LECTURE 17

Coordinate Representation for the Composite of Transformations

Coordinate Representative for the Composite of Transformations

The subject of Lecture 13 was the composition

$$S \circ T(u) \equiv S\left(T(u)\right)$$

of linear transformations. The theme was the linearity of this composition. The subject of this Lecture is the same, but the theme is different, namely the coordinate representation of the composition.

This is illustrated by the composite map, the example on page 13.6 and 13.7:

Example 17.1 (coord. Repres.'ve of a Composite Linear Map)

Given:

$T: U = \mathbb{R}^2 \longrightarrow V = \mathcal{P}_1$ $\qquad$ $S: V = \mathcal{P}_1 \rightarrow W = \mathcal{P}_2$

$\begin{bmatrix} a \\ b \end{bmatrix} \rightsquigarrow T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)x$ $\qquad$ $p(x) = a + bx \rightsquigarrow S(a+bx) = x\,p(x)$

$= ax + bx^2$

$S \circ T: U = \mathbb{R}^2 \longrightarrow W = \mathcal{P}_2$

$\begin{bmatrix} a \\ b \end{bmatrix} \rightsquigarrow S \circ T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ax + (a+b)x^2$

Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$

$C = \{1, x\}$ " " " P_1

$D = \{1, x, x^2\}$ " " " P_2

A) EXHIBIT the three coordinate representation matrices

(i) $[S]_{D \leftarrow C}$, (ii) $[T]_{C \leftarrow B}$, and (iii) $[S \circ T]_{D \leftarrow B}$ (*)

and

B) SHOW that

$$[S \circ T]_{D \leftarrow B} = [S]_{D \leftarrow C} \, [T]_{C \leftarrow B} \qquad (**)$$

Comments:

1. The to-be-exhibited matrices in (**) are the coordinate representatives Q (pages 16.5–16.6) for S, T, and $S \circ T$ respectively

2. In words one has: "The matrix of the composite is the product of the matrices."

3. Note how the inner subscripts C cancel each other, leaving the outer subscripts in the form $D \leftarrow B$.

A) Solution to part A:

(i) Calculation of $[T]_{C\leftarrow B}$:

$$\left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\right]_C = [1+x]_C = \begin{bmatrix}1\\1\end{bmatrix};\quad \left[T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]_C = [x]_C = \begin{bmatrix}0\\1\end{bmatrix}$$

Thus
$$[T]_{C\leftarrow B} = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$

(ii) Calculation of $[S]_{D\leftarrow C}$

$$[S(1)]_D = [x]_D = \begin{bmatrix}0\\1\\0\end{bmatrix};\quad [S(x)]_D = [x^2]_D = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Thus
$$[S]_{D\leftarrow C} = \begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix}$$

(iii) Calculation of $[S\circ T]_{D\leftarrow B}$

From page 13.7 we have
$$S\circ T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = ax + (a+b)x^2$$

consequently,

$$\left[S\circ T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\right]_D = \left[x+x^2\right]_D = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\left[S\circ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]_D = \left[x^2\right]_D = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Thus
$$\left[S\circ T\right]_{D\leftarrow B} = \begin{bmatrix}0 & 0\\1 & 0\\1 & 1\end{bmatrix}$$

B) Demonstration of part B:

$$[S]_{D\leftarrow C}\,[T]_{C\leftarrow B} = \begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\\1 & 1\end{bmatrix} = \left[S\circ T\right]_{D\leftarrow B}$$

Conclusion:

$$\boxed{[S]_{D\leftarrow C}\,[T]_{C\leftarrow B} = \left[S\circ T\right]_{D\leftarrow B}} \quad (\star)$$

Comment:

The line of reasoning leading to Eq. ($\star$) consisted of showing that the columns on the right hand side of Eq. (*) are the same as those on the left hand side.

The generalization of this line of reasoning gives rise to the following

Theorem 17.1 (Representation Thm for the composition of two maps)

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(i) Let U, V, W be finite dimensional vector spaces with bases B, C, and D.

