LECTURE 20

I) Liner transformation: Its coordinate representation

a) motivated

b) illustrated

c) constructed

d) similarity transformed by a transition matrix.

Change of Representation ("coordinate change") for Linear Transformations

Example 20.1 (Angular Momentum)

Consider the linear transformation

$$T_\omega : \quad E^3 \longrightarrow E^3$$

$$\vec{X} = \vec{e}_1 x^1 + \vec{e}_2 x^2 + \vec{e}_3 x^3 \rightsquigarrow T_{\vec{\omega}}(\vec{X}) = \underbrace{\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega^1 & \omega^2 & \omega^3 \\ x^1 & x^2 & x^3 \end{vmatrix}}_{\text{"}\vec{\omega}\times\vec{X}\text{"}} = \vec{e}_1 v^1 + \vec{e}_2 v^2 + \vec{e}_3 v^3 \equiv \vec{e}_j v^j \equiv \vec{v}$$

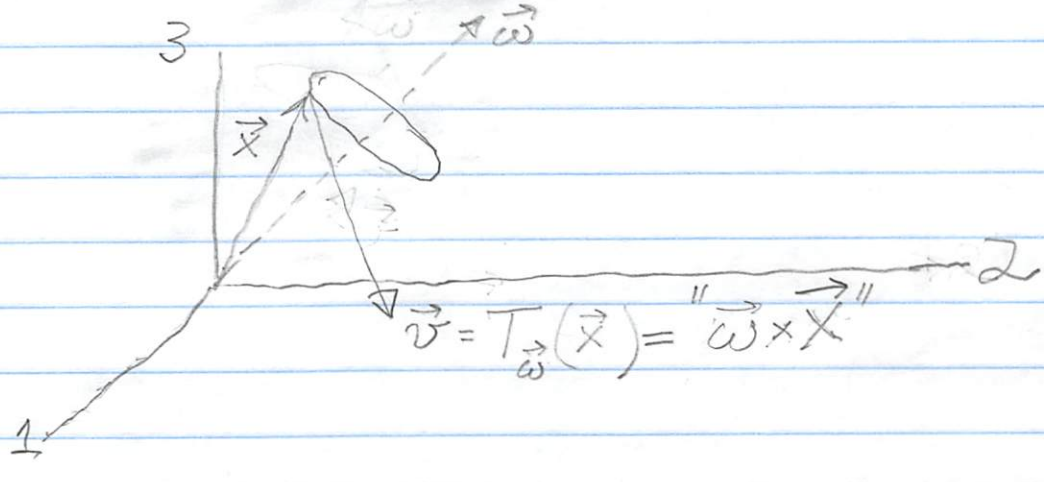
$$\equiv \vec{e}_i x^i$$

$$T(\vec{e}_1 x_1) = +\vec{e}_2 \omega^3 x^1 - \vec{e}_3 \omega^2 x^1$$

$$T(\vec{e}_2 x_2) = -e_1 \omega^1 x^3 + e_3 \omega^1 x^3$$

$$T(\vec{e}_3 x_3) = e_1 \omega^2 x^3 - e_2 \omega^1 x^3 \equiv \vec{e}_1 (\omega^2 x^3 - \omega^3 x^2) - \vec{e}_2 (\omega^1 x^3 - \omega^3 x^1) + \vec{e}_3 (\omega^1 x^3 + \omega^2 x^1)$$

from 3-dimensional Euclidean space E^3 into itself. This transformation expresses the instantaneous linear velocity $\vec{v}$ of a body at $\vec{x}$ with circular motion in plane whose normal is $\vec{\omega} = \omega^i \vec{e}_i$.



