LECTURE 21

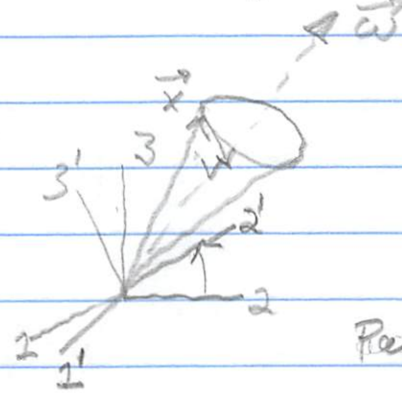
1. "Passive" vs "Active" Perspective
2. Rigid transformations mathematized by orthogonal matrices

For the purpose of concretizing matrix equations such as

$$[T_\omega]_C = P\,[T_\omega]_C\,P^{-1} \qquad (*)$$

in numerical terms, that represent actual observated measurements, one needs to refer to the individual matrix elements of a given representative.

Passive perspective: Basis changes but $\vec{\omega}$ stays fixed



$\vec{v} = T_{\vec{\omega}}\,\vec{x}$

Passive

$$[v]_C = P\,[v]_B \qquad [\vec{v}]_B = P^{-1}[v]_C$$
$$[x]_C = P\,[\vec{x}]_B \qquad [\vec{x}]_B = P^{-1}[x]_C$$

$$[v]_C = \underbrace{P[T_\omega]_B P^{-1}}_{[T_\omega]_C}\,\underbrace{P[x]_B}_{[x]_C}$$

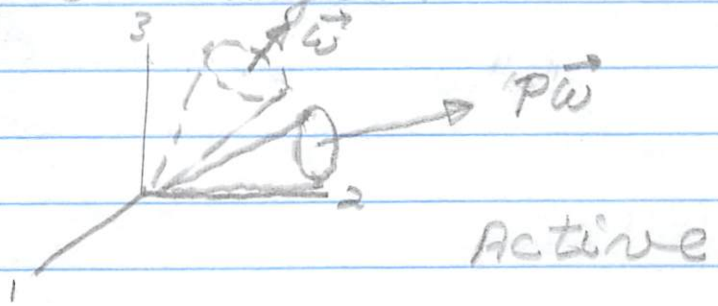
where $B=\{e_1, e_2, e_3\}$ is the old basis and $C=\{e_1', e_2', e_3'\}$ is the new basis

Conclusion:

The passive perspective is mathematized by the statement

$$[T_\omega]_C = P\,[T_\omega]_B\,P^{-1}$$

Active Perspective: Basis stays fixed, but $\vec{\omega}$ changed from $\vec{\omega}$ to $P\vec{\omega}$



a) Thus we have a new linear transformation

$$T_{p\vec{\omega}} : E^3 \to E^3$$

$$\vec{x} \rightsquigarrow \vec{w} = T_{p\vec{\omega}}(\vec{x}) = (p\vec{\omega}) \times \vec{x}$$

which is different from

$$T_{\vec{\omega}} : \quad \vec{x} \rightsquigarrow \vec{v} = T_{\vec{\omega}}(\vec{x}) = \vec{\omega} \times \vec{x}$$

b) Both are based on the same cross product rule:

$$\vec{v} = T_{\vec{\omega}}(\vec{x}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \omega^1 & \omega^2 & \omega^3 \\ x^1 & x^2 & x^3 \end{vmatrix} = \vec{\omega} \times \vec{x}$$

whose B-coordinate representative is

$$[v]_B = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}_B = \underbrace{\begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}}_{[T_\omega]_B} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}_B$$

and

$$\vec{w} = T_{p\vec{\omega}}(\vec{x}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ (p\vec{\omega})^1 & (p\vec{\omega})^2 & (p\vec{\omega})^3 \\ x^1 & x^2 & x^3 \end{vmatrix} = p\vec{\omega} \times \vec{x}$$

whose $\mathcal{B}$-coordinate representative is

$$[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}_{\mathcal{B}} = \underbrace{\begin{bmatrix} 0 & -(P\vec{\omega})^3 & (P\vec{\omega})^2 \\ (P\vec{\omega})^3 & 0 & -(P\vec{\omega})^1 \\ -(P\vec{\omega})^1 & (P\vec{\omega})^2 & 0 \end{bmatrix}}_{[T_{P\vec{\omega}}]_{\mathcal{B}}} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}_{\mathcal{B}}$$

c) Thus we need to answer two basic questions:

Q1: What is the relation between

$P\vec{\omega}$ and $\vec{\omega}$,

in particular between

$[P\vec{\omega}]_{\mathcal{B}}$ and $[\vec{\omega}]_{\mathcal{B}}$?

