LECTURE 23+1

1. Existence of eigenvectors on nontrivial subspaces invariant under a linear transformation

2. Two ways of solving $L\vec{x} = \vec{a}$

3. Commuting pairs of transformations:

   (i) Their mutual invariant subspaces,

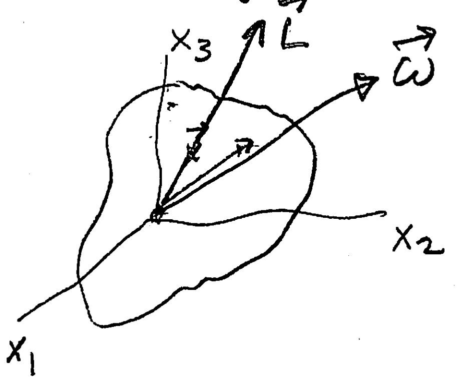
   (ii) Their common eigenvectors

Consigned to LECTURE 24

When a rigid body moves with one point fixed, the total angular momentum $\vec{L}$ has components relative to a given basis B:

$$[\vec{L}]_B = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}_B = \begin{bmatrix} I_{11} & I_{21} & I_{31} \\ I_{12} & I_{22} & I_{32} \\ I_{13} & I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

with $I_{ij} = I_{ji} = \iiint x_i x_j \rho(x_1, x_2 x_3)\, d^3x$



Note on mandatory reading parallel to

Lectures 22+1 and 23+1

1. For Lecture 22+1, the three theorems and the two corollaries on pages 57-61 in the handout from B. Friedman's "Principles and Techniques of Applied Mathematics" should have been chewed by now.

2. For Lecture 23+1 B. Friedman's pages 61-65 ff need to be chewed.

3. Know how to solve $Lx = a$ (on p 62) in two different ways. (Also see p23.8)

I. Existence of Eigenvectors.

One of the most important, i.e. consequential, if not *the* most important properties of a linear transformation

$$L : U \to U$$

is the set of solutions to its eigenvalue problem

$$L\vec{x} = \lambda\vec{x} \Longleftrightarrow (L - \lambda I)\vec{x} = \vec{0}.$$

If there exists a *non-zero* $\vec{x}$ which satisfies this equation for some scalar λ, then that scalar is called an *eigenvalue* (or *characteristic value*) and the vector(s) is/are the corresponding *eigenvector(s)* (or *characteristic vector(s)*) of L.

For a given eigenvalue λ there may be one or more eigenvectors. This is made precise by the following

Definition 23.1

The dimensionality of the null space of $L-\lambda I$ is called the multiplicity of the eigenvalue λ:

$$\dim \mathcal{N}(L-\lambda I) = \text{"multiplicity of } \lambda\text{"}$$

If $\dim \mathcal{N}(L-\lambda I) > 1$, then λ is said to be degenerate.

Comment

There are some matrices, namely "defective" matrices such as $L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which have no eigenvectors. But most linear transformations do. Indeed, if $\mathcal{M}$ is a subspace invariant under L and $\dim \mathcal{M} = k \geq 1$, then we have the following

Theorem 23.1 (Existence of eigenvectors)

Let $\mathcal{M} \subset U$ be a non-trivial, but finite dimensional, subspace invariant under L, i.e.

invariant under L

$$\boxed{\dim \mathcal{M} = k \geq 1.}$$

Then $\exists$ at least one eigenvector of L in $\mathcal{M}$:

$$\exists \vec{x} : \quad L x = \lambda x \quad \text{for some } \lambda.$$

Proof (in three steps):

Step I: For any $\vec{x} \in \mathcal{M}$ the representation theorem guarantees that L can be represented by its matrix representative A

$$[L\vec{x}]_c = A[\vec{x}]_c$$

Here A is the $k \times k$ representation matrix of L

relative to the k-dimensional basis C

in $\mathcal{M}$.

Step II

The fact that $\vec{x}$ is an eigenvector of $L \iff$

$\exists$ a non-zero $[\vec{x}]_c$ such that

$$(A - \lambda I)[x]_c = 0 \qquad (1)$$

or in index notation

$$\sum_{j=1}^{k} (a_{ij} - \lambda \delta_{ij}) x_j = 0 \qquad i = 1, \ldots, k$$

for some value of λ.

