LECTURE 21

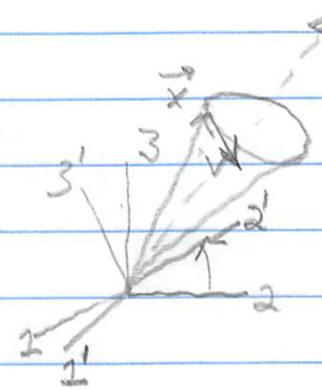
1. Passive vs Active Perspectives as Illustrated by the mathematization of rotations.

For the purpose of concretizing matrix equations such as

$$[T_\omega]_C = P\,[T_\omega]_C\,P^{-1} \qquad (*)$$

in numerical terms, that represent actual observated measurements, one needs to refer to the individual matrix elements of a given representative.

Passive perspective:
Basis changes but $\vec{\omega}$ stays fixed



"fixed gyroscope"

$\vec{v} = T_{\vec{\omega}}\,\vec{x}$

Passive

$$\overset{\text{NEW}}{[v]_C} = P\,\overset{\text{OLD}}{[v]_B} \qquad \overset{\text{NEW}}{[\vec{v}]_B} = P^{-1}\,\overset{\text{OLD}}{[v]_C}$$

$$[x]_C = P\,[\vec{x}]_B \qquad [\vec{x}]_B = P^{-1}\,[x]_C$$

$$[v]_C = P[v]_B = P[T_\omega]_B[x]_B$$

$$= P[T_\omega]_B P^{-1} P[x]_B = P[T_\omega]_B P^{-1}[\vec{x}]_C = [T_\omega]_C[\vec{x}]_C$$ which holds $\forall\, \vec{x}$

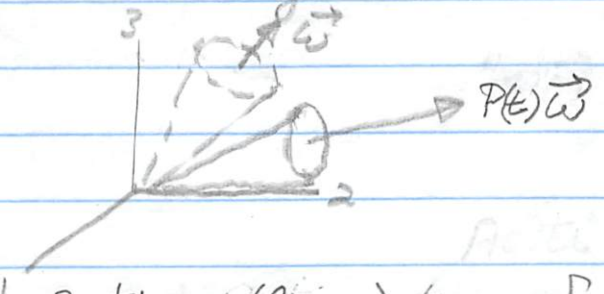
where $B=\{e_1, e_2, e_3\}$ is the old basis and $C=\{e_1', e_2', e_3'\}$ is the new basis

Conclusion:

The <u>passive</u> perspective is mathematized by the statement

$$\boxed{[T_\omega]_C = P\,[T_\omega]_B\,P^{-1}}$$

<u>Active</u> Perspective: Basis stays fixed, but $\vec{\omega}$ changed from $\vec{\omega}$ to $P\vec{\omega}$



"precessing gyroscope"

Active (linear) transformation

$$P(t):\ E^3 \to E^3$$
$$\vec{\omega} \to P(t)\,\vec{\omega}$$

(not to be confused with $T_{\vec{\omega}}$)

a) Thus we have a new linear transformation

$$T_{p\vec{\omega}} : E^3 \longrightarrow E^3$$

$$\vec{x} \rightsquigarrow \vec{w} = T_{p\vec{\omega}}(\vec{x}) = (p\vec{\omega}) \times \vec{x}$$

which is different from the original lin. xform'n

$$T_{\vec{\omega}} : \quad \vec{x} \rightsquigarrow \vec{v} = T_{\vec{\omega}}(\vec{x}) = \vec{\omega} \times \vec{x}$$

b) Both are based on the same cross product rule:

$$\underbrace{\vec{v} = T_{\vec{\omega}}(\vec{x})}_{\vec{e}_1 v^1 + \vec{e}_2 v^2 + \vec{e}_3 v^3} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega^1 & \omega^2 & \omega^3 \\ x^1 & x^2 & x^3 \end{vmatrix} = \vec{\omega} \times \vec{x}$$

whose B-coordinate representative is

$$[v]_B = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}_B = \underbrace{\begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}}_{[T_\omega]_B} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}_B$$

and

$$\vec{w} = T_{p\vec{\omega}}(\vec{x}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ (p\vec{\omega})^1 & (p\vec{\omega})^2 & (p\vec{\omega})^3 \\ x^1 & x^2 & x^3 \end{vmatrix} = p\vec{\omega} \times \vec{x}$$

whose $\mathcal{B}$-coordinate representative is

$$[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}_{\mathcal{B}} = \underbrace{\begin{bmatrix} 0 & -(P\vec{\omega})^3 & (P\vec{\omega})^2 \\ (P\vec{\omega})^3 & 0 & -(P\vec{\omega})^1 \\ -(P\vec{\omega})^1 & (P\vec{\omega})^2 & 0 \end{bmatrix}}_{[T_{P\vec{\omega}}]_{\mathcal{B}}} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}_{\mathcal{B}}$$

c) Thus we need to answer two basic questions:

Q1: What is the relation between the vectors

$$P\vec{\omega} \text{ and } \vec{\omega}$$

and between their coordinate representatives

$$[P(t)\vec{\omega}]_{\mathcal{B}} \text{ and } [\vec{\omega}]_{\mathcal{B}} \,?$$

Q2: What is the relation between the transformations

$$T_{P\vec{\omega}} \text{ and } T_{\vec{\omega}}$$

and between their coordinate representatives

$$[T_{P\vec{\omega}}]_{\mathcal{B}} \text{ and } [T_{\vec{\omega}}]_{\mathcal{B}} \,?$$

The answers to these questions are given by Eq. (*) on P 21.6, and more explicitly by Eq. (**) on P 21.10, and by the boxed Eq. (*) on P 22.8.

