LECTURE 22+1

Invariant subspaces

Direct Sum: A Reminder

Coordinate Representation Relative to Invariant Subspaces.

I, INVARIANT SUBSPACES

Let

$$L: \quad U \to U$$

be a linear transformation. Consider the set of all non-zero vectors $\vec{u}$ such that

$$L\vec{u} = \lambda \vec{u}$$

A non-zero vector which has the property that its transform is a mere scalar multiple of itself, i.e. whose direction does not get altered, is called an eigenvector of L. The shrinkage (or stretch) factor λ is called an eigenvalue of L.

The 1-dimensional subspace spanned by u is invariant under L. By this one means that

$$\vec{u} \in Sp(\{\vec{u}\}) \Rightarrow L\vec{u} \in Sp(\{\vec{u}\})$$

We now extend this concept from 1-D invariant subspaces to multi-dimensional subspaces

<u>Definition</u> 22.1 (Invariant Subspace)

Let

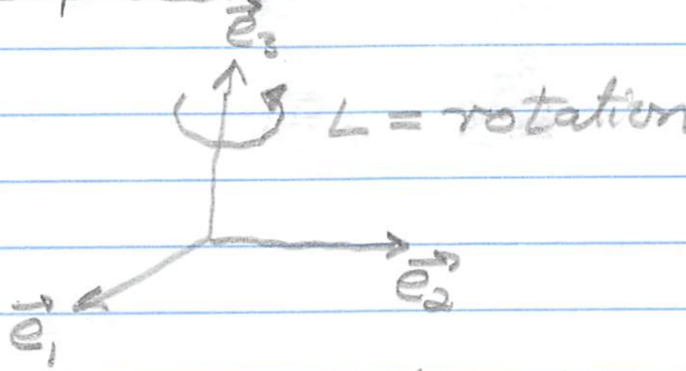
$$L: U \to U$$

be a linear transformation. A <u>subspace</u> $\mathcal{M} \subset U$ is said to be <u>invariant under L</u> if

$$\boxed{\underbrace{m \in \mathcal{M} \Rightarrow Lm \in \mathcal{M}}_{\equiv \ \boxed{L(\mathcal{M}) \subset \mathcal{M}}}}$$

i.e. the image of $\mathcal{M}$ under L is contained in $\mathcal{M}$.

Example 22.1 (Rotation)



Here $\mathcal{M}_1 = Sp(\{\vec{e}_3\})$ is invt under L
and
$\mathcal{M}_2 = Sp(\{\vec{e}_1, \vec{e}_2\})$ is invt under L.

There are at least four obvious subspaces which are invariant under any linear transformation. They are highlighted by the following

Theorem 22.1

Given $L : U \to U$
then
U, $\{\vec{0}\}$, $\mathcal{N}(L)$, and $\mathcal{R}(T)$ are invariant under L, i.e.

$$L(U) \subset U$$
$$L(\{\vec{0}\}) \subset \{\vec{0}\}$$
$$L(\mathcal{N}(L)) \subset \mathcal{N}(L)$$
$$L(\mathcal{R}(L)) \subset \mathcal{R}(T)$$

II. Direct Sum: A Reminder

Suppose that $\dim U = n$. Let $\mathcal{M}$ be a k-dimensional subspace of U, and let

$$\boxed{B = \{m_1, \ldots, m_k\}}$$

be a basis for $\mathcal{M}$. Being a set linearly independent in U, this set can be extended to a basis in U:

$$C = \overbrace{\{\underbrace{m_1, \ldots, m_k}_{\text{basis for } \mathcal{M}}, \underbrace{w_1, \ldots, w_{n-k}}_{\boxed{\{w_1, \ldots, w_{n-k}\}} = \text{ basis for } \mathcal{W}}\}}^{\text{basis for } U}$$

Let

$$\mathcal{W} = Sp(\{w_1, \ldots, w_{n-k}\})$$

<u>Note:</u>

(i) For any $\vec{u} \in U$ one has

$$\vec{u} = \vec{m} + \vec{w}, \quad = \sum_{i=1}^{k} c_i \vec{m}_i + \sum_{j=1}^{n-k} d_j \vec{w}_j$$

where $\vec{m} \in \mathcal{M}$ is unique and $\vec{w} \in \mathcal{W}$ is unique.