Q2: What is the relation between

$T_{P\vec{\omega}}$ and $T_{\vec{\omega}}$

in particular between

$[T_{P\vec{\omega}}]_{\mathcal{B}}$ and $[T_{\vec{\omega}}]_{\mathcal{B}}$?

The answers to these questions are given by Eq. (*) on P 21.6, and more explicitly by Eq. (**) on P 21.10.

andby Eq. ( ) on 21.7

(I) Answer to Q1:

A1: The action from vector $\vec{\omega}$ to vector $P\vec{\omega}$ is expressed mathematically by

$$[P\vec{\omega}]_{B} = [P_{C\leftarrow B}][\vec{\omega}]_{B}, \qquad (*)$$

$$\sum_{i=1}^{3} \vec{e}_i (P\vec{\omega})^i = \sum_i \sum_j e_i P_{ij} \omega^j$$

which is to say:

The numerical components of the new angular velocity $P\vec{\omega}$ are the numerical components of the old angular velocity $\vec{\omega}$ relative to the new basis C.

(II) Rigid Rotations

a) The action

$$P: \vec{\omega} \rightsquigarrow P\vec{\omega}$$

we are focussing on is a rigid rotation, an action which does not change the magnitude of the angular velocities:

$$\vec{\omega}\cdot\vec{\omega} = (P\vec{\omega})\cdot(P\vec{\omega}),$$

A rigid rotation has this property for all vectors; $\vec{w}_1, \vec{w}_2 \ldots$, including their sums one has

$$(\vec{w}_1+\vec{w}_2)\cdot(\vec{w}_1+\vec{w}_2)=(P\vec{w}_1+P\vec{w}_2)\cdot(P\vec{w}^1+P\vec{w}^2)$$

Consequently,

$$\boxed{\vec{w}_1\cdot\vec{w}_2 = P\vec{w}_1\cdot P\vec{w}_2} \qquad (*)$$

Explicitly, one finds that a rigid rotation does not change the inner product of any pair of vectors: <u>Inner products are invariant under rigid rotations.</u>

b) <u>Question</u>: Given an active rigid rotation

$$[P\vec{w}]_B = [P_{c\leftarrow B}][\vec{w}]_B$$

$$e_i (P\vec{w})^i = e_i P_{ij} w^j ,$$

what properties do the columns P_{ij} $j=1,2,3$ of $P_{c\leftarrow B}$ have?

Answer:

Our focus of attention is on E^3, the vector space, which accommodates orthonormal bases, such as

$$B = \{e_i\}_{i=1}^{3} \quad : \quad \vec{e}_k \cdot \vec{e}_\ell = \delta_{k\ell}$$

We let $\vec{\omega}_1$ and $\vec{\omega}_2$ coincide with two basis vectors:

$$\vec{\omega}_1 = \vec{e}_k$$
$$\vec{\omega}_2 = \vec{e}_\ell .$$

Applying the inner product invariance under rigid rotations, the boxed Eq. (*) on page 21.7, one finds

$$\delta_{k\ell} = e_k \cdot e_\ell = (P\vec{e}_k)\cdot(P\vec{e}_\ell)$$
$$= \vec{e}_i (P\vec{e}_k)^i \cdot \vec{e}_j (P\vec{e}_\ell)^j$$
$$= \vec{e}_i \cdot \vec{e}_j \; (P\vec{e}_k)^i (P\vec{e}_\ell)^j$$
$$= \delta_{ij} \; P_{im} (\vec{e}_k)^m \; P_{jn} (e_\ell)^n$$

$$\delta_{k\ell} = \delta_{ij}\, P^{i}_{\ m}\, \delta^{m}_{k}\, P^{j}_{\ n}\, \delta^{n}_{\ell}$$

$$= \delta_{ij}\, P^{i}_{\ k}\, P^{j}_{\ \ell}$$

$$\boxed{\delta_{k\ell} = P^{j}_{\ k}\, P^{j}_{\ \ell}} \qquad (**)$$

Thus one concludes that the columns of the matrix

$$P = \begin{bmatrix} \vdots & \vdots & \vdots \\ P^{j}_{\ 1} & P^{j}_{\ 2} & P^{j}_{\ 3} \\ \vdots & \vdots & \vdots \end{bmatrix} \qquad (***)$$

form an <u>orthonormal set</u>. The colums of a rigid rotation matrix are orthogonal and of unit length. One says that P is an "orthogonal matrix".

c) Q: What is the <u>inverse of P</u>?

