LECTURE 21

1. The Levi-Civita Epsilon
2. Algebraic Index Calculus
   applied to orthogonal transitions.

For the purpose of concretizing matrix equations such as

$$\left[T_\omega\right]_C = P\left[T_\omega\right]_c P^{-1}$$

in numerical terms, that represent actual observated measurements, one need to refer to the individual matrix elements of a given representative.

Introducing the indexed matrix elements of the two representatives

$$\left[T_\omega\right]_c = \left[T'_{ij}\right]$$

and

$$\left[T_\omega\right]_B = \left[T_{kl}\right]$$

one finds that Eq. (*) on P 20.10 becomes.

$$\boxed{T'_{ij} = P_{ik} T_{kl} P^{-1}_{lj}} \qquad (*)$$

Comment:

The relationship between a rotating position of a particle and its velocity exists on 4 levels of abstraction.

On the level closest to perceptual awareness this relationship is characterized by, say,

(3, -2, 1),

an array of numbers which refer

to the observed orientation of the plane and to the angular rate determined by the rotating particle. There is an unlimited number of such characterizations.

At next level of abstraction they are united into the concept of an algebraic triplet

$(\omega^1, \omega^2, \omega^3)$

whose elements may assume any values, but must assume particular numbers for any particular observed relationship.

At the third level this triplet is recharacterized from a vector to a matrix relative to a particular coordinate frame such as

$$[T_\omega]_B = [T_{k\ell}] \text{ or } [T_\omega]_C = [T^i_{\ j}],$$

depending on one's choice of basis.

The resulting set of particular matrices is unlimited in number. Nevertheless

they have a common conceptual denominator in that they are similar

$$[T_\omega]_C = P[T_\omega]_B P^{-1},$$

by virtue of the transition matrix P from the chosen basis B to basis C.

The fourth level of abstraction is is reached by omitting explicit reference to any particular P and thereby (mentally) blend all members of this equivalence class into a single new unit, namely the coordinate invariant position-velocity

relation, which is a linear transformation.
Mathematically it is designated by

$$T_\omega = \{ [T_\omega]_B \sim [T]_C : [T_\omega]_C = P_{C\leftarrow B} [T_\omega]_B (P_{C\leftarrow B})^{-1} \}$$

(End of Comment)

The change from the B-representation to the C-representation is still very general.

To concretize this change, let us focus on a transition

$$P = [P_{mn}]$$

which is characterized by

$$P^{-1} = P^{transpose} = [P^{-1}_{k\ell}] = [P^{transp}_{k\ell}] = [P_{\ell k}].$$

Side Comment!
Such an $n\times n$ matrix expresses a rotational change of coordinates: the coordinate frame gets rotated to a new coordinate basis. The algebraic and geometric properties of such

"orthogonal" matrices will be studied in Lecture 23 (or 24?)

In terms of such matrix the index-based matrix Eq. (*) on page 21.2 becomes

$$T'_{ij} = T_{kl} P_{ik} P_{jl} \qquad (1)$$

The particular choice T_ω for our linear transformation is such that its indexes B-representative, Eq. (*) on p 20.2 is very simple

$$[T_{kl}] = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix} \qquad (2)$$

We shall take advantage of this simplicity by expressing the elements of this

antisymmetric matrix in terms of the Levi-Civitate epsilon ϵ_{mkl}.

<u>Definition</u> (The <u>totally antisymmetric</u> L-C <u>epsilon</u>)

The Levi-Civita epsilon has three indeces and the following values.

$$\epsilon_{123} = 1$$

$$\epsilon_{mkl} = \pm 1 \text{ if } (mkl) \text{ is an } \binom{\text{even}}{\text{odd}} \text{ permutation of } 123$$

$$= 0 \text{ if any two subscripts repeat}$$

Thus

$$\epsilon_{mkl} = \epsilon_{klm} = \epsilon_{lmk} = -\epsilon_{lkm} = -\epsilon_{kml} = -\epsilon_{mlk}$$

<u>Comment</u>

The Levi-Civita epsilon (3-dimensional) is related to the Kronecker delta by means of the following equations

$$\boxed{\epsilon_{klm}\,\epsilon_{kln} = 2!\,\delta_{mn}}$$

and

$$\boxed{\epsilon_{ijt}\,\epsilon_{rst} = \delta_{ir}\,\delta_{js} - \delta_{is}\,\delta_{jr}}$$

One way of validating them is to exhibit the sums of the repeated indeces explicitly:

$$\epsilon_{kl1}\,\epsilon_{kl1} = \epsilon_{231}\,\epsilon_{231} + \epsilon_{321}\,\epsilon_{321} = 1 + (-1)(-1) = 2\,\delta_{11}$$

$$\epsilon_{kl1}\,\epsilon_{kl2} = \epsilon_{231}\,\epsilon_{232} + \epsilon_{321}\,\epsilon_{322} = 0$$

etc.