[Note: If $\vec{m} = \vec{w}$ then $\vec{m} \in \mathcal{M} \Rightarrow d_j = 0$; and $\vec{m} \in \mathcal{W} \Rightarrow c_i = 0$; $m = w \Rightarrow m = 0$]

(ii) $\mathcal{M} = Sp(\{m_1, \ldots, m_k\})$; $\mathcal{W} = Sp(\{w_1, \ldots, w_{n-k}\})$ imply

$$\mathcal{M} \cap \mathcal{W} = \{\vec{0}\}$$ because $v = c_i \vec{m}_i = d_j \vec{w}_j$, $c_i \vec{m}_i - d_j \vec{w}_j = \vec{0}$

(iii) $U = \mathcal{M} \oplus \mathcal{W}$, whenever (i) ... are satisfied.

III. COORDINATE REPRESENTATIVE RELATIVE TO BASIS FOR INVT SUBSPACE.

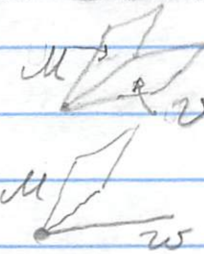
Suppose that U is finite-dimensional,

say,

a) $\boxed{\dim U = n}$,

and that U is the <u>direct sum</u> of

two subspaces:



$R^3 = \mathcal{M} + \mathcal{V}$

vs

$R^3 = \mathcal{M} \oplus \mathcal{W}$

b) $\boxed{U = \mathcal{M} \oplus \mathcal{W}}$

In other words, $\mathcal{M} \cap \mathcal{W} = \{\vec{0}\}$, and for

any $\vec{u} \in U$ $\exists$ vectors $\vec{m} \in \mathcal{M}$

and $\vec{w} \in \mathcal{W}$ such that

$$\vec{u} = \vec{m} + \vec{w}.$$

Furthermore, suppose that $\mathcal{M}$ invariant

under L,

c) $\boxed{L(\mathcal{M}) \subset \mathcal{M}}$.

(But <u>$\mathcal{W}$ is not necessarily invariant</u> under L)

d) Under such a circumstance, bases can be

chosen for $\mathcal{M}$ and $\mathcal{W}$ respectively yield a remarkable simple $n \times n$ coordinate representative for L. This simplicity is summarized by the following

Theorem 22.2

Given:

Suppose $\dim U = n$

If $\mathcal{M} \subset U$ is a subspace invariant under L,

$$L(\mathcal{M}) \subset \mathcal{M},$$

and

$$\dim(\mathcal{M}) = k.$$

Conclusion:

Then L can be represented by a matrix $Q_C = [L]_C$ whose form is

$$Q_C = \begin{array}{cc} \begin{array}{cc} \overbrace{\quad}^{k} & \overbrace{\quad}^{n-k} \end{array} & \\ \left[\begin{array}{c|c} A & B \\ \hline O & D \end{array}\right] & \begin{array}{l} \}k \\ \}n-k \end{array} \end{array}$$

where

A is $k \times k$, B is $k \times (n-k)$, D is $(n-k) \times (n-k)$

What can one say if both $\mathcal{M}$ and $\mathcal{W}$ are invariant under L? In that case one has

Corollary 22.1

If $\mathcal{M}$ and $\mathcal{W}$ are both invariant under L,

$$L(\mathcal{M}) \subset \mathcal{M}$$
$$L(\mathcal{W}) \subset \mathcal{W},$$

and if U is the direct sum of $\mathcal{M}$ and $\mathcal{W}$,

$$U = \mathcal{M} \oplus \mathcal{W},$$

then the coordinate representative of L has the form

$$Q_c \equiv [L]_c = \begin{array}{cc} \begin{array}{cc} \overbrace{}^{k} & \overbrace{}^{(n-k)} \end{array} & \\ \left[\begin{array}{c|c} A & O \\ \hline O & D \end{array}\right] & \begin{array}{l} \}k \\ \}(n-k) \end{array} \end{array}$$

<u>Comment.</u>

For the case described in the Corollary, one can decompose $[L]_c$ into the sum of two parts

$$[L_1]_c = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right] \text{ and } [L_2]_c = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D \end{array}\right]$$

with the property that

$$[L_1]_c + [L_2]_c = [L]_c$$

and

$$[L_1]_c [L_2]_c = [L_2]_c [L_1]_c = [0] \left(= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right] \right)$$

<u>proof</u> of Theorem 22.2 on P 22.6.

