

LECTURE 16

- 1.) Basis of a vector space as the mooring that ties abstract vector spaces and their linear transformations to the computationally real world.
- 2.) The Representation Theorem induced from an illustrative example.
- 3.) Computations via the Representation Theorem.

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The Representation Theorem

The most important link between abstract (finite-dimensional) vector space properties and linear transformations on one hand and their concrete manifestations in grasping the nature of the physical world on the other is the representation theorem of linear algebra.

The concretization of abstract vector space properties is done by choosing a particular basis. Lectures 4 and 5 illustrate this process in regard to linear independence and spanning properties among others. The concretization

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consisted of numerical calculations

formulated in terms of the properties of column vectors. The representation theorem extends this concretization to linear transformations in regard to their null spaces and range spaces.

The concept of a representation in regard to vector space properties as well as linear transformations starts with a choice of basis. Thus if one is given the linear transformation

$$T: U \rightarrow V, \quad u \mapsto T(u) = v$$

and one chooses the bases

$$B = \{u_1, \dots, u_n\} \text{ for } U$$

$$\text{and } C = \{v_1, \dots, v_m\} \text{ for } V,$$

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the corresponding representation

arising from this choice is related to

$T: U \rightarrow V$ by means of the following

constellation of relations:

$$U \xrightarrow{T} V$$

$$u \rightsquigarrow T(u) \in V$$

$$B = \{u_1, \dots, u_m\} \quad C = \{v_1, \dots, v_m\} \quad (*)$$

$$[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \rightsquigarrow [T(u)]_C = [v]_C = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} = Q \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \quad (*)$$

$$R^n \xrightarrow{Q} R^m$$

Question: What is Q ? What are its properties

Answer: Q and its properties are determined by applying Eq. (*) to all possible $u \in U$.

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Example 1: $D: \mathcal{P}_3 \rightarrow \mathcal{P}_3$

$$P(x) \rightsquigarrow D(P(x)) = P'(x)$$

Let $B = \{1, x, x^2, x^3\}$ basis for \mathcal{P}_3

$C = \{1, x, x^2\}$ basis for \mathcal{P}_2

FIND Q with respect to $B \in C: R^4 \rightarrow R^3$

Solution

Step I:

Note that

$$D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$$

Step II:

Take the C representative of both sides

$$\left[D(a + bx + cx^2 + dx^3) \right]_C = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

Step III:

D is linear. Consequently one obtains

$$\begin{bmatrix} D(1) a + D(x) b + D(x^2) c + D(x^3) d \\ 0 & 1 & 2x & 3x^2 \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

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$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} a + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} b + \begin{bmatrix} 0 \\ c \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} d = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

This is a linear combination of the columns
Step IV

Identify Q in terms of these columns

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

$$[D(x)]_c [D(x)]_c [D(x)]_c [D(x)]_c$$

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \end{bmatrix} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$$

Step IV Conclusion: The columns of Q are

the G coordinate representatives of

the images $D(u_i)$ of the basis elements
 $u_1 = 1, u_2 = x, u_3 = x^2, u_4 = x^3$

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Example 1 generalizes to the following

Representation Theorem

Let U and V be two finite dimensional vector spaces with bases

$$B = \{u_1, \dots, u_m\} \text{ and } C = \{v_1, \dots, v_n\}$$

respectively. If $T: U \rightarrow V$ is a linear transformation, then the $m \times n$ matrix

Q defined by

$$Q = [[T(u_1)]_C \quad \dots \quad [T(u_m)]_C]$$

satisfies

$$Q [u]_B = [T(u)]_C$$

for every vector $u \in U$.

Ex (Example 6 on p 356 in J&R)

Example: Let $T: P_3 \rightarrow V \rightarrow 2 \times 2$ matrices

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto T(p) = \begin{bmatrix} a_0 + a_2 & a_0 + a_3 \\ a_1 + a_2 & a_1 + a_3 \end{bmatrix}$$

Problem: Find a basis for $R(T)$

is T onto? one-to-one?

Solution:

$$R(T) = \text{sp} \{ T(1), T(x), T(x^2), T(x^3) \}$$

$$= \text{sp} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$A_1 \quad A_2 \quad A_3 \quad A_4$

Let $G = \{ (1, 0, 0, 0), (0, 1, 0, 0), (1, 0, 0, 0), (1, 0, 0, 0) \}$ be the natural basis for V .

$$\text{Let } Q = [A_1]_G [A_2]_G [A_3]_G [A_4]_G$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \quad Q^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(a) We do a column reduction on Q by doing a row reduction on Q^T . one obtains

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ Hence } D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{matrix} \uparrow \\ [B_1] \\ [B_2] \\ [B_3] \\ [B_4] \end{matrix}$

The non-zero columns of D form a basis for the 3-d subspace $\text{sp} \{ [A_i]_{B_j} \}^4 \subset \mathbb{R}^4$

This basis is given by

$$\{ B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \}$$

It is a basis for $R(T)$; $\dim R(T) = 3$; T is not onto

(b) We do a row reduction on Q to put it into echelon form

$$\begin{array}{l} \text{C row} \\ \text{reduction} \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x+z=0 \\ y+z=0 \\ z-w=0 \\ w=z \end{array} \quad \begin{array}{l} x=-z \\ y=-z \\ z=z \\ w=z \end{array}$$

This implies that $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \in N(T)$

and hence $T(1+x-x^2-x^3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$N(T) = \text{sp} \{ 1+x-x^2-x^3 \}$$

and $\dim N(T) = 1$ T is not 1-1

$\dim N(T) + \dim R(T) = 1 + 3 = 4$ which is $= \dim P_3$ (checks OK) what it should be.