The coordinate representation of this relation relative to the given basis

$$\mathcal{B} = \{e_1, e_2, e_3\}$$

is
$$[v]_{\mathcal{B}} = \underbrace{[T_{\vec{\omega}}]_{\mathcal{B}}}_{= \left[[T_{\vec{\omega}}(e_1)]_{\mathcal{B}} \; [T_{\vec{\omega}}(e_2)]_{\mathcal{B}} \; [T_{\vec{\omega}}(e_3)]_{\mathcal{B}} \right]} [\vec{x}]_{\mathcal{B}},$$

(← notation from 16.6)

i.e.

$$\begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}}^{Q_{\mathcal{B}} \equiv} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \qquad (\star)$$

$$[T(e_1)]_{\mathcal{B}} \quad [T(e_2)]_{\mathcal{B}} \quad [T(e_3)]_{\mathcal{B}}$$

Diagramatically one has

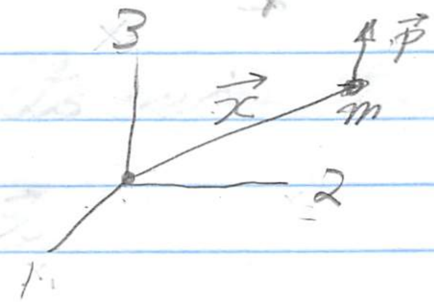
$$\begin{tikzcd}
 & E^3 \arrow[loop above, "T_{\vec{\omega}}"] \arrow[dl, "\mathcal{B}"'] \arrow[dr, "\mathcal{B}"] & \\
R^3 \arrow[rr, "Q_{\mathcal{B}} = {[T_{\vec{\omega}}]_{\mathcal{B}}}"'] & & R^3
\end{tikzcd}$$

$$[\vec{x}]_{\mathcal{B}} \rightsquigarrow [v]_{\mathcal{B}} = [T_{\vec{\omega}}]_{\mathcal{B}} [x]_{\mathcal{B}}$$

Example 20.2 (Moment of Inertia)

Consider
a) the momentum of a body of mass m:

$$\vec{p} = m\,\vec{v}$$



located at $\vec{x} \in V = E^3$

b) its moment of momentum, a.k.a. angular momentum, around the origin of E^3:

$$\vec{L} = \vec{x} \times \vec{p} = m\ \vec{x} \times (\vec{\omega} \times \vec{x})$$

$$= m\left(\vec{x}\cdot\vec{x}\ \vec{\omega} - \vec{x}\cdot\vec{\omega}\ \vec{x}\right)$$

In term of components

$$L_i = m\ \vec{x}\cdot\vec{x}\ \delta_{ij}\,\omega^j - x_i\, x_j\, \omega^j$$

$$= I_{ij}\ \omega^j$$

where

$$I_{ij} = m\left(x \cdot x\ \delta_{ij} - x_i x_j\right)$$

are the components of the moment of inertia tensor. It is the result of conceptually integrating the concept of the momentum $\boxed{\vec{p}}$ with the concept of the lever arm $\boxed{\vec{x}}$ to form the concept of moment of inerta I, which is mathematized by the symmetrix matrix

$$[I]_B = \begin{bmatrix} (x_2)^2+(x_3)^2 & -x_1x_2 & -x_1x_3 \\ -x_2x_1 & (x_3)^2+(x_1)^2 & -x_2x_3 \\ -x_3x_1 & -x_3x_2 & (x_1)^2+(x_2)^2 \end{bmatrix}$$

It is a transformation

$$I: \left[\vec{\omega}\right]_B \rightsquigarrow \left[\vec{L}\right]_B = \left[I\right]_B \left[\vec{\omega}\right]_B$$

PROBLEM:

What is the representation of this transformation relative to a new basis

$$C = \{\vec{e}\,'_1, \vec{e}\,'_2, \vec{e}\,'_3\},$$

which is related to the old basis B by mean of the transition matrix $[P]_{C \leftarrow B} \equiv P$?

$$\begin{tikzcd}
 & E^3 \arrow[dl, "C=\{e'_j\}"'] \arrow[dr, "B=\{e_i\}"] & \\
R^3 & & R^3 \arrow[ll, "{[P]_{C\leftarrow B} \equiv P}"]
\end{tikzcd}$$

$$\boxed{\begin{aligned} [\vec{x}]_C &= [P]_{C\leftarrow B}[X]_B \equiv P[X_B] \\ x'^j &= \sum_{j=1}^{3} P^j{}_i\, x^i \end{aligned}}$$