Step III

Equation (1) has a non-trivial solution

$$\iff \det |a_{ij} - \lambda \delta_{ij}| \equiv p(\lambda) = 0$$

$p(\lambda)$ is a k-order (≥ 1) polynomial in λ. Such

X a $p(\lambda)$ has at least one root. Consequently,

A, and therefore L, always has at least one eigenvector $\vec{x}$ in $\mathcal{M}$. Q.E.D.

Example: (Solution to eigenvalue problem and an important application)

Eigenvalue problem:

Consider $L = A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} : E^2 \to E^2$.

SKIP BUT Give only result 2.8 3 on p 23.7

1. Its eigen value equation $L\vec{x} = \lambda \vec{x}$ is

$$\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda I \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\boxed{\begin{aligned} (3-\lambda)x_1 + 4x_2 &= 0 \\ x_1 + (3-\lambda)x_2 &= 0 \end{aligned}} \qquad \left[L - \lambda I\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\dim(L - \lambda I)$

2. The characteristic polynomial is

$$p(\lambda) = (3-\lambda)^2 - 4 \qquad = \det \begin{vmatrix} 3-\lambda & 4 \\ 1 & 3-\lambda \end{vmatrix}$$
$$=$$
$$= \lambda^2 - 6\lambda + 5$$
$$= (\lambda - 1)(\lambda - 5).$$

There are two roots: $\lambda_1 = 1$ and $\lambda_2 = 5$.

3. There are two eigenvectors

For $\lambda_1 = 1$

$$2x_1 + 4x_2 = 0 \Rightarrow \boxed{\vec{X}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}} \text{ or any multiple thereof}$$

$\mathcal{N}(A - \lambda_1 I) = Sp(\{[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}]\})$

For $\lambda_2 = 5$

$$-2x_1 + 4x_2 = 0 \Rightarrow \boxed{\vec{X}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}} \text{ or any multiple thereof.}$$

$\mathcal{N}(A - \lambda_2 I) = Sp\left(\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}\right)$

23.7

II. Solving $L\vec{x}=\vec{a}$ by the Method of Eigenvectors; Application of the solution to the e.v. problem:

1. The eigenvectors form a lin. indep. set.

Thus any vector $\vec{x}$ is a linear combination of $\vec{x}_1$ and $\vec{x}_2$:

$$\boxed{\vec{x} = \eta_1 \vec{x}_1 + \eta_2 \vec{x}_2}$$

and $L\vec{x} = \eta_1 \vec{x}_1 + 5\eta_2 \vec{x}_2$

2. Given $\vec{a} = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2$, consider the problem of solving the inhomogeneous system of equations

$$L\vec{x} = \vec{a} = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2$$

for $\vec{x}$.

3. Solution:

$$\lambda_1 \eta_1 \vec{x}_1 + \lambda_2 \eta_2 \vec{x}_2 = L\vec{x} = \vec{a} = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2$$

$\{\vec{x}_1, \vec{x}_2\}$ is lin. indep. $\Longrightarrow$ $\eta_1 = \frac{\alpha_1}{\lambda_1} = \alpha_1$

$\eta_2 = \frac{\alpha_2}{\lambda_2} = \frac{\alpha_2}{5}$

Answer: $\boxed{\vec{x} = \frac{\alpha_1}{\lambda_1}\vec{x}_1 + \frac{\alpha_2}{\lambda_2}\vec{x}_2} = \alpha_1 \vec{x}_1 + \frac{\alpha_2}{5}\vec{x}_2$

Comment:

If none of the eigenvalues vanishes, and if the eigenvectors form a basis for U, then there are two ways of solving the system

$$L\vec{x} = \vec{a}.$$

(i) Find the inverse L^{-1} of L: $\vec{x} = L^{-1}\vec{a}$

(ii) Find the eigenvalues and eigenvectors of L: $\vec{x} = \frac{\alpha_1}{\lambda_1}\vec{x}_1 + \frac{\alpha_2}{\lambda_2}\vec{x}_2$