In terms of the Levi-Civita epsilon the that simple B-representative of T_ω, Eq.(2) on p 20.7

$$[T_{kl}] = -\omega^1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} - \omega^2 \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \omega^3 \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (3)$$

ϵ_{1kl} $\qquad\qquad$ ϵ_{2kl} $\qquad\qquad$ ϵ_{3kl}

has the following matrix elements

$$[T_\omega]_B : \quad T_{kl} = -\omega^m\,\epsilon_{mkl}$$

Consequently, the matrix elements, Eq(1) on p 20.9 are

$$[T_\omega]_C : \quad T'_{ij} = -\omega^m\,\epsilon_{mkl}\,P_{ki}\,P_{lj}$$

The right-hand side of this equation can be simplified considerable by noticing that T'_{ij} is antisymmetric:

$$T'_{ji} = -\omega^m \epsilon_{mk\ell} P_{kj} P_{\ell i}$$

$$= -\omega^m (-) \epsilon_{m\ell k} P_{kj} P_{\ell i}$$

$k = k'$ $\ell = \ell'$

$$= -\omega^m (-) \epsilon_{mk'\ell'} P_{\ell' j} P_{k' i}$$

rearrange the factors and drop the prime

$$= -\omega^m (-) \epsilon_{mk\ell} P_{\ell i} P_{kj}$$

$$= -T'_{ij}$$

Consequently, the matrix $[T'_{ij}]$ can also be expanded in term of the antisymmetric matrices as in Eq.(3) on page 21.10.

Thus one finds that there must exist coefficients $(\omega'^1, \omega'^2, \omega'^3)$ such that

$[\omega]_c$: $T'_{ij} = -\omega'^r \epsilon_{rij}$

Thus the task is to determine

ω'^r: $\omega'^1 = ?$, $\omega'^2 = ?$, $\omega'^3 = ?$

This task is consigned to the Appendix.

The major part of this task is an application of Algebraic Index Calculus. The calculated result, Eq.(★) on page A.21.9, is

$$\epsilon_{r23} P^{-1}_{rn} \omega^n = P^{-1}_{1n} \omega^n = \omega^n \epsilon_{nk\ell} P_{k2} P_{\ell 3} = -T'_{23} = \omega'^r \epsilon_{r23}$$

$$\epsilon_{r31} P^{-1}_{rn} \omega^n = P^{-1}_{2n} \omega^n = \omega^n \epsilon_{nk\ell} P_{k3} P_{\ell 1} = -T'_{31} = \omega'^r \epsilon_{r23}$$

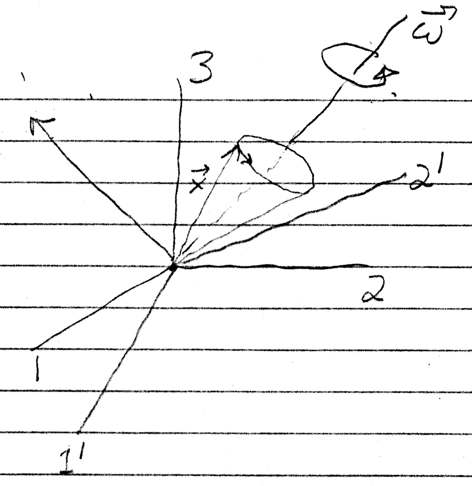
$$\epsilon_{r12} P^{-1}_{rn} \omega^n = P^{-1}_{3n} \omega^n = \omega^n \epsilon_{nk\ell} P_{k1} P_{\ell 2} = -T'_{12} = \omega'^r \epsilon_{r12}$$

or

$$\boxed{T'_{ij} = \underbrace{\left(P^{-1}_{rn} \omega^n\right)}_{\omega'^r} \epsilon_{rij}}$$

This result expresses the following fact:

The process of transitioning by means of an orthogonal coordinate transformation (which rotates only the coordinate axis forward, but leaves $\vec{v}$ and $\vec{x}$ uneffected)



yields measured values of $[\vec{v}]_C$ and $[\vec{X}]_C$ which are the same as those resulting from a new angular velocity $\vec{\omega}' = P^{-1}\vec{\omega}$ that has been rotated backward by P^{-1}

$$\vec{\omega}' = P^{-1}\vec{\omega}: [\vec{\omega}']_B = \left[\omega'^r = \sum_{n=1}^{3} P^{-1}{}_{rn}\,\omega^n\right] \qquad (1)$$

In other words, the relationship between the numerical values $[\vec{v}]_C$ and $[\vec{X}]_C$ in the new frame is the same as the relation between $[\vec{v}]_B$ and $[\vec{X}]_B$ in the old frame

but with the different angular velocity whose components in the original frame are given by Eq. (1) on p 21.13.

Mathematically one has

$$[v]_C = [T_{\vec{\omega}}]_C [\vec{X}]_C$$

is equivalent numerically to

$$[v]_C = [T_{P^{-1}\vec{\omega}}]_B [\vec{X}]_C$$

or

$$\boxed{P\,[T_{\vec{\omega}}]_B\,P^{-1} = [T_{P^{-1}\vec{\omega}}]_B}$$