LECTURE 2

The Subspace Theorem

Spanning set

Linear independence

Basis; coordinates

Basis-induced isomorphism

Reconsider the set of inventories of
a supermarket. But this time.
consider the inventories to consider of
fruit and vegetablas.

I. The Subspace Theorem

Q: How does one determine whether a set of objects forms a vector space?

A: This task is accomplished by verifying that they obey properties 3) a-d and 4) a-d of Definition 1 on pages 1.7-1.8 in Lecture 1.

However, under certain circumstances this 8-step task can be reduced to one of two step only. The key to success lies in identifying this set of objects as a subset of some vector space, and then merely checking closure under addition and multiplication by scalars.

One starts with

Definition 2 (subspace)

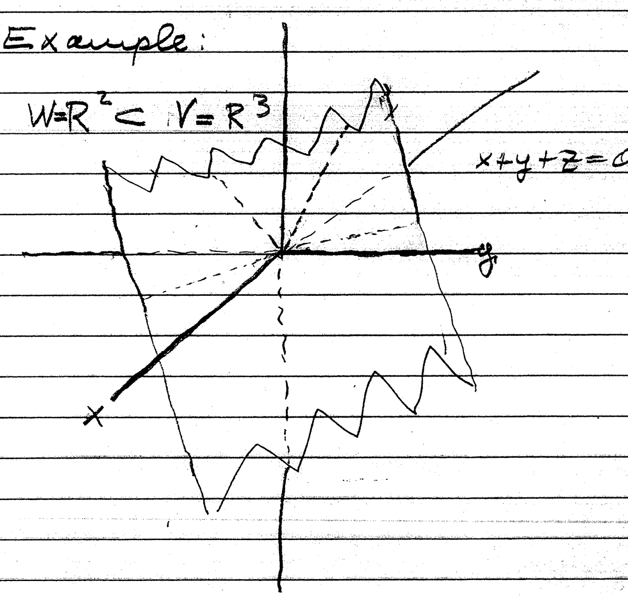
Let V, W both be vector spaces

$W \subset V$, i.e W is a subset of V

Then W is called a subspace of V.

Example:

$W = R^2 \subset V = R^3$



Sub(vector)spaces are easy to recognize

<u>because</u>

Theorem 1

Let a) $W \subset V$ = vector space
b) W is non-empty
i.e. W is a non-empty subset of V

Conclusion

$$\boxed{W \text{ is a subspace}} \Leftrightarrow \begin{cases} (1)\ u+v \in W \\ (2)\ cu \in W \end{cases} \quad \begin{matrix} u, v \in W \\ c \in F \end{matrix}$$

Comment:
More compactly, we have

W is a subspace $\Leftrightarrow cu+v \in W$

Proof: $\Rightarrow$ is obvious because W is a vector space.

$\Leftarrow$ 1. commutativity, associativity and distributivity are <u>inherited</u> from V

2. W contains the zero vector because $W \ni 0 \cdot \vec{u} = \vec{0}$ as shown in Lecture 1.

$\Leftarrow$ 1.

$$\left.\begin{array}{ll} \text{commutativity} & 3a \\ \text{associativity} & 3b \\ \text{distributivity} & 4c, 4d \\ 1\vec{u} = \vec{u} & 4a \\ (c_1 c_2)\vec{u} = c_1(c_2\vec{u}) & 4b \end{array}\right\} \text{inherited from } V$$

2. W contains the <u>zero vector</u> (3c):

$$W \ni \underset{\substack{\uparrow \\ \text{given}}}{} 0 \cdot \vec{u} \underset{\substack{\uparrow \\ \text{show in Lecture 1}}}{=} \vec{0}$$

3. <u>For any $u \in W$</u>, W contains the <u>additive inverse</u> (3d):

$$W \ni \underset{\substack{\uparrow \\ \text{given}}}{} (-1)\vec{u} \underset{\substack{\uparrow \\ \text{homework problem 1}}}{=} -\vec{u}$$

Comment:
Theorem 1 <u>simplifies the task</u> of <u>determining whether or not W is a vector space</u>: all one needs to do is verify $c\vec{u}+\vec{v} \in W$ whenever $\vec{u}, \vec{v} \in W$

Example.

For $m<n$

$$\mathcal{P}_m = \{ p(x) = a_m x^m + \cdots + a_1 x + a_0 : a_m, \ldots, a_0 \in R \}$$

is a subspace of $\mathcal{P}_n$. Indeed,

a) we know that $\mathcal{P}_n$ is a vector space

b) " " " $\mathcal{P}_m \subset \mathcal{P}_n$

Thus

c) $\boxed{\mathcal{P}_m \text{ is a subspace}}$

~~II Spanning Sets~~

~~We shall now give several defintions, and from these deduce several theorems.~~

~~The difficult part of this is to come up with the definitions; this is basically an inductive process whose hard work we owe to the 18th and 19th century mat~~

II Spanning Sets

We shall now state several definitions and from them deduce several theorems.

A definition states the distinguishing property(ies) of a concept ["span of Q"; "spanning set for V"].

The difficult part of this enterprise is to form the concept and to identify its distinguishing (i.e. essential) property. This done (largely) by a process of inductive reasoning, which is much more difficult than deductive reasoning, which is the formulation of the theorems.

In other words, arriving at these theorems depends crucially on the concepts identified by means of their definition. The quality and validity of the theorems depends on the quality and validity of the concepts which make the theorems possible.

The formation of concepts are greenlights for the statements of theorems, relations between the concepts.

