APPENDIX (for LECTURE 21)

The Algebraic Index Calculus Applied to

(i) Determinants

(ii) Calculation of the Inverse of a Matrix.

(iii) Orthogonal Coordinate Representative

Algebraic Index Calculus

Every matrix equation can be written in terms using index notation, as has already been exemplified in the boxed equations on pages 19.5 and 20.3

The advantage of the algebraic index calculus becomes evident, among others, in the context of defining and working with determinants. In terms of the Levi-Civita epsilon one has for any 3×3 matrix $\underline{P}$:

$$\det \underline{P} = \begin{vmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{vmatrix}$$

$$= \epsilon_{ijk} P_{1i} P_{2j} P_{3k} \quad \text{(row expansion)}$$

$$= \epsilon_{ijk} P_{i1} P_{j2} P_{k3} \quad \text{(column expansion)}$$

These two alternative expansions yield the immediate

Theorem 13.1

$$\boxed{\det P = \det P^{\text{transpose}}}$$

The properties (change of sign, vanishing value) of the epsilon expansion of $\det P$ are captured by means of the following useful identity

$$\boxed{\epsilon_{i_1 i_2 i_3} P_{i_1 j_1} P_{i_2 j_2} P_{i_3 j_3} = (\det P)\, \epsilon_{j_1 j_2 j_3}} \quad (1)$$

Note that we have introduced subscripted indeces. This is an obvious notational improvement for generalizing to higher dimensions.

Equation (1) on P A.21.3 has a number of useful properties. Consider the product of two 3×3 matrices

$$PR: \quad (PR)_{ik} = P_{ij} R_{jk}$$

Using Eq.(1) one obtains

$$(\epsilon_{i_1 i_2 i_3} P_{i_1 j_1} P_{i_2 j_2} P_{i_3 j_3}) R_{j_1 k_1} R_{j_2 k_2} R_{j_3 k_3}$$

$$= ((\det P)\, \epsilon_{j_1 j_2 j_3}) R_{j_1 k_1} R_{j_2 k_2} R_{j_3 k_3}.$$

Using it again on the r.h.s

$$\epsilon_{i_1 i_2 i_3} (P_{i_1 j_1} R_{j_1 k_1})(P_{i_2 j_2} R_{j_2 k_2})(P_{i_3 j_3} R_{j_3 k_3})$$

$$= (\det P)(\det R)\, \epsilon_{k_1 k_2 k_3}$$

Using it again on the l.h.s one finally obtains

$$(\det PR)\, \epsilon_{k_1 k_2 k_3} = (\det P)(\det R)\, \epsilon_{k_1 k_2 k_3}$$

Thus the index calculus furnishes us

with the

Theorem (Determinantal Product Theorem)

$$\boxed{(\det P)(\det R) = \det PR}$$

In particular, if $R = P^{-1}$

$$\boxed{\det(P^{-1}) = \frac{1}{\det P}}$$

If P is an orthogonal matrix, this becomes

$$\underbrace{\det P^{tr}}_{\overset{(P.12,13)}{\longrightarrow}\ \det P} = \frac{1}{\det P}$$

Thus

$$(\det P)^2 = 1$$

or

$$\boxed{\det P = \pm 1 \quad \text{for orthogonal matrices}}$$

The index calculus furnishes us also with explicit expressions for the inverse of a matrix whenever its determinant does not vanish.

Using the index calculus one obtains the inverse of a 3×3 matrix as follows:

Step I

$$\epsilon_{mk\ell} P_{1m} P_{2k} P_{3\ell} = \det P$$

$$\left.\begin{array}{l} \epsilon_{mk\ell} P_{2m} P_{2k} P_{3\ell} = 0 \\ \epsilon_{mk\ell} P_{3m} P_{2k} P_{3\ell} = 0 \end{array}\right\} \left(\begin{array}{c}\text{"repeated} \\ \text{rows of } P\text{"}\end{array}\right)$$

