LECTURE 15

Vector spaces: Retail vs. wholesale
juxtaposition vs. integration ("one in the many")
<u>Isomorphim</u>: What is it;
examples: MATLAB's kron function,

Isomorphism via equal dimension

Coordinate Representation of a Lin. Transformation.

So far our analysis has focussed on the nature of individual vector spaces ("retail"), their bases, dimension, relation to their duals, etc. for each one of them.

However, vector spaces are as different as the types of elements that comprise them (spaces of musical notes, spaces of polynomials, spaces of matrices, space of n-tuples, and others)

Although (as we have seen in Lecture 13 and in Homework 4 and 5) there exist linear transformations between these spaces, the disparateness in the comprising types runs the danger of shattering vector space theory into a mere juxtaposition of innumerable disconnected pieces.

Fortunately, because of the existence of certain types of linear transformations, one can claim that two vector spaces, however different in type, but having equal dimension, are essentially the same. This claim is based on the definition of isomorphim (meaning the same shape)

Definition (Isomorphism)

a) A linear transformation $T: U \to V$ is called an isomorphism if it is one-to-one and onto.

b) Two vector spaces U and V are said to be isomorphic whenever there exists an isomorphism between them. In this case one writes

$$U \cong V.$$

Example 15.1 SKIP

SHOW that $P_{n-1} \cong R^n$.

Solution:

1.) Let $B = \{1, x, \ldots, x^{n-1}\}$ be the standard basis for P_{n-1} and let

$$\left.\begin{array}{l} T(p(x)) = [p(x)]_B, \\ \text{i.e.} \\ T(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} \end{array}\right\} (*)$$

From Theorem 4 pages 4.5-4.7, 5.2 we know that

SKIP

$$T(p(x)+q(x)) = [p(x)+q(x)]_B = [p(x)]_B + [q(x)]_B = T(p(x)) + T(q(x))$$

and

$$T(cp(x)) = [c\,p(x)] = c[p(x)] = c\,T(p(x))$$

Thus T is a <u>linear transformation</u> (1)

2.) Let $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \in \mathcal{N}(T)$

Then $$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = T(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus $a_0 = \cdots = a_{n-1} = 0$ so that $p(x) = \vec{0}$.

Consequently $\mathcal{N}(T) = \{\vec{0}\}$, i.e

Thus T is <u>one-to-one</u> (2)

3.) The dimensional consistency theorem states

$$\dim \mathcal{N}(T) + \dim R(T) = n$$

SKIP.

But we know that

$$\dim \mathcal{N}(T) = 0$$

and

$$\dim R^n = n$$

Consequently,

$$\dim \mathcal{R}(T) = \dim R^n.$$

Thus T is <u>onto</u> (3)

4.) Equations (1)-(3) imply that

T is an isomorphism;

$$\mathcal{P}_{n-1} \cong R^n.$$

<u>Example 15.2</u>

Let M_{mn} be the vector space of $m \times n$ matrices.

SHOW that M_{mn} and R^{mn} are isomorphic

<u>Solution</u>: Introduce the standard basis for M_{mn} as follows

Let $e_i = \underbrace{\left.\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}\begin{matrix}\\ \\ \leftarrow i\\ \\ \\ \end{matrix}\right\} m}_{m\times 1};$

$f_j = \underbrace{\left.\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}\begin{matrix}\\ \\ \leftarrow j\\ \\ \\ \end{matrix}\right\} n}_{n\times 1};\quad f_j^t = \underbrace{\begin{array}{c} j \\ \downarrow \\ [0 \cdots 1 \cdots 0] \end{array}}_{n} \quad 1\times n$

a) Using MATLAB's Kronecker product kron(X,Y) (page 11-11 in MATLAB $\underset{\text{manual}}{\text{version 6}}$), generate

$$E_{ij} = kron(e_i, f_j^t) \in M_{mn}$$

$$= \underbrace{\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}}_{m\times 1}\underbrace{[0 \cdots 1 \cdots 0]}_{1\times n} = \underbrace{\begin{array}{cc} \begin{array}{ccc} & j & \\ & \downarrow & \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} \\ \leftarrow i \\ \\ \end{array} \end{array}}_{m\cdot 1\,\times\, 1\cdot n}$$

the standard basis

$$\{E_{ij} : i = 1 \cdots m;\ j = 1 \cdots n\}$$

of $m\times n$ matrices for M_{mn}

b) Again using MATLAB's Kronecker product

kron(X,Y), generate an <u>m·n dim'l col. vec.</u>

each is n-dimensional column

$$\mathrm{kron}(e_i, f_j) = \left.\begin{bmatrix} 0\cdot f_j \\ \vdots \\ 1\cdot f_j \\ \vdots \\ 0\cdot f_j \end{bmatrix}\right\} m\cdot n \in R^{mn}$$

