APPENDIX (for LECTURE 22)

The Algebraic Index Calculus Applied to

(i) Determinants

(ii) Calculation of the Inverse of a Matrix.

(iii) Orthogonal Coordinate Representative

Algebraic Index Calculus

Every matrix equation can be written in terms using index notation, as has already been exemplified in the boxed equations on pages 19.5 and 20.3

The advantage of the algebraic index calculus becomes evident, among others, in the context of defining and working with determinants. In terms of the Levi-Civita epsilon one has for any 3×3 matrix $\underline{P}$:

$$\det \underline{P} = \begin{vmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{vmatrix}$$

$$= \epsilon_{ijk} P_{1i} P_{2j} P_{3k} \qquad \text{(row expansion)}$$

$$= \epsilon_{ijk} P_{i1} P_{j2} P_{k3} \qquad \text{(column expansion)}$$

These two alternative expansions yield the immediate

Theorem 13.1

$$\boxed{\det P = \det P^{\text{transpose}}}$$

The properties (change of sign, vanishing value) of the epsilon expansion of det P are captured by means of the following useful identity

$$\boxed{\epsilon_{i_1 i_2 i_3} P_{i_1 j_1} P_{i_2 j_2} P_{i_3 j_3} = (\det P)\,\epsilon_{j_1 j_2 j_3}} \quad (1)$$

Note that we have introduced sub-scripted indeces. This is an obvious notational improvement for generalizing to higher dimensions.

Equation (1) on P A.22.3 has a number of useful properties. Consider the product of two 3×3 matrices

$$PR: \quad (PR)_{ik} = P_{ij} R_{jk}$$

Using Eq. (1) on P A.21.3 one obtains

$$\left(\epsilon_{i_1 i_2 i_3} P_{i_1 j_1} P_{i_2 j_2} P_{i_3 j_3}\right) R_{j_1 k_1} R_{j_2 k_2} R_{j_3 k_3}$$

$$= \left((\det P)\, \epsilon_{j_1 j_2 j_3}\right) R_{j_1 k_1} R_{j_2 k_2} R_{j_3 k_3}.$$

Using it again on the r.h.s

$$\epsilon_{i_1 i_2 i_3} \left(P_{i_1 j_1} R_{j_1 k_1}\right)\left(P_{i_2 j_2} R_{j_2 k_2}\right)\left(P_{i_3 j_3} R_{j_3 k_3}\right)$$

$$= (\det P)(\det R)\, \epsilon_{k_1 k_2 k_3}$$

Using it again on the l, h, s one finally obtains

$$(\det PR)\, \epsilon_{k_1 k_2 k_3} = (\det P)(\det R)\, \epsilon_{k_1 k_2 k_3}$$

Thus the index calculus furnishes us with the

Theorem (Determinantal Product Theorem)

$$\boxed{(\det P)(\det R) = \det PR}$$

In particular, if $R = P^{-1}$

$$\boxed{\det(P^{-1}) = \frac{1}{\det P}}\,,$$

If P is an orthogonal matrix, for which $P^{-1} = P^{tr}$, this becomes

$$\underbrace{\det P^{tr}}_{\det P} = \frac{1}{\det P}$$

(PA.22.3)

Thus

$$(\det P)^2 = 1$$

or

$$\boxed{\det P = \pm 1 \quad \text{for orthogonal matrices}}$$

The index calculus furnishes us also with explicit expressions for the inverse of a matrix whenever its determinant does not vanish.

Applying the index notation to the totally antisymmetric Levi-Civita symbol ϵ_{mjk}, one calculates the inverse of any invertible 3×3 matrix P by the following 4 steps:

I. $P_{1m} P_{2j} P_{3j} \epsilon_{mjk} = \det P$

$P_{2m} P_{2j} P_{3j} \epsilon_{mjk} = 0$

$P_{3m} P_{2j} P_{3j} \epsilon_{mjk} = 0$

("repeated rows of P")

or more briefly

$$P_{im} P_{2j} P_{3k} \epsilon_{mjk} = \delta_{i1} \times \det P \qquad (1)$$

II. Similarly (i.e. expansion in term of the 2^{nd} and 3^{rd} row of P) one obtains

$$P_{im} P_{3j} P_{1k} \epsilon_{mjk} = \delta_{i2} \times \det P \qquad (2)$$

$$P_{im} P_{1j} P_{2k} \epsilon_{mjk} = \delta_{i3} \times \det P \qquad (3)$$

III. Divide by $\det P$, and Eqs (1)-(3) become

$$\underbrace{P_{im} \cdot \frac{1}{\det P} \epsilon_{mjk} P_{2j} P_{3k}}_{P^{-1}_{m1}} = \delta_{i1}$$

$$\underbrace{P_{im} \cdot \frac{1}{\det P} \epsilon_{mjk} P_{3j} P_{1k}}_{P^{-1}_{m2}} = \delta_{i2}$$

$$\underbrace{P_{im} \frac{1}{\det P} \epsilon_{mjk} P_{1j} P_{2k}}_{P^{-1}_{m3}} = \delta_{i3}$$

IV. These expresions for

$$P^{-1} = \left[P^{-1}_{m1} \; P^{-1}_{m2} \; P^{-1}_{m3} \right]$$

can be given by a single formula. By applying

$$\epsilon_{i j_2 j_3} \epsilon_{n j_2 j_3} = 2! \, \delta_{in}$$

to:

$$\epsilon_{mjk} P_{im} P_{j_2 j} P_{j_3 k} = \det P \, \epsilon_{i j_2 j_3}$$

one obtains

$$P_{im} \epsilon_{mjk} P_{j_2 j} P_{j_3 k} \epsilon_{n j_2 j_3} = 2!(\det P)\, \delta_{in}$$

Comparing this with

$$P_{im} P^{-1}_{mn} = \delta_{in}$$

yields

$$P^{-1}_{mn} = \frac{1}{2!\,\det P} \epsilon_{mjk} P_{j_2 j} P_{j_3 k} \epsilon_{n j_2 j_3}$$

which are the matrix elements of P^{-1}.