Lecture 22

Passive and Active Perspectives; Mathematized and Related.

The Levi-Civita Symbol
The relation between $PT_{\omega}P^{-1}$ and $T_{\omega}$

9) Given a basis $B = \{e_1, e_2, e_3\}$, the coordinate representative $[T_{\omega}]_B$ of the transformation $T_{\omega} : E^3 \to E^3$ with $T_{\omega}(x) = \omega \times x$

is defined by

$T_{\omega}(x) = [e_1, e_2, e_3] [T_{\omega}]_B [x]_B$.

Apply this definition to the formula for $T_{\omega}$. One obtains

$T_{\omega}(x) = \omega \times x$

$= \begin{bmatrix} e_1 & e_2 & e_3 \\ \omega^1 & \omega^2 & \omega^3 \\ x^1 & x^2 & x^3 \end{bmatrix} = e_1(\omega^2 x^3 - \omega^3 x^2) + e_2(\omega^3 x^1 - \omega^1 x^3) + e_3(\omega^1 x^2 - \omega^2 x^1)$

$= \begin{bmatrix} e_1 & e_2 & e_3 \\ 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix} [x]_B$

$[T_{\omega}]_B = [T_{\omega}]_B$
The coordinate representative \([T_{\mathbf{w}}]_B\) of the new transformation \(T_{\mathbf{w}}\) is defined in the same way, and is also expressed by an antisymmetric 3x3 matrix:

\[
[T_{\mathbf{w}}]_B = \begin{bmatrix}
0 & -(P\mathbf{w})^3 & (P\mathbf{w})^2 \\
(P\mathbf{w})^3 & 0 & -(P\mathbf{w})^2 \\
-(P\mathbf{w})^2 & (P\mathbf{w})^1 & 0
\end{bmatrix} \equiv [T'_{\mathbf{w}}]_B
\]

Note: For typographical reasons we refer to the matrix elements of \([T_{\mathbf{w}}]_B\) as \(T_{ij}\), while those of \([T_{\mathbf{w}}]_B\) simply as \(T'_{ij}\).

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b) The Levi-Civita symbol is every efficient way of achieving the goal of mathematizing the relation between the active and the passive perspective (Question 2 on page 21.4) is by means of the index calculus applied to the totally anti-symmetric Levi-Civita symbol \( \varepsilon_{i j k l} \) for \( \mathbb{E}^3 \). It is defined as follows:

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Definition ("The totally antisymmetric
L.-C. epsilon")

The Levi-Civita epsilon (for \( E^3 \)) has
three indices and the following values:

\[
\varepsilon_{123} = 1
\]
\[
\varepsilon_{mke} = \pm 1 \text{ if } (mke) \text{ is an (even)}
\]
\[
\text{permutation of } 123
\]
\[
= 0 \text{ if any two indices repeat.}
\]

Thus

\[
\varepsilon_{mke} \varepsilon_{k \ell m} = \varepsilon_{k \ell m} \varepsilon_{mke} = -\varepsilon_{ellm} = -\varepsilon_{km} = -\varepsilon_{kle} = -\varepsilon_{mke}
\]

Comment:
The 3-dimensional L-C epsilon is related
to the Kronecker delta by means
of the following equations:

\[
\sum_{k=1}^{3} \varepsilon_{k \ell m} \varepsilon_{k \ell m} = 2! \delta_{mm}
\]
\[
\sum_{t=1}^{3} \varepsilon_{ijt} \varepsilon_{kt} = \delta_{i}^{t} \delta_{j}^{t} - \delta_{i}^{j} \delta_{j}^{t}
\]
Apply the L-C. symbol to the antisymmetric coordinate representative

\[
[T_{ij}]_B = \begin{bmatrix} 0 & -\omega^2 & \omega^1 \\ \omega^2 & 0 & -\omega^1 \\ -\omega^1 & \omega^1 & 0 \end{bmatrix}
\]