I Answer to Q 1:

A1: The linear action P moves vector $\vec{\omega}$ and the vectors e_i to the vector $P\omega$ and the vectors Pe_i

$$\vec{\omega} \rightsquigarrow P\vec{\omega} = \sum_i \vec{e}_i \left[P\vec{\omega}\right]_B^i$$

$$\vec{e}_1 \rightsquigarrow P\vec{e}_1 = \sum_i e_i P_{i1}$$

$$\vec{e}_2 \rightsquigarrow P\vec{e}_2 = \sum e_i P_{i2}$$

$$\vec{e}_3 \rightsquigarrow P\vec{e}_3 = \sum e_i P_{i3}$$

Hence the effect of the action $P(t)$ ("precession") on $\vec{\omega}$ is

$$\vec{\omega} \rightsquigarrow P(t)\vec{\omega} = \vec{e}_i \left(P_{ij}(t)\omega^j\right)$$

initial final

$$[\vec{\omega}]_B \rightsquigarrow \boxed{[P\vec{\omega}]_B = \left[P_{C\leftarrow B}(t)\right] \left[\vec{\omega}\right]_B} * \left(= [\omega]_C\right)$$

which is to say that

the $\boxed{\text{numerical}}$ components of the new angular velocity $P(t)\vec{\omega}$ are the $\boxed{\text{numerical}}$ components of the old angular velocity relative to the new basis C

II Answer to Q 2 (How is $[T_{P\vec{\omega}}]_B$ related to $[T_{\vec{\omega}}]_B$?)

The precession $P(t)$ effects all parts of the gyroscope

$$\vec{\omega} \rightsquigarrow P\vec{\omega} \qquad [\vec{\omega}]_B \rightsquigarrow [P_{C\leftarrow B}][\vec{\omega}]_B \quad (1a)$$
$$\vec{x} \rightsquigarrow P\vec{x} \qquad [\vec{x}]_B \rightsquigarrow [P_{C\leftarrow B}][\vec{x}]_B \quad (1b)$$
$$\vec{v} \rightsquigarrow P\vec{v} \qquad [\vec{v}]_B \rightsquigarrow [P_{C\leftarrow B}][\vec{v}]_B \quad (1c)$$

The gyroscopic relation between $\vec{v}$ and $\vec{x}$ is

$$\vec{v} = \vec{\omega} \times \vec{x} = T_{\vec{\omega}}\vec{x}$$

$$P\vec{v} = (P\vec{\omega}) \times (P\vec{x}) = T_{P\vec{\omega}}\vec{x}$$

Its B representation is

$$(2) \quad [\vec{v}]_B = \underbrace{[T_{\vec{\omega}}]_B}_{\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}} [X]_B \rightsquigarrow [P_{C\leftarrow B}][v]_B = [P_{C\leftarrow B}][T_{\vec{\omega}}]_B[X]_B$$

Same using (1)

$[P_{C\leftarrow B}]$

$$(3) \quad [Pv]_B = [T_{P\vec{\omega}}]_B [PX]_B$$

21, 8

we have

$$[P\vec{v}]_B = [P_{C\leftarrow B}][v]_B \qquad \text{using (1c)}$$

$$= [P_{C\leftarrow B}][T_{\vec{\omega}}]_B[x]_B \qquad \text{using (2)}$$

$$= [P_{C\leftarrow B}][T_{\vec{\omega}}]_B \underbrace{[P^{-1}_{C\leftarrow B}][P_{C\leftarrow B}]}_{\text{identity matrix}}[x]_B$$

$$= [P_{C\leftarrow B}][T_{\vec{\omega}}]_B[P^{-1}_{C\leftarrow B}][P\vec{x}]_B \quad \text{using (1b)}$$

$$= [T_{P\omega}]_B [P\vec{x}]_B \qquad \text{using (3)}$$

This holds $\forall\ \vec{x}$.

CONCLUSION:

$$\boxed{[T_{P\vec{\omega}}]_B = P\,[T_\omega]_B\,P^{-1}}$$

OR

$$\begin{bmatrix} 0 & -(P\vec{\omega})^3 & (P\vec{\omega})^2 \\ (P\vec{\omega})^3 & 0 & -(P\vec{\omega})^1 \\ -(P\vec{\omega})^2 & (P\omega)^1 & 0 \end{bmatrix} = P \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix} P^{-1}$$

where

$$[P\vec{\omega}]_B = [P_{C\leftarrow B}][\omega]_B \qquad \text{using (1a)}$$

$(P\omega)^i$ $i = 1, 2, 3$ are the components of $[P\omega]_B$