Pictorially the problem is the following scheme:

$$\begin{tikzcd}[column sep=large, row sep=huge]
 & U=E^3 \arrow[loop above, "T_{\vec{\omega}}"] \arrow[dl, "C"'] \arrow[d, "B"'] \arrow[dr, "B"] \arrow[drr, "C"] & & \\
\mathbb{R}^3 \arrow[r, shift right, "(P_{C\leftarrow B})^{-1}=P_{B\leftarrow C}"'] \arrow[rrr, bend right=40] & \mathbb{R}^3 \arrow[l, shift right, "P_{C\leftarrow B}"'] \arrow[r, "{[Q]_B \equiv [T_{\vec{\omega}}]_B}"'] & \mathbb{R}^3 \arrow[r, "P_{C\leftarrow B}"] & \mathbb{R}^3
\end{tikzcd}$$

$[X_c]_c$ $\qquad [X]_B \rightsquigarrow [v]_B \rightsquigarrow [v]_c$

$$[Q]_C \equiv [T_\omega]_C = ?$$

$[v]_c = [Q]_c [X]_c$

How is the representative $[T_\omega]_B$ related to the representative $[T_\omega]_C$ relative to the new basis $C = \{\vec{e}'_i\}$? The picture suggests

$$Q_C = P_{C\leftarrow B}\, Q_B \left(P_{C\leftarrow B}\right)^{-1} = P_{C\leftarrow B}\, Q_B\, P_{B\leftarrow C}$$

$$\boxed{[T_\omega]_C = P\,[T_\omega]_B\,P^{-1} = P_{C\leftarrow B}\, Q_B\, P_{B\leftarrow C}} \quad (*)$$

Is this relation valid or invalid?
A suggestion is merely that; namely a mere suggestion. At this stage of our process of reasoning one must say that is neither valid nor invalid. This is because this relation is a mere floating abstraction: it is disconnected and detached from the meaning of B, C, P and P^{-1}. The unanswered questions are: what properties of the given information compels us into accepting this relation? How does this relation use the basis vectors e_i? or e_j'? What about the columns of $[T]_B$?

of $[T_w]_C$? Why does one multiply the matrices in the order as specified in this relation and not in the reverse order, for example?

None of these questions have been answered. Consequently, without their answers one cannot claim to have an understanding of this relation.

Let us give a line of reasoning that at least in part bridges our cognitive gap.

Second Line of Reasoning

We start the process by applying

the representation theorem (P16.6, 16.6)

to $\qquad \vec{v} = T_\omega(\vec{x})$.

According to it one has the B-representative $[T_\omega]_B$, of T_ω,

$$\left([\vec{v}]_B\right) = [T(\vec{x})]_B = [T_\omega]_B [\vec{x}]_B \quad (\star\star)$$

Next we apply the coordinate trans-formation

$$P = [P]_{C \leftarrow B}.$$

Following the definition (P19.5) the result is

$$[\vec{v}]_C = P[\vec{v}_B] = P[T_\omega]_B \, [X]_B$$

Next we insert the identity matrix

$$I = P^{-1}P \text{ between } [T]_B \text{ and } [X]_B$$

one obtains

$$[v]_C = P[T_\omega]_B P^{-1} P[\vec{X}]_B$$

(Def'n on P19.5)

$$\boxed{[\vec{v}]_C = \underbrace{P[T_\omega]_B P^{-1}}_{[T_\omega]_C} [\vec{x}]_C} \qquad \forall [\vec{x}]_C \qquad (\star\star\star)$$

Recall that the C-representative $[T_\omega]_C$ is determined by the equation

$$[\vec{v}]_C = [T_\omega(\vec{x})]_C = [T_\omega]_C [\vec{x}]_C$$

Compare this expression with the l.h.s. of Eq. (***). One obtains

$$[T_\omega]_C [\vec{x}]_C = P [T_\omega]_B P^{-1} [\vec{x}]_C$$

This holds for all vectors $\vec{x}$. Thus the C and B representatives of T_ω are related by

$$\boxed{[T_\omega]_C = P [T_\omega]_B P^{-1}}.$$

The line of reasoning leading to this equation is obviously better than the one leading to Eq. (*)

at the bottom of p 20.4.