$(in+j)^{th}$ place $\longrightarrow$ $1\cdot f_j$

$m\cdot n\times 1$ with a 1 in the $(n\cdot i+j)^{th}$ place

the standard basis.

c) The mapping T:

$$\mathrm{span}\{\mathrm{kron}(e_i, f_j^t)\} = M_{mn} \xrightarrow{T} \mathrm{span}\{\mathrm{kron}(e_i, f_j)\} = R^{mn}$$

$$A = \sum_{i}^{m}\sum_{j=1}^{n} A_{ij}\,\mathrm{kron}(e_i, f_j^t) \quad (\in M_{mn}) \xrightarrow{T}$$

$$T(A) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\,\mathrm{kron}(e_i, f_j) \quad (\in R^{mn})$$

CLAIM: T is one-to-one and onto.

Thus $M_{mn} \cong R^{mn}$

The easiest way to tell whether two vector spaces are isomorphic is to check their dimension. This is achieved by the following

Theorem 15.1 (Isomorphism via equal dimension)

Let U and V be two finite-dimensional vector spaces. Then

$\dim U = \dim V \iff U$ is isomorphic to V.

Comment

This easily used theorem unites all the collection of n-dimensional vector spaces into a single type, namely R^n. It mirrors exactly their algebraic and geometric structure in its own conceptual simplicity.

proof: $\Leftarrow$ (very easy; see P15.11)

Proof; $\Longrightarrow$

Let $B = \{u_1, ..., u_n\}$ be a basis for U and

$C = \{v_1, ..., v_n\}$ be a basis for V

We need to construct T and then show that it is
(i) linear,
(ii) one-to-one
(iii) onto.

Let $u \in U$. One can write

$$u = c_1 u_1 + ... + c_n u_n$$

where $\{c_1, ..., c_n\}$ are uniquely determined.

Construct $T: U \xrightarrow{T} V$
by letting

$$u \overset{T}{\rightsquigarrow} T(u) = c_1 v_1 + ... + c_n v_n$$

Note T is linear;

$$T(\alpha u + \alpha' u') = T\left(\alpha(c_1 u_1 + ... c_n u_n) + \alpha'(c_1' u_1 + ... + c_n' u_n)\right)$$

$$= (\alpha c_1 + \alpha' c_1') v_1 + \cdots + (\alpha c_n + \alpha c_n') v_n$$
$$= \alpha(c_1 v_1 + ... + c_n v_n) + \alpha'(c_1' v_1 + ... + c_n' v_n)$$

$= \alpha T(u) + \alpha' T(u')$

Thus T is <u>linear</u>

(ii)

Let $u \in \mathcal{N}(T)$

$$u = c_1 u_1 + \cdots + c_n u_n$$

Hence $0 = T(u) = c_1 v_1 + \cdots + c_n v_n$

Thus $c_1 = \cdots = c_n = 0$

Consequently

$$u = \vec{0},$$

$$\mathcal{N}(T) = \{\vec{0}\}$$

Thus T is <u>one-to-one</u>

(iii) The dimensional consistency theorem implies

$$\dim \mathcal{N}(T) + \dim R(T) = \dim U \quad (1)$$

But

$$\dim \mathcal{N}(T) = 0 \quad (2)$$

$$\dim U = \dim V \quad (3)$$

$(1),(2),(3) \Rightarrow \dim R(T) = \dim V \quad (4)$

$$R(T) \subseteq V \quad (5)$$

$(4),(5) \Rightarrow R(T) = V$ i.e. T is <u>onto</u>

(i)-(iii) Imply that T is an isomorphism: $U \cong V$.

Proof: $\Leftarrow$ 15,11

U is isomorphic to V means that $\exists$
$T : U \to V$

such that T is one-to-one and onto.
Consequently, the dimensional consistency relation

$$\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim U$$

becomes

$$0 + \dim \mathcal{R}(T) = \dim U.$$

That T is onto implies $\mathcal{R}(T) = V$.

Thus $\dim V = \dim U$ Q.E.D.