LECTURE 18

I. Representations: A Review
Basis Representation of an Invertible

II. Transformation

III. Example

IV. Epistemological Role of

the Representation Concept and

the Representation Theorem

I. Representations: A Review

Let $B = \{u_1 \ldots u_n\}$ be a basis for U

$C = \{v_1 \ldots u_m\}$ " " for V

$T : B \to V$

$u \rightsquigarrow T(u) = v$

$$[u]_B = [u_1 c_1 + \cdots + u_n c_n]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}_B$$

(see p 16.3)

is the B-induced representation of u

$$[v]_C = [v_1 d_1 + \cdots v_m d_m]_C = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}$$

(see p 16.3)

is the C-induced representation of v

Basis C implies

$$v = T(u) \iff [v]_C = [T(u)]_C$$

The Representation Theorem 16.6 states

$$\begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}_C = [v]_C = [T(u)]_C = [T(u_1 c_1 + \cdots + u_n c_n)]_C$$

$$= [T(u_1)]_C c_1 + \cdots + [T(u_n)]_C c_n$$

$$= \underbrace{\Big[[T(u_1)]_C \; \cdots \; [T(u_n)]_C \Big]}_{\equiv Q} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

II. Representation for Invertible Maps

The representation theorem for composite maps (on page 17.6) can be used to find the inverse T^{-1} of an invertible transformation T

$$T^{-1} \circ T = I \qquad T \circ T^{-1} = I,$$

i.e. (Theorem 14.2) a linear transformation which is onto and one-to-one. The following theorem allows one to use matrix methods to do this.

Theorem (Representation of the Inverse)

Let $T: U \to V$ be a linear xformation between n-dimensional vector spaces. U and V.

Let B and C be bases for U and V.

Conclusion: a) T is invertible

$\Leftrightarrow [T]_{C \leftarrow B}$ is invertible.

b) In this case

$$[T^{-1}]_{B \leftarrow C} = \left([T]_{C \leftarrow B}\right)^{-1}$$

<u>proof</u>: $\Longrightarrow$

If T is invertible then

$$T^{-1} \circ T = I_V$$

Apply the representation theorem 17.1 (Page 17.6) for composite maps:

$$I_n = [I_V]_B = [T^{-1} \circ T]_B$$
$$= [T^{-1}]_{B \leftarrow C} [T]_{C \leftarrow B} \qquad (*)$$

Notice that all matrices are $n \times n$ matrices.

Equation $(*)$ shows that $[T]_{C \leftarrow B}$ is invertible and that its inverse is

$$\boxed{[T^{-1}]_{B \leftarrow C} = \left([T]_{C \leftarrow B}\right)^{-1}}$$

Comment: What would we have obtained if we had started with $[I_V]_C = [T \circ T^{-1}]_C$ instead?

<u>proof</u>: $\Longleftarrow$

Conversely, assume that $Q = [T]_{C \leftarrow B}$ is invertible. Q is an $n \times n$ matrix: $\mathbb{R}^n \longrightarrow \mathbb{R}^n$

To show that T is invertible, we show that

$\mathcal{N}(T) = 0$ and $\mathcal{R}(T) = V$

Step I

For any $v \in V$ consider $T(u) = v$.

C is a basis for V implies

$$(*) \quad v = T(u) \Leftrightarrow [v]_C = [T(u)]_C \overset{\text{Rep'n (*) Theorem (P16.6)}}{=} \underbrace{[T]_{C \leftarrow B}}_{Q} [u]_B$$

Step II

Q is invertible $\Rightarrow$

$\exists \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \equiv [u]_B$ s.t. $Q[u]_B = [v]_C$, i.e. $[u]_B$

is a solution to $Q[u]_B = [v]_C$.

Step III

B is a basis for U implies

$$[u]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Leftrightarrow u = b_1 u_1 + \cdots + b_n u_n$$

(*) implies

Step IV u is a solution to $v = T(u)$.

$\boxed{\text{Hence } T \text{ is } \underline{\text{onto}}}$ i.e. $\boxed{\mathcal{R}(T) = V}$

Step V $\quad T$ is onto $\Longleftrightarrow \dim R(T) = \dim V = n$

The dimensional consistency theorem states

$$\dim \mathcal{N}(T) + \underbrace{\dim R(T)}_{n} = n$$

Hence

$$\dim \mathcal{N}(T) = 0$$

Consequently, $\boxed{T \text{ is onto } \underline{\underline{\text{and}}} \text{ one-to-one,}}$

i.e. $\boxed{T \text{ is invertible.}}$

III. Example

GIVEN:

(i) Let $\mathcal{D}$ be the space of differentiable functions.

