LECTURE 22+1

Invariant Subspaces

Direct Sum: A Reminder

Coordinate Representation Relative to Invariant Subspaces
I, INVARIANT SUBSPACES

Let
\[ L : \mathbb{V} \rightarrow \mathbb{V} \]
be a linear transformation. Consider the set of all non-zero vectors \( \vec{u} \) such that
\[ L \vec{u} = \lambda \vec{u} \]

A non-zero vector which has the property that its transform is a mere scalar multiple of itself, i.e. whose direction does not get altered, is called an eigen vector of \( L \). The shrinkage (or stretch) factor \( \lambda \) is called an eigenvalue of \( L \).

The 1-dimensional subspace spanned by \( \vec{u} \) is invariant under \( L \). By this...
one means that
\[ \mathbf{u} \in \text{Sp}(E \mathbb{A}^3) \Rightarrow \mathbf{L}\mathbf{u} \in \text{Sp}(E \mathbb{A}^3) \]

We now extend this concept from 1-D invariant subspaces to multi-dimensional subspaces.

**Definition 22.1 (Invariant Subspace)**

Let
\[ \mathbf{L} : \mathbf{u} \rightarrow \mathbf{u} \]
be a linear transformation. A subspace \( \mathcal{M} \subseteq \mathbf{U} \) is said to be invariant under \( \mathbf{L} \) if
\[ \mathbf{m} \in \mathcal{M} \Rightarrow \mathbf{Lm} \in \mathcal{M} \]

i.e. the image of \( \mathcal{M} \) under \( \mathbf{L} \) is contained in \( \mathcal{M} \).
Example 22.1 (Rotation)

\[ L = \text{rotation} \]

Here \( M_1 = \text{Sp}(\{e_1\}) \) is invariant under \( L \) and \( M_2 = \text{Sp}(\{e_1, e_2, e_3\}) \) is invariant under \( L \).

There are at least four obvious subspaces which are invariant under any linear transformation. They are highlighted by the following:

**Theorem 22.1**

Given \( L : U \to U \), then

- \( U, \{0\} \), \( N(L) \), and \( R(T) \) are invariant under \( L \), i.e.,
  - \( L(U) = U \)
  - \( L(\{0\}) = \{0\} \)
  - \( L(N(L)) = N(L) \)
  - \( L(R(T)) = R(T) \)
II. Direct Sum: A Reminder

Suppose that \( \dim U = n \). Let \( \mathcal{U} \) be a \( k \)-dimensional subspace of \( U \), and let

\[
\mathcal{B} = \{ \mathbf{w}_1, \ldots, \mathbf{w}_k \}
\]

be a basis for \( \mathcal{U} \). Being a set linearly independent in \( U \), this set can be extended to a basis in \( U \):

\[
\mathcal{C} = \{ \mathbf{m}_1, \ldots, \mathbf{m}_k, \mathbf{w}_1, \ldots, \mathbf{w}_{n-k} \}
\]

Let

\[
\mathcal{W} = \text{sp} \{ \mathbf{w}_1, \ldots, \mathbf{w}_{n-k} \}
\]

Note: (i) For any \( \mathbf{u} \in U \) one has

\[
\mathbf{u} = \overline{\mathbf{m}} + \overline{\mathbf{w}} = \sum_{i=1}^{k} c_i \mathbf{m}_i + \sum_{j=1}^{n-k} d_j \mathbf{w}_j
\]

where \( \overline{\mathbf{m}} \in \mathcal{U} \) is unique and \( \overline{\mathbf{w}} \in \mathcal{W} \) is unique.

(Note: If \( \overline{\mathbf{m}} = \overline{\mathbf{w}} \) then \( \mathbf{m} \in \mathcal{U} \Rightarrow d_j = 0 \); and \( \mathbf{w} \in \mathcal{W} \Rightarrow c_i = 0 \), \( \mathbf{m} = \mathbf{w} = \mathbf{0} \)

(ii) \( \mathcal{U} = \text{sp} \{ \mathbf{m}_1, \ldots, \mathbf{m}_k \} \), \( \mathcal{W} = \text{sp} \{ \mathbf{w}_1, \ldots, \mathbf{w}_{n-k} \} \)

empty

\[
\mathcal{M} \cap \mathcal{W} = \{ \mathbf{0} \}
\]

because \( \mathbf{u} = c_i \mathbf{m}_i + d_j \mathbf{w}_j \)

\[
c_i \mathbf{m}_i - d_j \mathbf{w}_j = \mathbf{0}
\]

(iii') \( U = \mathcal{M} \oplus \mathcal{W} \)

whenever (i), (ii) are satisfied.
Suppose that $U$ is finite-dimensional, say, \[ \dim U = n \]
and that $U$ is the direct sum of two subspaces:
\[ U = M + W \]

In other words, \[ M \cap W = \{0\} \], and for any \( \vec{u} \in U \) there exist \( \vec{m} \in M \) and \( \vec{w} \in W \) such that
\[ \vec{u} = \vec{m} + \vec{w} \]

Furthermore, suppose that $M$ invariant under $L$,
\[ L(M) = M \]
(but $W$ is not necessarily invariant under $L$).