However, it is still deficient. Why? Because it did not address the role of the individual basis vector $\vec{e}_i$ and $\vec{e}\,'_j$ comprising the bases B and C.

Consequently, we give a <u>third line of reasoning</u> that bridges that gap <u>completely</u>. This line of reasoning is the most demanding.

It requires that one recall and use the whole context surrounding the representation theorem, including the construction of the various matrices from the individual basis vectors comprising the bases, $B = [e_1\, e_2\, e_3]$ and $C = [e_1'\, e_2'\, e_3']$

We start with the following constructive definitions

$$[T]_C = Q_C = \left[[T_{\vec{\omega}}(e_1')]_C, \dots, [T_{\vec{\omega}}(e_3')]_C \right]$$

$$[T]_B = Q_B = \left[[T_{\vec{\omega}}(e_1)]_B, \dots, [T_{\vec{\omega}}(\vec{e}_3)]_B \right]$$

and

$$P \equiv [P]_{C \leftarrow B} = \left[[e_1]_C, \dots, [e_3]_C \right].$$

$$P^{-1} \equiv [P]_{B \leftarrow C} = \left[[e_1']_B, \dots, [e_3']_B \right]$$

We have

$$Q_C = \Big[\underbrace{[T_{\vec{\omega}}(\vec{e}_1')]_C}_{\text{vector} \in R^3}, \dots, \underbrace{[T_{\vec{\omega}}(\vec{e}_3')]_C}_{\text{vector} \in R^3} \Big]$$

$$Q_C = \left[P_{C \leftarrow B} [T_{\vec{\omega}}(\vec{e}_1')]_B, \dots, P[T_{\vec{\omega}}(\vec{e}_2')]_B \right] \qquad \text{Def'n 19.1 of } P \text{ on P19.5}$$

$$Q_C = \left[P_{C \leftarrow B} [T_\omega]_B [\vec{e}_1']_B, \dots, P_{C \leftarrow B} [T_\omega]_B [\vec{e}_3']_B \right] \qquad \text{Rep'n Thm on P16.6}$$

$$Q_C = \left[P_{C \leftarrow B} [T_\omega]_B \underset{\substack{\| \\ P_{B \leftarrow C}}}{P^{-1}} [\vec{e}_1']_C, \dots, P[T_\omega]_B P^{-1} [e_3'] \right]$$

because $[u]_B = P^{-1}[u]_C$

$$Q_C = \left[P[T_\omega]_B P^{-1} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, P[T_\omega]_B P^{-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \right]$$

Thus

$$Q_C \equiv [T_\omega]_C = P \underbrace{[T_\omega]_B}_{Q_B} P^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{[T_\omega]_C = P [T_\omega]_B P^{-1}} \qquad (\star)$$

or

$$[T_\omega]_C = P_{C\leftarrow B} [T_\omega]_B P_{B\leftarrow C}$$

where we used $\boxed{P_{B\leftarrow C} = (P_{C\leftarrow B})^{-1}}$.

Indeed, we have

$$\left.\begin{aligned} [X]_C &= P_{C\leftarrow B}[X_B] \\ [X]_B &= P_{B\leftarrow C}[X_C] \end{aligned}\right\} [X]_C = P_{C\leftarrow B} P_{B\leftarrow C} [X]_C$$

$$\downarrow$$

$$\therefore I = P_{C\leftarrow B} P_{B\leftarrow C}$$

"Right inverse" $\longrightarrow \boxed{P_{B\leftarrow C} = (P_{C\leftarrow B})^{-1}}$

Alternatively: $[X]_B = P_{B\leftarrow C} [X]_C$

$$= P_{B\leftarrow C} P_{C\leftarrow B} [X]_B$$

$$\therefore I = \underbrace{P_{B\leftarrow C}} P_{C\leftarrow B}$$

"Left inverse" $\longrightarrow \boxed{P_{B\leftarrow C} = (P_{C\leftarrow B})^{-1}}$

Comment: Of course, "Right inverse" = "Left inverse"