Lecture 22

Passive and Active Perspectives:
Mathematized and Related

The Levi-Civita Symbol

Active and Passive Perspectives:

The Relation between $PT_{\vec{\omega}}P^{-1}$ and $T_{P\vec{\omega}}$

a) Given a basis $B=\{e_1, e_2, e_3\}$, the coordinate representative $[T_{\vec{\omega}}]_B$ of the transformation

$$T_{\omega} : \quad E^3 \longrightarrow E^3$$
$$\vec{x} \rightsquigarrow T_{\vec{\omega}}(\vec{x}) \equiv \vec{\omega}\times\vec{x}$$

is defined by

$$T_{\vec{\omega}}(\vec{x}) = [e_1\; e_2\; e_3]\,[T_{\omega}]_B\,[\vec{x}]_B\,.$$

Apply this definition to the formula for $T_{\vec{\omega}}$. One obtains

$$T_{\vec{\omega}}(\vec{x}) = \vec{\omega}\times\vec{x}$$
$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ \omega^1 & \omega^2 & \omega^3 \\ x^1 & x^2 & x^3 \end{vmatrix} = e_1(\omega^2x^3-\omega^3x^2) + e_2(\omega^3x^1-\omega^1x^3) + e_3(\omega^1x^2-\omega^2x^1)$$

$$= [e_1\; e_2\; e_3]\underbrace{\begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}_B}_{[T_{\vec{\omega}}]_B \equiv [T_{ij}]}\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}_B$$

The coordinate representative $[T_{p\vec{\omega}}]_B$ of the new transformation $T_{p\vec{\omega}}$ is defined in the same way, and is also expressed by an antisymmetric 3×3 matrix,

$$[T_{p\vec{\omega}}]_B = \begin{bmatrix} 0 & -(p\vec{\omega})^3 & (p\vec{\omega})^2 \\ (p\vec{\omega})^3 & 0 & -(p\vec{\omega})^1 \\ -(p\vec{\omega})^2 & (p\vec{\omega})^1 & 0 \end{bmatrix} \equiv [T'_{ij}]$$

Note: For typographical reasons we will refer to the matrix elements of $[T_{p\vec{\omega}}]_B$ as T'_{ij}, while those of $[T_{\vec{\omega}}]_B$ simply as T_{ij}.

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b) The Levi-Civita symbol

A very efficient way of achieving the goal of mathematizing the relation between the active and the passive perspective (Question 2 on page 21.4) is by means of the index calculus applied to the totally anti-symmetric Levi-Civita symbol ϵ_{mkl} for E^3. It is defined as follows:

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Definition ("The totally antisymmetric L.-C. epsilon)

The Levi-Civita epsilon (for E^3) has three indices and the following values

$\epsilon_{123} = 1$

$\epsilon_{mk\ell} = \pm 1$ if $(mk\ell)$ is an $\binom{\text{even}}{\text{odd}}$ permutation of 123

$= 0$ if any two indices repeat.

Thus

$\epsilon_{mk\ell} = \epsilon_{k\ell m} = \epsilon_{\ell mk} = -\epsilon_{\ell km} = -\epsilon_{km\ell} = -\epsilon_{m\ell k}$

Comment:

The 3-dimensional L-C epsilon is related to the Kronecker delta by means of the following equations:

$$\boxed{\begin{aligned} &\sum_{k=1}^{3}\sum_{\ell=1}^{3} \epsilon_{k\ell m}\epsilon_{k\ell n} = 2!\,\delta_{mn} \\ &\sum_{t=1}^{3} \epsilon_{ijt}\epsilon_{rst} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr} \end{aligned}}$$

Apply the L-C. symbol to the anti-symmetric coordinate representative

$$[T_{\vec{\omega}}]_B = [T_{kl}]_B = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}$$

There are only three non-zero elements

$$\begin{aligned} T_{23} &= -\omega^1 \quad (= -T_{32}) \\ T_{31} &= -\omega^2 \quad (= -T_{13}) \\ T_{12} &= -\omega^3 \quad (= -T_{21}) \end{aligned}$$

