LECTURE 16

1.) Basis of a vector space as the mooring that ties abstract vector spaces and their linear transformations to the computationally real world.

2.) The Representation Theorem induced from an illustrative example.

3.) Computations via the Representation Theorem.

The Representation Theorem

The most important link between abstract (finite-dimensional) vector space properties and linear transformations on one hand and their concrete manifestations in grasping the nature of the physical world on the other is the representation theorem of linear algebra.

The concretization of abstract vector space properties is done by chosing a particular basis. Lectures 4 and 5 illustrate this process in regard to linear independence and spanning properties, among others. The concretization consisted of numerical calculations formulated in terms of the properties of column vectors. The representation theorem extends this concretization to linear transformations in regard to their null spaces and range spaces.

The concept of a representation in regard to vector space properties as well as linear transformations starts with a choice of basis. Thus, if one is given the linear transformation

$$T: U \to V, \quad u \mapsto T(u) = v$$

and one chooses the bases

$$B = \{u_1, \ldots, u_n\} \quad \text{for } U$$

and

$$C = \{v_1, \ldots, v_m\} \quad \text{for } V,$$

the corresponding representation arising from this choice is related to $T: U \to V$ by means of the following constellation of relations:

$$\begin{tikzcd}[column sep=large]
U \arrow[r, "T"] & V \\
u \arrow[r, squiggly, "T"'] \arrow[d, squiggly, "B=\{u_1,\dots,u_n\}:"'] & T(u) \equiv v \arrow[d, squiggly, "C=\{v_1,\dots,v_m\}"] \\
{[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}} \arrow[r, squiggly, "Q=?"] & {[T(u)]_C = [v]_C \equiv \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} \overset{(*)}{=} Q \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}} \\
R^n \arrow[r, "Q=?"] & R^m
\end{tikzcd}$$

(*)

Question: What is Q? What are its properties

Answer: Q and its properties are determined, by applying Eq.(*) to all possible $u \in U$.

Example 1: $D : \mathcal{P}_3 \to \mathcal{P}_2$

$$p(x) \rightsquigarrow D(p(x)) = p'(x)$$

Let $B = \{1, x, x^2, x^3\}$ basis for $\mathcal{P}_3$

$C = \{1, x, x^2\}$ basis for $\mathcal{P}_2$

FIND Q with respect to B & C : $R^4 \to R^3$

Solution

Step I:

Note that

$$D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$$

Step II:

Take the C representative of both sides

$$\left[D(a + bx + cx^2 + dx^3)\right]_C = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

Step III:

D is linear. Consequently one obtains

$$\Big[\underbrace{D(1)}_{0}\,a + \underbrace{D(x)}_{1}\,b + \underbrace{D(x^2)}_{2x}\,c + \underbrace{D(x^3)}_{3x^2}\,d\Big]_C = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

$$\begin{bmatrix}0\\0\\0\end{bmatrix}a+\begin{bmatrix}1\\0\\0\end{bmatrix}b+\begin{bmatrix}0\\2\\0\end{bmatrix}c+\begin{bmatrix}0\\0\\3\end{bmatrix}d=\begin{bmatrix}b\\2c\\3d\end{bmatrix}$$

This is a linear combination of the columns

Step IV

Identify Q in terms of these columns

$$\begin{array}{c}\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}\\ \underbrace{[D(1)]_C\ [D(x)]_C\ [D(x^2)]_C\ [D(x^3)]_C}_{Q}\end{array}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}b\\2c\\3d\end{bmatrix}$$

$$\begin{bmatrix}q_{11}&q_{12}&q_{13}&q_{14}\\q_{21}&q_{22}&q_{23}&q_{24}\\q_{31}&q_{32}&q_{33}&q_{34}\end{bmatrix}:R^4\to R^3$$

Step V

Conclusion: The columns of Q are the C coordinate representatives of the images $D(u_i)$ of the basis elements $u_1=1,\ u_2=x,\ u_3=x^2,\ u_4=x^4$.

Example 1 generalizes to the following

Representation Theorem

Let U and V be two finite dimensional vector spaces with bases

$B = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_m\}$

respectively. If $T: U \to V$ is a linear transformation, then the $m \times n$ matrix Q defined by

$$\boxed{Q = \left[[T(u_1)]_C \ \cdots \ [T(u_n)]_C \right]}$$

satisfies

$$Q[u]_B = [T(u)]_C = [T(c_1 u_1 + \cdots + c_n u_n)]_C$$
$$= [T(u_1)c_1 + \cdots + T(u_n)c_n]_C$$
$$= \left[[T(u_1)]_C \ \cdots \ [T(u_n)]_C \right] \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}_B}_{[u]_B}$$

for every vector $u \in U$.

Example 2: Let $T: \mathcal{P}_3 \to V = 2\times 2$ matrices

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto T(p) = \begin{bmatrix} a_0+a_2 & a_0+a_3 \\ a_1+a_2 & a_1+a_3 \end{bmatrix}$$

Problem: Find a basis for $\mathcal{R}(T)$

" " " " $\mathcal{N}(T)$

Is T onto? one-to-one?

Solution:

1.) $\mathcal{R}(T) = Sp\{T(1), T(x), T(x^2), T(x^3)\}$

$$= Sp\left\{ \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}_{A_3}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{A_4} \right\}$$

2.) Let $C = \left\{ \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right\}$ be the natural basis for V.

Let $Q = \begin{bmatrix} [A_1]_C & [A_2]_C & [A_3]_C & [A_4]_C \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}; \quad Q^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(a) We do a column reduction on Q by doing a row reduction on Q^T. One obtains

$$D^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Hence } D = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \\ [B_1]_C & [B_2]_C & [B_3]_C & \end{array} \end{array}$$

The non-zero columns of D form a basis for the 3-d subspace $Sp\{[A_i]_B\}_1^4 \subset R^4$.

This basis is given by

$$\left\{ B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

It is a basis for $\mathcal{R}(T)$; $\boxed{\dim \mathcal{R}(T) = 3}$ T is <u>not</u> onto

(b) We do a <u>row reduction</u> on Q to put it into echelon form

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[\text{reduction}]{\text{row}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x + z = 0 \\ y + z = 0 \\ z - w = 0 \end{cases} \quad \begin{aligned} x &= -z \\ y &= -z \\ z &= z \\ w &= z \end{aligned}$$

This implies that $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \in \mathcal{N}(T)$

and hence

$$T(1 + x - x^2 - x^3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{N}(T) = Sp\{1 + x - x^2 - x^3\}$$

and $\boxed{\dim \mathcal{N}(T) = 1}$ T is not 1-1

$\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = 1 + 3 = 4 = \dim \mathcal{P}_3$ which checks OK