<u>Step I</u> Let $C = \{m_1, \ldots, m_k, w_1, \ldots, w_{n-k}\}$ be a basis such that

$$U = Sp(\{m_1, \ldots, m_k\}) \oplus Sp(\{w_1, \ldots, w_{n-k}\})$$

as depicted on P 22.4

<u>Step II</u> Let $\vec{u} \in U$. Its coordinate representative relative to C is

$$[u]_c = \begin{bmatrix} [m] \\ [w] \end{bmatrix} \begin{matrix} \} \ k\text{-tuple} \\ \} \ (n-k)\text{-tuple} \end{matrix}$$

<u>Step III</u> Apply the Representation Theorem on P 16.6 to L:

$$[L(\vec{u})]_c = [L]_c \, [u]_c$$

$$\begin{matrix} k \{ \\ n-k \{ \end{matrix} \quad = \begin{array}{c} \begin{matrix} \overbrace{}^{k} & \overbrace{}^{n-k} \end{matrix} \\ \left[\begin{array}{c|c} A & B \\ \hline E & D \end{array}\right] \\ \underbrace{}_{k} \end{array} \begin{bmatrix} [m] \\ [w] \end{bmatrix} = \left[\begin{array}{c} A[m] + B[w] \\ \hline \underset{\uparrow \ (n-k)\times k}{E}\,\underset{\uparrow \ k}{[m]} + D\,\underset{\uparrow \ (n-k)}{[w]} \end{array}\right] \begin{matrix} \} \ k \\ \} \ (n-k) \end{matrix}$$

$k \times (n-k)$

<u>Step IV</u>: Let $\vec{u} \in \mathcal{M} \Rightarrow [\vec{u}]_c = \begin{bmatrix} [m] \\ [0] \end{bmatrix}$ i.e. $[w] = [0]$

$$\therefore [L(\vec{u})]_c = \begin{bmatrix} A[m] \\ E[m] \end{bmatrix}$$

$L(\mathcal{M}) \subset \mathcal{M}$ means

Step V $L(\vec{u}) \in \mathcal{M} \Rightarrow [L(\vec{u})]_C = \begin{bmatrix} [L(\vec{u})] \\ [0] \end{bmatrix} \}(n-k)$

← zero column,

Using Step IV by letting $\vec{u} = \vec{m}_1, \vec{m}_2, \ldots, \vec{m}_k$ one obtains

$$\begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1k} \\ E_{21} & E_{22} & & E_{2k} \\ \vdots & & & \vdots \\ E_{n-k\,1} & E_{n-k\,2} & & E_{n-k\,k} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & & \\ 0 & 0 & & 0 \end{bmatrix}}_{k} \Big\} (n-k)$$

$[m_1]_B \quad [m_2]_B \quad [m_k]_B$

Thus E = zero matrix.

Conclusion:

$$[L(\vec{u})]_C = \left[\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right]$$

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proof of Corollary $L(\mathcal{W}) \subset \mathcal{W}$

Let $\vec{u} \in \mathcal{W}$ so that $[\vec{u}]_C = \begin{bmatrix} [0] \\ [w] \end{bmatrix}$.

Thus $[L(\vec{u})]_C = \begin{bmatrix} B[w] \\ D[w] \end{bmatrix}$

$L(\vec{u}) \in \mathcal{W} \Rightarrow [L(\vec{u})]_C = \begin{bmatrix} [0] \\ D[w] \end{bmatrix}$

$\Rightarrow B[w] = [0]$

$\Downarrow$

B = zero matrix.