There are only three non-zero elements

\[
T_{23} = -\omega^1 (=-T_{32}) \\
T_{31} = -\omega^2 (=-T_{13}) \\
T_{12} = -\omega^3 (=-T_{21})
\]

One obtains

\[
[T_{kl}]_B = \omega^1 \epsilon_{k1l} + \omega^2 \epsilon_{k2l} + \omega^3 \epsilon_{k3l}
\]

\[= -\omega^m \epsilon_{mkl} \quad (3 \times 3)
\]

Thus any antisymmetric matrix \([T_{kl}]_B\) can be expanded in terms of the standard Levi-Civita basis

\[
[T_{kl}]_B = -\omega^1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \omega^2 \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \omega^3 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
\epsilon_{1kl} \quad \epsilon_{2kl} \quad \epsilon_{3kl}
\]

namely \(\{\epsilon_{1kl}, \epsilon_{2kl}, \epsilon_{3kl}\}\).
c) The introduction of Levi-Civita into

\[ [T^c_w]_b = P_{c \leftarrow b} [T^c_w]_b \quad P_{b \leftarrow c} = P [T^c_w]_b P \]

yields

\[ T^i_{ij} = [P [T^c_w] P']_{ij} = P_{ik} T^c_{kl} P_{lj} \]

\[ = T^i_{kj} P_{ik} P_{lj} \]

\[ = -\omega^m E_{m kl} P_{ik} P_{lj} \quad (= -T^i_{k j}) \]

From the appendix to this lecture on page 7.22.7 we have that the inverse \( P' \) of any 3x3 matrix \( P \) has matrix elements

\[ E_{m kl} P_{ik} P_{lj} = (\det P) P'_{m1} = (\det P) P_{m1} \]

\[ E_{m kl} P_{ik} P_{lj} = (\det P) P'_{m2} = (\det P) P_{m2} \]

\[ E_{m kl} P_{ik} P_{lj} = (\det P) P'_{m3} = (\det P) P_{m3} \]

(only for orthogonal matrices)
This one has

\[ T'_{23} = \left[ P \left[ T_w \right]_B P^{-1} \right]_{23} = -\omega^m (\det P) P_{1m} = -T'_{32} \quad (1) \]

\[ T'_{31} = \left[ P \left[ T_w \right]_B P^{-1} \right]_{31} = -\omega^m (\det P) P_{2m} = -T'_{13} \quad (2) \]

\[ T'_{12} = \left[ P \left[ T_w \right]_B P^{-1} \right]_{12} = -\omega^m (\det P) P_{3m} = -T'_{21} \quad (3) \]

d) Compare these non-zero matrix elements of \( P \left[ T_w \right]_B P^{-1} \) induced by the passive perspective with those of \( \left[ T_{Pw} \right]_B \) induced by the active perspective. The latter matrix is

\[
\left[ T_{Pw} \right]_B = \begin{bmatrix}
0 & -(p\omega)^3 & (p\omega)^2 \\
(p\omega)^3 & 0 & -(p\omega)^1 \\
-(p\omega)^2 & (p\omega)^1 & 0
\end{bmatrix}_B
\]
Its non-zero elements are

\[
\begin{align*}
[T_{p,w}]_{23} &= -(P_w)^2 = -P_{1,m} \omega^m \\
[T_{p,w}]_{31} &= -(P_w)^2 = -P_{2,m} \omega^m \\
[T_{p,w}]_{12} &= -(P_w)^3 = -P_{3,m} \omega^m
\end{align*}
\]  

(4)  

(5)  

(6)

The comparison with Eqs (1) - (3) on p. 21, 17 shows that, except for the scalar factor \(\text{det} P\), they coincide with the equations (4) - (6) above. Indeed, one has

\[
P \begin{bmatrix} T_{p,w} \end{bmatrix}_P = (\text{det} P) \begin{bmatrix} T_{p,w} \end{bmatrix}_B
\]

where \(P\) is an orthogonal matrix

\[
P P^\text{transpose} = I,
\]

The boxed equation mathematizes the relation between the passive and the active perspective.