A: The boxed Eq. (**) at the top of this page reads

$$\delta_{k\ell} = \sum_j P_{jk} P_{j\ell}$$

$$= \sum_j \left(P^{\text{traspose}}\right)_{kj} P_{j\ell} \qquad k^{th} \text{ row} \times \ell^{th} \text{ column}$$

$$= \left(P^T P\right)_{k\ell}$$

Thus

$$I = P^T P \iff \boxed{P^{-1} = P^T}$$

whenever P is an orthogonal matrix

d) The more explicit answer to Q 1 on page 21,4 is that

$$\boxed{\left[P\vec{w}\right]_B = P\left[\vec{w}\right]_B}$$

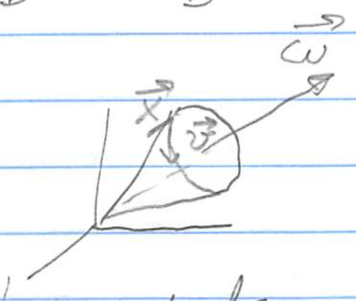
where $P = P_{C \leftarrow B}$ is an orthogonal matrix; $P^t = P^{-1}$.

Example

GIVEN:

(1) $[\vec{v}]_B = [T_{\vec{\omega}}]_B [\vec{X}]_B$:

$$\begin{bmatrix} v_x \\ v_y \\ v_y \end{bmatrix}_B = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_B$$



(2) A rotation around the z-axis

$$\begin{bmatrix} v_x' \\ v_y' \\ v_z' \end{bmatrix}_C = \overbrace{\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{P} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_B ; \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}_C = \overbrace{\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{P} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_B$$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_B = \underbrace{\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P^{-1}} \begin{bmatrix} v_x' \\ v_y' \\ v_z' \end{bmatrix}_C ; \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}_B = \underbrace{\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P^{-1}} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}_C$$

where $c = \cos\alpha$ and $s = \sin\alpha$ and $c^2 + s^2 = 1$

FIND: $\bar{\omega}^1 = ?$ in terms of $\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix}$

$\bar{\omega}^2 = ?$

$\bar{\omega}^3 = ?$

Solution;

Step I) Introduce the given rotation P into the given transformation equation

$$[\vec{v}]_B = [T_{\vec{\omega}}]_B [\vec{X}]_B$$

$$P[\vec{v}]_B = P[T_{\vec{\omega}}]P^{-1}P[\vec{X}]_B$$

$$[\vec{v}]_C = P[T_{\vec{\omega}}]P^{-1}[\vec{X}]_C$$

Explicitly we have

$$\begin{bmatrix} v_x' \\ v_y' \\ v_z' \end{bmatrix}_C = \underbrace{\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}}_{[T_\omega]_B} \underbrace{\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P^{-1}} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}_C$$

Step II) Do the matrix mplication

$$= \begin{bmatrix} s\omega^3 & -c\omega^3 & c\omega^2 - s\omega^1 \\ c\omega^3 & s\omega^3 & -s\omega^2 - c\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}_C$$

$$= \underbrace{\begin{bmatrix} 0 & -\omega^3 & -c\omega^2 - s\omega^1 \\ \omega^3 & 0 & -s\omega^2 - c\omega^1 \\ -c\omega^2 + s\omega^1 & s\omega^2 + c\omega^1 & 0 \end{bmatrix}} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}_C$$

Step III) compare this with $\begin{bmatrix} 0 & -\bar{\omega}^3 & \bar{\omega}^2 \\ \bar{\omega}^3 & 0 & -\bar{\omega}^1 \\ -\bar{\omega}^2 & \bar{\omega}^1 & 0 \end{bmatrix}$

and obtain

$$\boxed{\begin{bmatrix} \bar{\omega}^1 = c\omega^1 + s\omega^2 \\ \bar{\omega}^2 = -s\omega^1 + c\omega^1 \\ \bar{\omega}^3 = \omega^3 \end{bmatrix} = P \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix}_B}$$

Thus $\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix}$ gets rotated around the

z-axis by the rotation

$$P = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

thereby illustrating the mathematical fact that

$$P[T_{\vec{\omega}}]_B P^{-1} = 1\,[T_{P\vec{\omega}}]_B$$

with prefactor

$$\det P = 1$$

because P does not refer to any reflection,