: \( U \cong V \).
Proof: \[ U \text{ is isomorphic to } V \text{ means that } \exists \; \mathbf{T} : U \rightarrow V \\
\mathbf{T} \; \text{is one-to-one and onto,} \\
\text{consequently, the dimensional} \\
\text{consistency relation} \\
\dim \mathcal{N}(\mathbf{T}) + \dim \mathcal{R}(\mathbf{T}) = \dim U \\
\text{becomes} \\
0 + \dim \mathcal{R}(\mathbf{T}) = \dim U. \\
\text{That } \mathbf{T} \text{ is onto implies } \mathcal{R}(\mathbf{T}) = V, \\
\text{Thus } \dim V = \dim U \; \text{ Q.E.D.} \]