(ii) Consider the subspace W of $\mathcal{D}$ given by

$$W = \operatorname{Span}(e^{3x}, xe^{3x}, x^2e^{3x})$$

(iii) Consider

$$B = \{e^{3x}, xe^{3x}, x^2e^{3x}\},$$ (basis for W)

which is a basis for W.

(iv) Let $D = \frac{d}{dx} : W \to W$

a) DETERMINE the matrix $Q = [D]_{W \leftarrow W}$ with respect to B

b) COMPUTE the derivative of
$$5e^{3x} + 2xe^{3x} - x^2e^{3x} = f(x)$$
using Q and the Representation Theorem (p 16.6) and compare the result with $Df(x)$.

c) SHOW that D is invertible, i.e. D^{-1} exists.

d) FIND $\int x^2 e^{3x}\,dx = D^{-1}(xe^{3x})$ using the Representation Thm (p16.6)

SOLUTION.

a) Apply D to a general element of W

$D\left(ae^{3x} + bxe^{3x} + cx^2e^{3x}\right)$

(*) $= (3a+b)e^{3x} + (3b+2c)xe^{3x} + 3cx^2e^{3x}$ (*)

$D(e^{3x}) = 3e^{3x}$ $\quad D(xe^{3x}) = e^{3x} + 3xe^{3x}$ $\quad D(x^2e^{3x}) = 2xe^{3x} + 6x^2e^{3x}$

$\left[D(e^{3x})\right]_B = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ $\quad \left[D(xe^{3x})\right]_B = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ $\quad \left[D(x^2e^{3x})\right]_B = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

Hence

$$Q = [D]_{B \leftarrow B} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

b) $[f(x)]_B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$

$$[D(f(x))]_B = [D]_{B \leftarrow B}[f]_B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \\ -3 \end{bmatrix}$$

Answer: $D\,f(x) = 17e^{3x} + 4xe^{3x} - 3x^2e^{3x}$

which agrees with Eq.(*) with
$a = 5 \quad b = 2 \quad c = -1$

c) We have

$$[D]_{B \leftarrow B} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Using the Inverse Representation Theorem on p 18.1 one has

$$[D^{-1}]_{B \leftarrow B} = \left([D]_{B \leftarrow B}\right)^{-1}$$

$$= \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}^{-1}$$

row-reduction to echelon form of augmented matrix

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$= \begin{bmatrix} 1/3 & -1/9 & 2/27 \\ 0 & 1/3 & -2/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$[x^2e^{3x}]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\int x^2e^{3x}\,dx\right]_B = \left[D^{-1}(x^2e^{3x})\right]_B$$

Rep'n thm

$$= [D^{-1}]_{B \leftarrow B}\,[x^2e^{3x}]_B$$

$$\left[\int x^2 e^{3x}\,dx\right]_B = \underbrace{\begin{bmatrix} 1/3 & -1/3 & 2/27 \\ 0 & 1/3 & -2/9 \\ 0 & 0 & 1/3 \end{bmatrix}}_{D_B^{-1}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/27 \\ -2/9 \\ 1/3 \end{bmatrix}_B$$

Thus $\boxed{\int x^2 e^{3x}\,dx = \frac{2}{27}e^{3x} - \frac{2}{9}xe^{3x} + \frac{1}{3}x^2e^{3x}}$

Comment

Integration constant is zero because, being a linear xformation, D^{-1} necessarily sends a zero element into a zero element.

IV. Representation Concept } : Their Epistemic Role.
Representation Theorem }

The Representation Theorem on page 16.6 is of fundamental importance to linear mathematics in finite dimensions. It is the means—the vehicle—by which one shuttles back and forth between (a) abstract vector spaces, abstract linear transformation and their properties on one hand and (b) their computational concretes (spaces of n-tuples, matrices and their properties) on the other.

Without the concept of a representation and the representation theorem, general vector spaces, general linear transformations etc, would be mere <u>floating abstractions</u> <u>disconnected</u> and <u>detached</u> from the physical world, disconnected and detached from relations that exist in it and are quantified in terms of particular n-tuples and matrices.

Furthermore, without the concept of a representation and the representation theorem basis independent concepts and relations would not even be conceivable.