One obtains

$$[T_{kl}]_B = \omega^1 \epsilon_{k1l} + \omega^2 \epsilon_{k2l} + \omega^3 \epsilon_{k3l}$$

$$= -\omega^m \epsilon_{mkl} \, .$$

Thus any antisymmetric 3×3 matrix $[T_{kl}]_B$ can be expanded in term of the standard Levi-Civita basis

ϵ_{123} ϵ_{213} ϵ_{312}

$$[T_{kl}]_B = -\omega^1 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\epsilon_{1kl}} - \omega^2 \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\epsilon_{2kl}} - \omega^3 \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\epsilon_{3kl}}$$

namely $\{\epsilon_{1kl}, \epsilon_{2kl}, \epsilon_{3kl}\}$.

c) The introduction of Levi-Civita

into

$$[T_\omega]_C = P_{C\leftarrow B}[T_\omega]_B P_{B\leftarrow C} = P[T_\omega]_B P^{-1}.$$

yields

$$T'_{ij} = \left[P[T_\omega]_B P^{-1}\right]_{ij} = P_{ik}T_{kl}P^{-1}_{lj} \qquad P^{-1} = P^{tr}$$
$$= T_{kl}P_{ik}P_{jl}$$
$$= -\omega^m \epsilon_{mkl} P_{ik}P_{jl} \quad (= -T'_{ji})$$

From the appendix to this lecture on page A.22.7 we have that the inverse P^{-1} of any 3×3 matrix P has matrix elements

$$\epsilon_{mkl}P_{2k}P_{3l} = (\det P)P^{-1}_{m1} = (\det P)P_{1m}$$
$$\epsilon_{mkl}P_{3k}P_{1l} = (\det P)P^{-1}_{m2} = (\det P)P_{2m}$$
$$\epsilon_{mkl}P_{1k}P_{2l} = (\det P)P^{-1}_{m3} = (\det P)P_{3m}$$

(only for orthogonal matrices)

This one has

$$T'_{23} = \left[P [T_{\vec{\omega}}]_B P^{-1} \right]_{23} = -\omega^m (\det P)\, P_{1m} = -T'_{32} \quad (1)$$

$$T'_{31} = \left[P [T_{\vec{\omega}}]_B P^{-1} \right]_{31} = -\omega^m (\det P)\, P_{2m} = -T'_{13} \quad (2)$$

$$T'_{12} = \left[P [T_{\vec{\omega}}]_B P^{-1} \right]_{12} = -\omega^m (\det P)\, P_{3m} = -T'_{21} \quad (3)$$

d) Compare these non-zero matrix elements of $P[T_{\vec{\omega}}]_B P^{-1}$ induces by the passive perspective with those of $[T_{P\vec{\omega}}]_B$ induced by the active perspective. The latter matrix is

$$[T_{P\vec{\omega}}]_B = \begin{bmatrix} 0 & -(P\vec{\omega})^3 & (P\vec{\omega})^2 \\ (P\vec{\omega})^3 & 0 & -(P\vec{\omega})^1 \\ -(P\vec{\omega})^2 & (P\vec{\omega})^1 & 0 \end{bmatrix}_B$$

Its non-zero elements are

$$\left[T_{P\vec{\omega}}\right]_{23} = -\left(P\vec{\omega}\right)^1 = -P_{1m}\,\omega^m \qquad (4)$$

$$\left[T_{P\vec{\omega}}\right]_{31} = -\left(P\vec{\omega}\right)^2 = -P_{2m}\,\omega^m \qquad (5)$$

$$\left[T_{P\vec{\omega}}\right]_{12} = -\left(P\vec{\omega}\right)^3 = -P_{3m}\,\omega^m \qquad (6)$$

The comparison with Eqs (1)-(3) on p 21.17 shows that, except for the scalar factor det P, they coincide with the equation (4)-(6) above. Indeed, one has

$$\boxed{P\left[T_{\vec{\omega}}\right]_{B} P^{-1} = (\det P)\cdot\left[T_{P\vec{\omega}}\right]_{B}}$$

where P is an orthogonal matrix

$$P\,P^{\text{transpose}} = I.$$

The boxed equation mathematizes the relation between the passive and the active perspective.