LECTURE 22 (Revised)

Active-Passive Relation
via
the Levi Civita symbol

Active and Passive Perspectives:

The Relation between $P T_{\vec{\omega}} P^{-1}$ and $T_{P\vec{\omega}}$

a) Given a basis $B = \{e_1, e_2, e_3\}$, the coordinate representative $[T_{\vec{\omega}}]_B$ of the transformation

$$T_{\omega} : \quad E^3 \longrightarrow E^3$$
$$\vec{x} \rightsquigarrow T_{\vec{\omega}}(\vec{x}) \equiv \vec{\omega} \times \vec{x}$$

is defined by

$$T_{\vec{\omega}}(\vec{x}) = [e_1\ e_2\ e_3]\,[T_{\omega}]_B\,[\vec{x}]_B \,.$$

Apply this definition to the formula for $T_{\vec{\omega}}$. One obtains

$$T_{\vec{\omega}}(\vec{x}) = \vec{\omega} \times \vec{x}$$
$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ \omega^1 & \omega^2 & \omega^3 \\ x^1 & x^2 & x^3 \end{vmatrix} = e_1(\omega^2 x^3 - \omega^3 x^2) + e_2(\omega^3 x^1 - \omega^1 x^3) + e_3(\omega^1 x^2 - \omega^2 x^1)$$

Note: $\boxed{\epsilon_{ikl}\,\epsilon_{imn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}}$

$$T_{\vec\omega}(\vec x) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \omega^1 & \omega^2 & \omega^3 \\ x^1 & x^2 & x^3 \end{vmatrix} = e_1\omega^2x^3 - e_1\omega^3x^2 + e_2\omega^3x^1 - e_2\omega^1x^3 + e_3\omega^1x^2 - e_3\omega^2x^1$$

Def'n (The totally antisymm. L-C. symbol)

$$\epsilon_{123} = 1$$

$$\epsilon_{mkl} = \pm 1 \quad \text{if } mkl \text{ is an } \begin{matrix}\text{even}\\ \text{odd}\end{matrix} \text{ permutation of } 123$$

$$= 0 \quad \text{if any two indices repeat}$$

$$\epsilon_{mkl} = \epsilon_{klm} = \epsilon_{lmk} = -\epsilon_{lkm} = -\epsilon_{kml} = -\epsilon_{mlk}$$

Thus, in terms of the Levi-Civita symbol we have

a) $T_{\vec\omega}(\vec x) = \vec\omega \times \vec x = e_i\,\epsilon_{ijk}\,\omega^j x^k$ — index notation

$$= [e_1\ e_2\ e_3] \underbrace{\begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}_{\mathcal{B}}}_{[T_{kl}]_{\mathcal{B}}} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}_{\mathcal{B}}$$ — matrix notation

$$\overbrace{-\omega^1 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\epsilon_{1kl}} - \omega^2 \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\epsilon_{2kl}} - \omega^3 \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\epsilon_{3kl}}}^{[T_{kl}]_{\mathcal{B}}} = -\omega^m \epsilon_{mkl}$$

ϵ_{123} ϵ_{213} ϵ_{312}

and

b) $$T_{p\vec{\omega}}(\vec{x}) = p\vec{\omega}\times\vec{x} = [e_1\ e_2\ e_3]\underbrace{\begin{bmatrix} 0 & -(p\vec{\omega})^3 & (p\vec{\omega})^2 \\ (p\vec{\omega})^3 & 0 & -(p\vec{\omega})^1 \\ -(p\vec{\omega})^2 & (p\vec{\omega})^1 & 0 \end{bmatrix}_B}_{[T'_{k\ell}]_B \equiv [T_{p\vec{\omega}}]_B}\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}_B$$

Explicitly we have

$$\left.\begin{aligned} [T'_{23}]_B &= -(p\vec{\omega})^1 \\ [T'_{31}]_B &= -(p\vec{\omega})^2 \\ [T'_{12}]_B &= -(p\vec{\omega})^3 \end{aligned}\right\} (\star)$$

The "prime" on $[T'_{k\ell}]_B$ serves as a reminder that the transformation $T_{p\vec{\omega}}$ is induced by the vector $p\vec{\omega}$, and not by $\vec{\omega}$, which induces the transformation $T_{\vec{\omega}}$.

Our goal is to establish a relation between the numerical matrix elements of $[T_{p\vec{\omega}}]_B$ and those of $[T_{\vec{\omega}}]_C \equiv P[T_{\vec{\omega}}]_B P^{-1}$.

Thus let us exhibit those of $[T_{\vec{\omega}}]_C$.

$$[T_{ij}]_C = \left[P[T_\omega]_B P^{-1}\right]_{ij} = P_{ik}\, T_{k\ell}\, P^{-1}_{\ell j} \quad P^{-1} = P^{tr}$$

$$= T_{k\ell}\, P_{ik}\, P_{j\ell}$$

$$= -\omega^m \epsilon_{mk\ell}\, P_{ik}\, P_{j\ell} \qquad (\star\star)$$

From Appendix 22 <u>the inverse</u> P^{-1} of any 3×3 matrix P has matrix elements

$$P_{1m} = \frac{1}{(\det P)} \epsilon_{mk\ell} P_{2k} P_{3\ell} = \frac{(-1)}{\det P} \epsilon_{mk\ell} P_{3k} P_{2\ell}$$

$$P_{2m} = \quad '' \quad 3 \quad 1 \quad = (-1) \quad 1 \quad 3$$

$$P_{3m} = \quad '' \quad 1 \quad 2 \quad = (-1) \quad 2 \quad 1$$

$$= (0) \quad 1 \quad 1$$

$$= (0) \quad 2 \quad 2$$

$$= (0) \quad 3 \quad 3$$

Thus the r.h.s. of Eq. $(\star\star)$ becomes

$$[T_{23}]_C = \left[P[T_{\vec\omega}]_B P^{-1}\right]_{23} = -\omega^m (\det P)\, P_{1m} = -(\det P)(P\vec\omega)^1$$

$$[T_{31}]_C = \quad '' \quad 31 = - \quad '' \quad P_{2m} = -(\det P)(P\vec\omega)^2$$

$$[T_{12}]_C = \quad '' \quad 12 = - \quad '' \quad P_{3m} = -(\det P)(P\vec\omega)^3$$

$$(\star\star\star)$$

Compare (★ ★★) with (★) on P-2- and obtain the relation

$$\boxed{P\left[T_{\vec{\omega}}\right]_B P^{-1} = (\det P)\left[T_{P\vec{\omega}}\right]_B}$$

between the passive transformation matrix elements and the active transformation matrix elements.

In the first case it is the basis vectors (but not $\vec{\omega}$) which get rotated by P, while in the second case it is the vector $\vec{\omega}$ (and not the basis vectors) which get rotated by P.

The matrix P is understood to be an orthogonal matrix: $P^{tr} = P^{-1}$.