APPENDIX to Lecture 20

Solving the confluent hypergeometric (=Kummer) differential equation via the method of the transition matrix.

General Comment:

The problem of finding a basis for the null space of a linear transformation can be simplified by changing the matrix representation. In terms of the notation of Lecture 20 one has

Problem: Solve $[T]_B[\vec{x}]_B = \vec{0}$

Solution: Let $[\vec{x}]_B = P^{-1}[\vec{x}]_C$ with $[\vec{x}]_C = P[\vec{x}]_B$

Let $P[T]_B P^{-1} = [T]_C$.

Consequently,

$$[T]_B[\vec{x}]_B = [T]_B P^{-1}[x]_C = P^{-1}[T]_C[\vec{x}]_C.$$

Thus $$\boxed{[T]_B [\vec{x}]_B = 0 \Leftrightarrow [T]_C[\vec{x}]_C = 0} \qquad (1)$$

If P^{-1} was chosen appropriately, then $[T]_C [\vec{x}]_C = 0$ may be readily solved and the solution is

$$[\vec{x}]_B = P^{-1}[\vec{x}]_C$$

For the purpose of typographical efficiency we make the following replacements

$$[\vec{x}]_B \rightarrow y$$

$$[T]_B \rightarrow A$$

$$P^{-1} \rightarrow K$$

$$P[\vec{x}]_B = [x]_C \rightarrow v$$

$$P[T]_B P^{-1} = [T]_C \rightarrow B$$

With this notation the equivalence, Eq. (1) on page A.20.1, is the same as

$$\boxed{Ay = 0 \Longleftrightarrow Bv = 0} \qquad (1')$$

In terms of this notation we now focus on the following problem and its solution.

Problem (Confluent hypergeometric functions via linear algebra)

Consider the nullspace $N(T)$ as determined by

$$Ay \equiv \left[z\frac{d^2}{dz^2} + (c-z)\frac{d}{dz} - a \right] y(z) = 0$$

Introduce the transition matrix K as defined by

$$y = Kv \equiv \int_\alpha^\beta e^{zt}\, v(t)\, dt$$

Here α and β, which are part of the definition of K, are-as-yet-unspecified.

a) FIND a linear operator B such that

$$AKv = KBv \qquad \forall v$$

b) FIND a solution v to the equation

$$Bv = 0$$

c) Suppose $\operatorname{Re} c > \operatorname{Re} a > 0$ (here $\operatorname{Re} \cdots \equiv$ real part of $\cdots$), DETERMINE α and β such that $y = Kv$ is a solution to $Ay = 0$ which is well-defined for all z in the complex plane

d) EXHIBIT such a solution.

Comment

From the perspective of linear algebra the task of solving the differential

$$\left[z\frac{d^2}{dz^2} + (c-z)\frac{d}{dz} - a\right] y(z) = 0$$

is the problem of finding non-trivial elements of the null space $\mathcal{N}(T)$ of the linear transformation

$$T = z\frac{d^2}{dz^2} + (c-z)\frac{d}{dz} - a \, .$$

Its domain are functions $y(z)$ which are 2^{nd} differentiable.

The constants c and a are (in general) complex constants, while the independent variable is also complex in general.

Comment: The solution(s) to this differential

are the hypergeometric a.k.a. Kummer functions. They are the fountainhead for numerous special functions in physics, mathematics, and statistics, among others.

Solution (solution to confluent hypergeometric equation via linear algebra)

The solution method is a four or five step process

Step I

Let

$$y(z)=\int_\alpha^\beta K(z,t)\,v(t)\,dt\equiv\int_\alpha^\beta e^{zt}v(t)\,dt=Kv$$

Consequently

$$Ay=AKv=\int_\alpha^\beta\left[z\frac{d^2}{dz^2}+(c-z)\frac{d}{dz}-a\right]\left(e^{zt}\right)v(t)\,dt$$

$$=\int_\alpha^\beta\left[zt^2+(c-z)t-a\right]\left(e^{zt}\right)v(t)\,dt$$

$$=\int_\alpha^\beta\left[t^2\frac{d}{dt}+t\left(c-\frac{d}{dt}\right)-a\right]\left(e^{zt}\right)v(t)\,dt$$

$$=\int_\alpha^\beta\left[(t^2-t)\frac{d}{dt}+(tc-a)\right]\left(e^{zt}\right)v(t)\,dt$$

Step II

Do an integration by parts to bring the integral into the form $KBv\ (=AKv)$.