Under such a circumstance, bases can be chosen for $M$ and $W$ respectively.
yield a remarkable simple $n \times n$ coordinate representative for $L$. This simplicity is summarized by the following

**Theorem 22.2**

**Given**

Suppose $\dim U = n$

If $M \subseteq U$ is a subspace invariant under $L$, $L(M) \subseteq M$, and $\dim(M) = k$.

**Conclusion**

Then $L$ can be represented by a matrix

$$Q_c = [L]_c$$

whose form is

$$Q_c = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where $A$ is $k \times k$, $B$ is $k \times (n-k)$, $D$ is $(n-k) \times (n-k)$. 
What can one say if both $M$ and $W$ are invariant under $L$? In that case one has

**Corollary 22.1**

If $M$ and $W$ are both invariant under $L$,

\[
L(M) = M, \quad L(W) = W,
\]

and if $U$ is the direct sum of $M$ and $W$

\[
U = M \oplus W,
\]

then the coordinate representative of $L$ has the form

\[
Q_c = [L]_c = \begin{bmatrix}
A & 0 \\
0 & D
\end{bmatrix}_{3k \times (2n-k)}
\]
Comment: For the case described in the corollary, one can decompose $[L]_c$ into the sum of two parts:

$$[L_1]_c = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad [L_2]_c = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$$

with the property that

$$[L_1]_c + [L_2]_c = [L]_c$$

and

$$[L_1]_c [L_2]_c = [L_2]_c [L_1]_c = [0 \ 0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
proof of Theorem 22.2 on p 22.6.

**Step I** Let \( C = \{ m_1, \ldots, m_k, w_1, \ldots, w_{n-k} \} \) be a basis such that
\[
U = \text{sp} \left( \{ m_1, \ldots, m_k \} \right) \oplus \text{sp} \left( \{ w_1, \ldots, w_{n-k} \} \right)
\]
as depicted on p 22.4.

**Step II** Let \( \vec{u} \in U \). Its coordinate representative relative to \( C \) is
\[
[u]_C = \begin{bmatrix} [m] \end{bmatrix} \text{ }_{k \text{- tuple}} \begin{bmatrix} [w] \end{bmatrix} \text{ }_{(n-k) \text{- tuple}}
\]

**Step III** Apply the Representation Theorem on p.16.6 to \( U \):
\[
[L(\vec{u})]_C = [L]_C [U]_C = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} [m] \end{bmatrix} = \begin{bmatrix} A[m] + B[w] \end{bmatrix} \text{ }_{k \text{- tuple}} \begin{bmatrix} [w] \end{bmatrix} = \begin{bmatrix} E[m] + D[w] \end{bmatrix} \text{ }_{(n-k) \text{- tuple}}
\]

**Step IV** Let \( \vec{u} \in M \Rightarrow [U]_C = \begin{bmatrix} [m] \end{bmatrix} \text{ i.e. } [w] = [0] \)
\[
[L(\vec{u})]_C = \begin{bmatrix} A[m] \end{bmatrix} \begin{bmatrix} E[m] \end{bmatrix}
\]
Step IV. \( L(\bar{u}) \in M \Rightarrow [L(\bar{u})]_c = \begin{bmatrix} [L(\bar{u})]_c \end{bmatrix}_{n-k} \)

Using Step IV by letting \( \bar{u} = \bar{m}_1, \bar{m}_2, \ldots, \bar{m}_k \) one obtains

\[
\begin{bmatrix}
E_{11} & E_{12} & \cdots & E_{1k} \\
E_{21} & E_{22} & \cdots & E_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
E_{n-k1} & E_{n-k2} & \cdots & E_{n-kk}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
\bar{m}_1 \\
\bar{m}_2 \\
\vdots \\
\bar{m}_k
\end{bmatrix}
\]

Thus \( E = \text{zero matrix} \).

Conclusion:

\[
[L(\bar{u})]_c = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

Proof of Corollary: \( L(\bar{w}) \subset W \)

Let \( \bar{u} \in W \) so that \([\bar{u}]_c = \begin{bmatrix} [B] \\ [w] \end{bmatrix} \).
Thus \([L(\bar{u})]_c = \begin{bmatrix} [B[w] ] \\ [D[w]] \end{bmatrix} \) \( \Rightarrow [B[w]] = [0] \) \( \Rightarrow B[w] = [0] \) \( \Rightarrow B = \text{zero matrix} \).