For

(Lecture 2)

In the first lecture we considered vectors wholesale. In this lecture we shall consider them retail. In the first lecture we where interested in recognizing whether the whole set is indeed a vector space and how one does this.

In this lecture we will single out finite number of vectors and use them to coordinatize the vector space. It turns out that there is an enourmous amount of flexibility in establishing a coordinate system, and this flexibility is a very powerful tool. This is the nature a vector space.

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II. Spanning Set

A spanning set is a set of vectors which, given any vector $\vec{u}$ in V, guarantees the existence of a linear combination which is equal to this given vector $\vec{u}$.

More formally one has

Definition 3a (Span of a set of vectors in V)

Let $Q = \{v_1, \ldots, v_k : v_i \in V\} \subset V$ be a set of vectors in V. Then

$$Sp(Q) \equiv \text{span}\, Q = \{ v : v = a_1 v_1 + \cdots + a_k v_k \,;\, a_i \in R \}$$

is called the "span of Q"

= "The set of linear combinations of the v_i's."

Theorem 2 $Sp(Q)$ is a subspace of V.

1. $Sp(Q) \subseteq V$
2. We have closure under addition

$$v + w = (a_1 + b_1)v_1 + \cdots + (a_k + b_k)v_k \in Sp(Q)$$

3. We have closure under scalar mplication

$$cv = ca_1 v_1 + \cdots ca_k v_k \in Sp(Q)$$

Hence $Sp(Q)$ is a subspace of V indeed.

Definition 3 b (Spanning set for V)

$Q = \{v_1, \ldots, v_k\} \subset V$ is said to be a "spanning set for V" if $\boxed{V \subseteq Sp(Q)}$,

i.e. for any $v \in V$ $\exists$ constants $a_1, \ldots, a_k$ such that

$$v = a_1 v_1 + \cdots + a_k v_k$$

It is worth while to note that Def. 3b is equivalent to $V = Sp(Q)$, namely

Theorem 3

Q is a spanning set for $V \Leftrightarrow V = Sp(Q)$

Proof:

$\Rightarrow$:

$$\left.\begin{array}{l} V \subseteq Sp(Q) \\ \quad \uparrow \\ \text{Def. 3b} \\ Sp(Q) \subseteq V \\ \quad \uparrow \\ \text{closure under linear combination.} \end{array}\right\} V = Sp(Q)$$

$\Leftarrow$:

$V = Sp(Q) \Rightarrow V \subseteq Sp(Q)$, i.e

$v = a_1 v_1 + \cdots + a_k v_k$ for some $\{a_1, \ldots, a_k\}$

$\therefore$ $Q = \{v_1, \ldots, v_k\}$ is a spanning set for V.

<u>Example 1</u>

Let $\mathcal{P}_n$ = set of real polynomials of degree n or less

$$\{1, x, \dots, x^n\} \equiv Q \quad (= \text{spanning set for } \mathcal{P}_n)$$

$$\mathcal{P}_n = Sp\{1, x, \dots, x^n\}$$

<u>Example 2</u>

Let

$$W = \{p(x) : p(x) \in \mathcal{P}_2,\ p(0) = 0\} \subset \mathcal{P}_2$$

$$= \{a_2x^2 + a_1x + a_0 : p(0) = 0\}$$

$$\Downarrow$$

$$a_0 = 0$$

$$\therefore W = \{a_2x^2 + a_1x\}$$

$$= Sp\{x, x^2\}$$

$Q = \{x, x^2\}$ is the spanning set

<u>Example 3</u> : Spanning set may be <u>non-finite</u>!

Let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$ = set of all polynomials

$$Q = \{1, x, \dots, x^k, \dots\}\ ;\ Sp(Q) = \mathcal{P}$$

Deferred to Lecture 3

III <u>Linear Independence</u>

<u>Besides the spanning property of a set of vector there is its other key property, namely its linear independence (or dependence). These concepts are identified by means of the following definitions.</u>

<u>Definition 4</u> (Linear dependence / independence)

<u>Let V be a vector space</u>

Let $\{v_1, \dots, v_p\} \subset V$

Consider the equation

$$a_1v_1 + \dots + a_pv_p = \vec{0}$$

If there exists a <u>nontrivial</u> solution, i.e. $\exists\ a_1, \dots, a_p$ not all zero, then one says that $\{a_1, \dots, a_p\}$ is a <u>linearly dependent set</u>

Deferred to Lecture 3

2,13

Put differently,

a) the set $\{v_1, \dots, v_p\}$ is said to be linearly dependent whenever

$$a_1 v_1 + \cdots + a_p v_p = \vec{0}$$

has a non-trivial solution,

b) we have

$$\boxed{\begin{array}{l} a_1, \dots, a_p \text{ not all zero} \\ \text{is a solution to} \\ a_1 v_1 + \quad + a_p v_p = \vec{0} \end{array}} \iff \{v_1, \dots, v_p\} \text{ is a } \underline{\text{lin. dep. set}}$$

Stated still more differently:

c) $\{v_1, \dots, v_p\}$ is a <u>linearly independent set</u> if it is <u>not</u> linearly dependent

i.e given $a_1 v_1 + \cdots + a_p v_p = \vec{0}$, then

$$\boxed{\{v_1, \dots, v_p\} \text{ is a linearly independent set}} \iff \begin{array}{l} a_1 = \cdots = a_p = 0 \\ \text{is the } \underline{\text{only}} \text{ (unique!) solution.} \end{array}$$