(ii) Let

$$T: U \to V \quad \text{and} \quad S: V \to W$$

be linear transformations.

Conclusion

$$[S \circ T]_{D \leftarrow B} = [S]_{D \leftarrow C}[T]_{C \leftarrow B}$$

Nota bene: This is Theorem 20 in J R&A.

This theorem says that if (i) $Q = T_{C \leftarrow B}$ is the representative of $T : U \to V$ relative to the bases B and C,

$$Q[u]_B = [T(u)]_C$$

and (ii) $P = S_{D \leftarrow C}$ is the representative of $S : V \to W$ realative to the bases C and D,

$$P[v]_C = [S(v)]_D,$$

then $PQ = S_{D \leftarrow C} T_{C \leftarrow B}$ is the representative of $S \circ T : U \to W$ relative to the bases B and D

$$PQ[u]_B = [S \circ T(u)]_D$$

This theorem, the matrix of the composition is equal to the composition of two matrices, is illustrated by the following Figure 17.1

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & V \arrow[r, "S"] & W \end{tikzcd} : \begin{tikzcd} \mathbb{R}^n \arrow[r, "{[T]_{C\leftarrow B}}"] \arrow[rr, bend right] & \mathbb{R}^m \arrow[r, "{[S]_{D\leftarrow C}}"] & \mathbb{R}^k \end{tikzcd}$$

$$[S]_{D\leftarrow C}[T]_{C\leftarrow B} = [S\circ T]_{D\leftarrow B}$$

$$u \rightsquigarrow T(u) \rightsquigarrow S(T(u)) \;:\; [u]_B \rightsquigarrow \underbrace{[T]_{C\leftarrow B}}_{Q}[u]_B \rightsquigarrow \underbrace{[S]_{D\leftarrow C}}_{P}\underbrace{[T]_{C\leftarrow B}}_{Q}[u]_B$$

Figure 17.1 Coordinate representatives of T, S, and $S\circ T$.

proof:

Step I: Let u_i be the i^{th} basis vector of B.

The i^{th} column of $[S \circ T]_{D \leftarrow B}$ is

$$[S \circ T]_{D \leftarrow B} [u_i]_B = [S \circ T(u_i)]_D \equiv r.h.s. \quad (\star\star)$$

This is an application of the Representation Theorem on page 16.6

$$Q[u]_B = [T(u)]_C$$

with

$[S \circ T]_{D \leftarrow B}$	as Q
u_i	as u
basis B	as basis B
basis D	as basis C
$S \circ T$	as T

Step II:

Taking advantage of the definition of $S \circ T$ one finds that the r.h.s. of Eq. $(\star\star)$ is

$$r.h.s = [S \circ T(u_i)]_D$$

$$= [S(T(u_i))]_D$$

Step III

Again using the Representation Theorem with

$[S]_{D\leftarrow C}$	as Q
$T(u_i)$	as u
basis C	as basis B
basis D	as basis C
S	as T

one obtains

$$
\begin{aligned}
r.h.s &= \left[S\left(T(u_i)\right)\right]_D \\
&= [S]_{D\leftarrow C}\,[T(u_i)]_C \\
&= [S]_{D\leftarrow C}\,[T]_{C\leftarrow B}\,[u_i]_B \qquad (\star\star\star)
\end{aligned}
$$

Use Rep'n Thm

The Representation Theorem was used a 2nd time to obtain Eq. $(\star\star\star)$.

Step IV

Compare Eq. $(\star\star\star)$ with Eq. $(\star\star)$ at the top of page 17.7 and obtain

$$\left[S\circ T\right]_{D\leftarrow B}\left[u_i\right]_B = \left[S\right]_{D\leftarrow C}\left[T\right]_{C\leftarrow B}\left[u_i\right]_B \quad \forall\, u_i \in B$$

Thus

$$\boxed{\left[S\circ T\right]_{D\leftarrow B} = \left[S\right]_{D\leftarrow C}\left[T\right]_{C\leftarrow B}}$$

which had to be demonstrated

Quid Erat Demonstrantum = Q.E.D.