Or more briefly

$$P_{jm}\, \epsilon_{mk\ell}\, P_{2k} P_{3\ell} = \delta_{j1} \det P \quad (1)$$

Step II

Similarly

$$P_{jm}\, \epsilon_{mk\ell}\, P_{3k} P_{1\ell} = \delta_{j2} \det P \quad (2)$$

$$P_{jm}\, \epsilon_{mk\ell}\, P_{1k} P_{2\ell} = \delta_{j3} \det P \quad (3)$$

Step III

Divide by $\det P$ and Eqs(1)-(3)

become

$$\underbrace{P'_{jm}\,\frac{1}{\det P}\,\epsilon_{mkl}\,P_{2k}P_{3l}}_{\equiv\; P^{-1}_{m1}} = \delta_{j1} \qquad (1')$$

$$\underbrace{P'_{jm}\,\frac{1}{\det P}\,\epsilon_{mkl}\,P_{3k}P_{1l}}_{\equiv\; P^{-1}_{m2}} = \delta_{j2} \qquad (2')$$

$$\underbrace{P'_{jm}\,\frac{1}{\det P}\,\epsilon_{mkl}\,P_{1k}P_{2l}}_{\equiv\; P^{-1}_{m3}} = \delta_{j3} \qquad (3')$$

In light of the cyclic nature of the appearance of the indices 1, 2, 3, these three equations can be written more compactly as

$$\boxed{P'_{jm}\,\underbrace{\frac{1}{\det P}\,\epsilon_{mkl}\,P_{rk}P_{sl}\,\frac{1}{2}\,\epsilon_{rst}}_{P^{-1}_{mt}} = \delta_{jt}}$$

Multiply Eqs. (1') - (3') by P^{-1}_{nj} (and, of course, sum over j); use

$$P^{-1}_{nj}\,P_{jm} = \delta_{nm}$$

and obtain

$$P^{-1}_{n1} = \frac{1}{\det P}\,\epsilon_{nkl}\,P_{2k}P_{3l} = -\frac{1}{\det P}\,\epsilon_{nkl}\,P_{3k}P_{2l}$$

$$P^{-1}_{n2} = \frac{1}{\det P}\,\epsilon_{nkl}\,P_{3k}P_{1l} = -\frac{1}{\det P}\,\epsilon_{nkl}\,P_{1k}P_{3l}$$

$$P^{-1}_{n3} = \frac{1}{\det P}\,\epsilon_{nkl}\,P_{1k}P_{2l} - \frac{1}{\det P}\,\epsilon_{nkl}\,P_{2k}P_{1l},$$

Multiplying by ω^n and summing over n yields

$$\omega^n P^{-1}_{n1} = \frac{1}{\det P}\,\omega^n\epsilon_{nkl}\,P_{2k}P_{3l} = -(2\leftrightarrow 3)$$

$$\omega^n P^{-1}_{n2} = \frac{1}{\det P}\,\omega^n\epsilon_{nkl}\,P_{3k}P_{1l} = -(3\leftrightarrow 1)$$

$$\omega^n P^{-1}_{n3} = \frac{1}{\det P}\,\omega^n\epsilon_{nkl}\,P_{1k}P_{2l} = -(1\leftrightarrow 2)$$

These equations apply when one replaces P with P^{-1}. Furthermore, for orthogonal matrices

$$P^{-1}_{2k} = P_{k2},\quad P^{-1}_{3\ell} = P_{\ell 3},\ \text{etc.}$$

and

$$\det P = 1.$$

Consequently,

$$\left.\begin{aligned} P^{-1}_{1n}\,\omega^n &= \omega^n \epsilon_{nk\ell} P_{k2} P_{\ell 3} \\ P^{-1}_{2n}\,\omega^n &= \omega^n \epsilon_{nk\ell} P_{k3} P_{\ell 1} \\ P^{-1}_{3n}\,\omega^n &= \omega^n \epsilon_{nk\ell} P_{k1} P_{\ell 2} \end{aligned}\right\} (\star),$$