This will define the transformed linear B such that $AKv=KBv$ for all v:

$AKv=KBv=$

$$=\int_\alpha^\beta e^{zt}\left[(-)\frac{d}{dt}\{t(t-1)\,v(t)\}+(tc-a)\,v(t)\right]dt \qquad (2)$$
$$+(t^2-t)e^{zt}v(t)\Big|_\alpha^\beta$$

Step III

Applying Eq. (1′) on page A.20.2 to the above Eq. (2) on page A.20.8, one finds

$$Ay=0 \iff Bv=0 \iff (i)\quad t(t-1)\frac{dv}{dt}+(2t-1+a-tc)v=0$$
$$(ii)\quad t(t-1)e^{zt}v(t)\Big|_\alpha^\beta=0$$

Thus

$$Bv=\left[t(t-1)\frac{d}{dt}+(2t-1+a-tc)\right]v$$

provided the limits α and β are chosen such that the boundary term,

$$t(t-1)e^{zt}v(t)\Big|_\alpha^\beta\equiv P\left[e^{zt},v\right]_\alpha^\beta=0 \qquad (3)$$

Step IV

The equation $Bv=0$ is a first order linear equation which is separable.

One finds

$$-\frac{dv}{v} = \frac{2-c}{t-1} + (a-1)\left(\frac{1}{t-1} - \frac{1}{t}\right)$$

$$-\ln v = (1-c+a)\ln(t-1) - (a-1)\ln t - \ln c_1$$

Thus one has

$$v(t) = c_1 (t-1)^{c-a-1}\, t^{a-1} \qquad (3a)$$

or

$$v(t) = c_2 (1-t)^{c-a-1}\, t^{a-1} \qquad (3b)$$

This solution is related to $y(z)$ in step I, namely

$$\boxed{y(z) = \text{const.} \int_{\alpha=0}^{\beta=1} e^{zt}\, t^{a-1} (1-t)^{c-a-1}\, dt}$$

This is because the boundary term in step III vanishes when $\alpha=0$ and $\beta=1$.

There are other possibilities for the integration limits α and β. They are determined for various constants a & c and values for z by applying Eqs (4) on page A.20.9 to $P[e^{zt}, v]$, Eq.(3) on page A.20.8; indeed, on has

$$\boxed{P[e^{zt}, v] = c_1 (t-1)^{c-a}\, t^{a}\, e^{zt}} \longrightarrow 0$$

as $t \to 0$ for $\mathrm{Re}\, a > 0$
as $t \to 1$ for $\mathrm{Re}\,(c-a) > 0$
as $t \to \infty$ for $\mathrm{Re}\, z < 0$
as $t \to -\infty$ for $\mathrm{Re}\, z > 0$

These determine the limits in the complex t-plance.

Step V

One of the solutions to the confluent hypergeometric equations has therefore the following integral representations

$$y(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} (t-1)^{c-a-1} t^{a-1} \, dt \quad \text{for } \operatorname{Re} c > \operatorname{Re} a > 0$$

$$\equiv F(a, c, z).$$

This is function entire in z, i.e. analytic in the finite complex z-plane.

Comment

(i) This function is the Kummer (or confluent hypergeometric) function.

The normalization factor has been chosen so that

$$F(a, c, 0) = 1.$$

In fact, one can show that

$$F(a,c,z) = 1 + \frac{a}{c} z + \frac{a(a+1)}{c(c+1)} \frac{z^2}{2!} + \cdots$$

$$= \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

Here

$$\Gamma(c) = \int_0^\infty e^{-t} t^{c-1} \, dt$$

$$\Gamma(n) = (n-1)!$$

is the <u>gamma function.</u>

(ii) Normalization factor

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad p = a,\ q = c-a,$$

which is responsible for $F(a, c, 0) = 1$, is also known as the <u>beta function</u>